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A passivity-based nonsingular terminal sliding mode controller for mechanical port-Hamiltonian systems

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Abstract— **This paper proposes a novel nonsingular terminal sliding mode controller for mechanical systems based on passivity-based control. In the authors' previous study, passivity-based sliding mode control is realized with kinetic potential energy shaping (KPES), which allows us to construct a wider class of energy-based Lyapunov function candidates. This paper extends KPES to deal with a special class of Lyapunov function candidates whose arguments depend nonlinearly on the momentum. Based on this extension, we propose a nonsingular terminal sliding mode controller that achieves finite time convergence of the closed-loop system with an energy-based Lyapunov function. Due to the passivity-based approach, the proposed controller guarantees Lyapunov stability of the closed-loop system even if the discontinuous control input is replaced with a continuous one to alleviate chattering. A numerical example demonstrates the effectiveness of the proposed method.**

I. Introduction

Passivity-based control is a natural control method that utilizes conserved quantities of the plant systems as Lyapunov function candidates. Such a control method is often used for a port-Hamiltonian system, which is one of the frameworks representing physical systems such as mechanical systems[1], electro-mechanical systems [2], and nonholonomic systems [3]. For them, the Hamiltonian function representing the total energy of the system is employed as a Lyapunov function candidate. To construct a desired Lyapunov function, a potential function may be reshaped by feedback input, which is called energy shaping. In particular, for mechanical systems, many methods have been proposed to construct a desired Lyapunov function based on the passivity-based approach, e.g., [4], [5].

On the other hand, sliding mode control is a nonlinear control method classified as variable structure control. See, e.g., [6], [7] for details. This method constrains the state variable of the plant system to a subspace called a sliding surface so that the state evolves along the desired dynamics. This phase is called a sliding mode. The controller includes a discontinuous function so that the sliding variable, which defines the sliding surface, converges to zero in a finite time. This phase is called a reaching mode. Normally, the sliding surface is characterized by linear equations of the state variable, and as a result, the state converges exponentially in the sliding mode. Terminal sliding mode control [8], on the other hand, has been proposed so far to achieve finite time convergence in the sliding mode as well as in the reaching mode by constructing a nonlinear sliding surface. It is also known that the control input may diverge due to singular terms, which is called a singular problem. Some solutions to this problem have been proposed, e.g., [9], [10]. In particular, designing a nonsingular sliding variable approach [11] has been proposed to avoid the singular problem without changing the dynamics in the sliding mode by modifying only the representation of the sliding surface.

The authors' previous results unify passivity-based control and sliding mode control for mechanical systems [12], [13]. These methods are based on kinetic potential energy shaping (KPES) [14], and related works to KPES are studied, e.g., [15], [16]. Using KPES, we can select a sharp-bottom function of the sliding variable as a potential function and realize sliding mode control with Lyapunov stability. However, in the previous studies, the derivative of the sliding variable with respect to the momentum needs to be a nonsingular matrix. This prevents one from designing a nonsingular terminal sliding surface in the port-Hamiltonian framework.

In this paper, we propose a new passivity-based controller for mechanical systems that achieves nonsingular terminal sliding mode control with Lyapunov stability. First, we modify the KPES technique to handle a class of artificial potential functions whose arguments include nonlinear functions of the momentum such that their derivative can be a singular matrix. Such modification allows one to construct a nonlinear sliding surface where the state converges to zero in a finite time without singularity problems. Then, we realize terminal sliding mode control with Lyapunov stability by selecting a sharp-bottom potential function. The sharpness of the potential function corresponds to a high gain discontinuous input, resulting in the state variable being constrained to the sliding surface. Moreover, since the proposed method is based on passivity-based control, the closed-loop system remains Lyapunov stable if the sharp-bottom potential function is replaced with a smooth approximation for alleviating chattering. As opposed to conventional terminal sliding mode controllers, a continuous approximation of input does not lose asymptotic stability of the closed-loop system. A numerical example shows how the proposed controller works.

The remainder of this paper is organized as follows. Section II briefly refers to the background of mechanical port-Hamiltonian systems and terminal sliding mode control. Section III gives the main result of the paper, a new passivity-based controller based on modified KPES is proposed. Sec-

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tion IV demonstrates the effectiveness of the proposed controller by a numerical example. Finally, Section V concludes the paper.

Notation The symbols I_n and 0_n denote the $n \times n$ identity matrix and the $n \times n$ matrix of zeros, respectively. For a vector $x \in \mathbb{R}^n$ and a symmetric matrix $A \in \mathbb{R}^{n\times n}$, we denote Euclidean norm by $\|x\| \equiv \sqrt{x^\top x}$ and the weighted-norm by $\|x\|_A^2 \equiv x^\top Ax$. The general p-norm is defined by $\|x\|_p \equiv (|x_1|^p + \ldots, +|x_n|^p)^{1/p}$ with $p \geq 1$. The symbols $\mathbb{R}_{>0}$ and $\mathbb{R}_{\geq 0}$ denote the set of positive real numbers and that of non-negative real numbers, respectively. For a function $f : \mathbb{R}^n \to \mathbb{R}$, the symbol $\nabla_x f$ denotes the gradient of f with respect to x, that is,

$$\nabla_x f \equiv \frac{\partial f}{\partial x}^\top = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right)^\top.$$

II. Preliminary results

This section introduces the background of the mechanical port-Hamiltonian systems and terminal sliding mode control.

A. Port-Hamiltonian systems

In this paper, we consider fully-actuated mechanical systems described in a port-Hamiltonian form as follows:

$$\begin{pmatrix}\dot q\\ \dot p\end{pmatrix} = \underbrace{\begin{pmatrix}0_m & I_m\\ -I_m & -D_0(q,p)\end{pmatrix}}_{J_0(x)} \begin{pmatrix}\nabla_q H_0\\ \nabla_p H_0\end{pmatrix} + \begin{pmatrix}0_m\\ I_m\end{pmatrix} u,$$
$$H_0(q,p) = \frac{1}{2} p^\top M(q)^{-1} p. \tag{1}$$

Here, $q, p \in \mathbb{R}^m$ are the position and the momentum respectively, and $x = (q^\top, p^\top)^\top$ is the state variable of the system. The symbol u denotes the input force or torque. The matrix $M(q) = M(q)^\top \in \mathbb{R}^{m\times m}$ represents the inertia matrix of the system that is uniformly positive definite, and $D_0(q,p) \succeq 0 \in \mathbb{R}^{m\times m}$ denotes the damping coefficient. The symbol $H_0(q,p)$ is the Hamiltonian function which represents the total physical energy of the system. The matrix $J_0(x) \in \mathbb{R}^{2m\times 2m}$ is called the structure matrix of the port-Hamiltonian system (1).

B. Momentum transformation

A coordinate transformation is often applied for port-Hamiltonian systems (1) so that the kinetic energy does not depend on the position q. See, e.g., [15], [16]. First, let us apply Cholesky factorization of $M(q)^{-1}$ as

$$M(q)^{-1} = T(q)T(q)^\top, \tag{2}$$

where $T(q) \in \mathbb{R}^{m\times m}$ is a nonsingular matrix. Then the change of coordinates

$$\chi \equiv \begin{pmatrix}q\\ \eta\end{pmatrix} = \begin{pmatrix}q\\ T(q)^\top p\end{pmatrix} \tag{3}$$

converts the system (1) into a new port-Hamiltonian system

$$\begin{pmatrix}\dot q\\ \dot \eta\end{pmatrix} = \underbrace{\begin{pmatrix}0_m & T(q)\\ -T(q)^\top & -D(q,\eta)\end{pmatrix}}_{J(\chi)} \begin{pmatrix}\nabla_q H(\eta)\\ \nabla_\eta H(\eta)\end{pmatrix} + \begin{pmatrix}0_m\\ T(q)^\top\end{pmatrix} u,$$
$$H(\eta) = \frac{1}{2}\|\eta\|^2, \tag{4}$$

where $D(q,\eta) \in \mathbb{R}^{m\times m}$ is defined by

$$D(q,\eta) = \sum_{i=1}^{m}\Big[(T^\top e_i)(\eta^\top T^{-1}\nabla_{q_i} T) - (\eta^\top T^{-1}\nabla_{q_i} T)^\top (T^\top e_i)^\top\Big] + T^\top D_0 T, \tag{5}$$

with $e_i \in \mathbb{R}^m$ which is the i-th Euclidean basis vector of $\mathbb{R}^m$. The transformed system has a new Hamiltonian function H which is independent of the position q. Then, by modifying the structure matrix $J(\chi)$ appropriately, we can choose a potential function $U_{\rm kp}$ that depends on both the position q and the momentum η, which is called kinetic-potential energy shaping (KPES). For example, $U_{\rm kp} = (\alpha q + \eta)/2$ with $\alpha > 0$ is selected in [14] and $U_{\rm kp} = U(\phi(q) + \eta)$ is selected in [12], where ϕ is a diffeomorphism. Using this technique, we can select potential functions more freely, but their arguments depend linearly on the momentum η in the previous results.

C. Terminal sliding mode control

Sliding mode control is known as a robust nonlinear control method. See, e.g., [6], [7] for details. The feature of the method is to constrain the state variable to a subspace called sliding surface where the state evolves along the desired dynamics.

Let us consider a second-order scalar system

$$\begin{aligned}\dot x_1 &= x_2,\\ \dot x_2 &= f(x) + g(x)u.\end{aligned} \tag{6}$$

The functions $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ are known and $g(x) \neq 0,\ \forall x \in \mathbb{R}^2$. In order to make the state variable reach a sliding surface in a finite time and enforce it to the surface, discontinuous high gain feedback input is employed. As a result, the closed-loop system includes the following dynamics

$$\dot\sigma = -\beta \operatorname{sgn}\sigma, \quad \beta > 0. \tag{7}$$

Here, $\operatorname{sgn}(\cdot)$ is the signum function defined by

$$\operatorname{sgn}(z)\begin{cases} = 1, & z > 0,\\ \in [-1,1], & z = 0,\\ -1, & z < 0.\end{cases} \tag{8}$$

For a vector $x \in \mathbb{R}^m$, $\operatorname{sgn}(x)$ is defined by $\operatorname{sgn}(x) = (\operatorname{sgn}(x_1), \ldots \operatorname{sgn}(x_m))^\top$. The sliding variable $\sigma(x)$ is usually designed by a linear combination of x such as $\sigma(x) = x_2 + 0.7x_1$ as shown in Fig. 1. Then, the state variable converges exponentially on the surface.

Terminal sliding mode control [8] realizes finite time convergence of the state variable in the sliding mode. To achieve this, the sliding surface is selected as

$$\sigma(x) = x_2 + k\lfloor x_1 \rceil^{\frac{1}{\alpha}} = 0, \quad 1 < \alpha, \quad k > 0. \tag{9}$$

Here, the function $\lfloor\cdot\rceil^y$ is defined by $\lfloor z\rceil^y = |z|^y \operatorname{sgn} z$ for $y, z \in \mathbb{R}$ ([17]). By designing such a nonlinear sliding surface,

$$x_2 = -k\lfloor x_1\rceil^{\frac{1}{\alpha}} \tag{10}$$

holds in the sliding mode and the state converges to the origin in a finite time. The nonlinear surface and the responses of

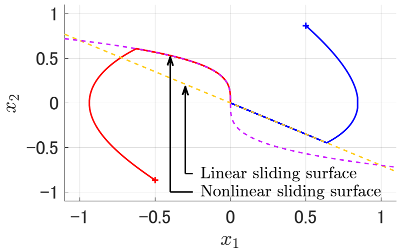


Fig. 1. The trajectory of the states via sliding mode control with a linear sliding surface and a nonlinear (terminal) sliding surface.

the state variable in the phase plane are also shown in Fig. 1. To achieve terminal sliding mode, the input is designed as

$$u = g(x)^{-1}(-f(x) - k\alpha^{-1}|x_1|^{\frac{1}{\alpha}-1}x_2 - \beta \operatorname{sgn}\sigma), \quad (11)$$

but it may diverge because of the singular term $|x_1|^{\frac{1}{\alpha}-1}$, which is referred to as a singular problem.

Nonsingular terminal sliding mode control [11] can avoid the singular problem. The idea of the method is to employ the sliding surface which is nonlinear with respect to x_2 as

$$\sigma(x) = \lfloor x_2 \rceil^{\alpha} + kx_1 = 0, \quad 1 < \alpha < 2, \quad (12)$$

which implies that the same dynamics as (10) is realized in the sliding mode ($\sigma(x) = 0$). The control input is given by

$$u = g(x)^{-1}(-f(x) - k\alpha^{-1}\lfloor x_2 \rceil^{2-\alpha} - \beta \operatorname{sgn}\sigma), \quad (13)$$

and the singular term vanishes. In this way, the state variable converges in a finite time without the singular problem.

III. Passivity-Based Terminal Sliding Mode

This section gives the main result of the paper. A new terminal sliding mode controller for port-Hamiltonian systems is proposed.

A. KPES for a special class of potential functions

This subsection proposes a new KPES method where we can select potential functions whose arguments consist of nonlinear functions of η such that their derivatives can be singular matrices. The advantage of using KPES is to make the symmetric part of $J(\chi)$ negative definite, and as a result, the time derivative of the Hamiltonian function is strictly negative definite. In the proposed method, on the other hand, the time derivative of the Hamiltonian function is negative semi-definite because of the selection of another class of potential functions. Although asymptotic stability of the closed-loop system cannot directly be guaranteed by the Hamiltonian function, the following lemma proves asymptotic stability of the closed-loop port-Hamiltonian system with a special class of potential function and a negative semi-definite structure matrix under some assumptions.

Lemma 1: Consider the system (4). For any function $U_{\mathrm{kp}} : \mathbb{R}^m \to \mathbb{R}$, any matrix $D_{\mathrm{d}}(q,\eta) \in \mathbb{R}^{m\times m}$, any diffeomorphism $\phi : \mathbb{R}^m \to \mathbb{R}^m$ satisfying $\phi(0) = 0$, and a function $\psi : \mathbb{R}^m \to \mathbb{R}^m$ satisfying $\psi(0) = 0$ and

$$\frac{\partial \psi(\eta)}{\partial \eta} = \operatorname{diag}\big(\mu_1(\eta_1), \ldots, \mu_m(\eta_m)\big) \quad (14)$$

with functions μ_i's where $\mu_i : \mathbb{R} \to \mathbb{R}_{\geq 0}$, the feedback input

$$u = T(q)^{-\top}\Big\{\big(D(q,\eta) - D_{\mathrm{d}}(q,\eta)\big)\eta - \Big(T(q)^{\top} + D_{\mathrm{d}}(q,\eta)\frac{\partial \psi(\eta)}{\partial \eta}\frac{\partial \phi(q)}{\partial q}^{-\top}\Big)\nabla_q U_{\mathrm{kp}}(\phi(q) + \psi(\eta))\Big\} \quad (15)$$

converts (4) into the closed-loop port-Hamiltonian system

$$\begin{pmatrix} \dot q \\ \dot\eta \end{pmatrix} = \underbrace{\begin{pmatrix} -T(q)\frac{\partial \psi(\eta)}{\partial \eta}\frac{\partial \phi(q)}{\partial q}^{-\top} & T(q) \\ -T(q)^{\top} & -D_{\mathrm{d}}(q,\eta) \end{pmatrix}}_{J_{\mathrm{kp}}(\chi)} \begin{pmatrix} \nabla_q H_{\mathrm{kp}} \\ \nabla_\eta H_{\mathrm{kp}} \end{pmatrix},$$

$$H_{\mathrm{kp}}(q,\eta) = \frac{1}{2}\|\eta\|^2 + U_{\mathrm{kp}}(\phi(q) + \psi(\eta)). \quad (16)$$

Furthermore, if U_{kp} is positive definite and smooth and its derivative satisfies $\nabla_\sigma U_{\mathrm{kp}}(\sigma) = 0$ if and only if $\sigma = 0$, and if

$$\frac{\partial \phi(q)}{\partial q}T(q) = \operatorname{diag}(a_1(q), \ldots, a_m(q)) \succ 0, \quad (17)$$

$$D_{\mathrm{d}}(q,\eta) + D_{\mathrm{d}}(q,\eta)^{\top} \succ 0, \quad (18)$$

hold with functions a_i's where $a_i : \mathbb{R}^m \to \mathbb{R}_{>0}$, then the origin of the closed-loop system (16) is asymptotically stable with the Lyapunov function H_{kp}.

Proof. First, the transformation of the system (4) into the one (16) by the feedback input (15) can be proved by a direct calculation with the following equation

$$\nabla_\eta U_{\mathrm{kp}}(\phi(q)+\psi(\eta)) = \frac{\partial \psi(\eta)}{\partial \eta}\frac{\partial \phi(q)}{\partial q}^{-\top}\nabla_q U_{\mathrm{kp}}(\phi(q)+\psi(\eta)).$$

Next, we show asymptotic stability of the closed-loop system (16). Let us consider the Hamiltonian function H_{kp} as a Lyapunov function candidate. Its time derivative along the dynamics (16) is derived as

$$\begin{aligned}\dot H_{\mathrm{kp}} &= \nabla_\chi H_{\mathrm{kp}}(\chi)^{\top} J_{\mathrm{kp}}(\chi)\nabla_\chi H_{\mathrm{kp}}(\chi) \\ &= -\sum_{i=1}^{m} a_i(q)\mu_i(\eta_i)(\nabla_{\sigma_i}U(\sigma))^2 \\ &\quad -\frac{1}{2}\Big\|\frac{\partial \psi(\eta)}{\partial \eta}^{\top}\nabla_\sigma U(\sigma) + \eta\Big\|^2_{(D_{\mathrm{d}}+D_{\mathrm{d}}^{\top})} \leq 0,\end{aligned}$$

which implies with the assumptions (17) and (18) that the closed system (16) is Lyapunov stable. In particular, if $\dot H_{\mathrm{kp}} = 0$ holds identically, we have

$$a_i(q)\mu_i(\eta_i)(\nabla_{\sigma_i}U(\sigma))^2 = 0, \ \forall i, \quad (19)$$

$$\mu_i(\eta_i)\nabla_{\sigma_i}U(\sigma) + \eta_i = 0, \ \forall i. \quad (20)$$

Equation (19) implies that $\mu_i(\eta_i) = 0$ and/or $\nabla_{\sigma_i}U(\sigma) = 0$ holds for all i, so it follows from (20) that $\eta = 0$. In addition, $\dot\eta = 0$ also holds, so from (16) we have

$$\dot\eta = -T(q)^{\top}\frac{\partial \phi(q)}{\partial q}^{\top}\nabla_\sigma U(\sigma) = 0. \quad (21)$$

As the matrices $T(q)$ and $\partial\phi(q)/\partial q$ are nonsingular, $\nabla_\sigma U(\sigma) = 0$ holds, which implies $\sigma = \phi(q) + \psi(0) = 0$ and then $q = 0$. Therefore, it follows from LaSalle's invariance principle that the origin of the closed-loop system is asymptotically stable. This completes the proof. □

Compared with the previous works, this lemma introduces a free parameter $U_{\rm kp}$ whose argument consists of a nonlinear function $\psi(\eta)$ such that its derivative may be a singular matrix. Such additional freedom allows us to construct terminal sliding mode controllers in the port-Hamiltonian framework.

Remark 1: For mechanical systems, the condition (17) is restrictive in general due to the requirement of solving a partial differential equation such as

$$\frac{\partial \phi(q)}{\partial q} = T(q)^{-1}. \tag{22}$$

As proposed in [12, Lemma 2], it can be satisfied by applying another coordinate and feedback transformations to the system (1). By utilizing the method, the matrix $T(q)$ can be replaced with any nonsingular matrix including a constant one. Therefore, the condition (17) can always be fulfilled by employing this modification.

B. Passivity-based terminal sliding mode control

In the previous subsection, the KPES method is extended to obtain the closed-loop system with a special class of potential functions. By using the extension, nonsingular terminal sliding mode control is realized in the port-Hamiltonian framework.

Inspired by (12), the sliding variable σ is defined as

$$\sigma = \phi(q) + \lfloor \eta \rceil^{\alpha},\ 1 < \alpha < 2, \tag{23}$$

where a function $\lfloor \cdot \rceil^{\alpha} : \mathbb{R}^m \to \mathbb{R}^m$ is defined by $\lfloor x \rceil^{\alpha} = (|x_1|^{\alpha} \operatorname{sgn} x_1, \ldots, |x_m|^{\alpha} \operatorname{sgn} x_m)^{\top}$ for $x \in \mathbb{R}^m$. In addition, the derivative of $\lfloor \cdot \rceil^{\alpha}$ is calculated as

$$\frac{\partial \lfloor \eta \rceil^{\alpha}}{\partial \eta} = \alpha \operatorname{diag}(|\eta_1|^{\alpha-1}, \ldots, |\eta_m|^{\alpha-1}). \tag{24}$$

Notice that the function $\lfloor \cdot \rceil^{\alpha}$ satisfies the condition (14) required in Lemma 1. The proposed nonsingular terminal sliding mode controller is given by the following theorem.

Theorem 1: Consider the system (4). Suppose there exists a diffeomorphism $\phi : \mathbb{R}^m \to \mathbb{R}^m$ satisfying $\phi(0) = 0$ and

$$A(q) \equiv \frac{\partial \phi(q)}{\partial q} T(q) \tag{25}$$

$$= \operatorname{diag}(a_1(q), \ldots, a_m(q)) \succeq \varepsilon I_m \succ 0 \tag{26}$$

with scalar functions $a_i(q)$'s. Then, the feedback input

$$u = -\frac{\partial \phi(q)}{\partial q}^{\top} \left(2\beta \operatorname{sgn} \sigma + \frac{\lfloor \eta \rceil^{2-\alpha}}{\alpha} \right) + T(q)^{-\top} D(q, \eta)\eta \tag{27}$$

with $1 < \alpha < 2$ and $\beta > 0$ converts (4) into the following closed-loop system

$$\begin{pmatrix} \dot{q} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} -T(q) \frac{\partial \lfloor \eta \rceil^{\alpha}}{\partial \eta} \frac{\partial \phi(q)}{\partial q}^{-\top} & T(q) \\ -T(q)^{\top} & -\left(\frac{\partial \lfloor \eta \rceil^{\alpha}}{\partial \eta} \right)^{-1} A(q) \end{pmatrix} \begin{pmatrix} \nabla_q H_{\rm smc} \\ \nabla_{\eta} H_{\rm smc} \end{pmatrix},$$

$$H_{\rm smc}(q, \eta) = \frac{1}{2} \|\eta\|^2 + \beta \|\phi(q) + \lfloor \eta \rceil^{\alpha}\|_1. \tag{28}$$

Moreover, along the closed loop system (28), the sliding variable defined by (23) is enforced to converge to zero in a finite time, and the state variable also converges to the origin in a finite time with a Lyapunov function $H_{\rm smc}$.

Proof. The first claim is directly proved by Lemma 1 with the properties that the matrices $(\partial \phi(q)/\partial q)T(q)$ and $\partial \lfloor \eta \rceil^{\alpha}/\partial \eta$ are diagonal. Note that (28) contains the singular term $(\partial \lfloor \eta \rceil^{\alpha}/\partial \eta)^{-1}$, which cannot be defined when $\eta_i = 0$. In this paper, we denote the matrix satisfying

$$I_m = B \frac{\partial \lfloor \eta \rceil^{\alpha}}{\partial \eta} = \frac{\partial \lfloor \eta \rceil^{\alpha}}{\partial \eta} B, \quad \frac{1}{\alpha} \lfloor \eta \rceil^{2-\alpha} = B\eta. \tag{29}$$

by

$$B = \left(\frac{\partial \lfloor \eta \rceil^{\alpha}}{\partial \eta} \right)^{-1} \tag{30}$$

To simplify the notation, we continue to use $(\partial \lfloor \eta \rceil^{\alpha}/\partial \eta)^{-1}$ in what follows.

Next, to prove finite time convergence of σ, let us apply the following coordinate transformation to the system (28):

$$\xi = \begin{pmatrix} \sigma \\ \eta \end{pmatrix} = \begin{pmatrix} \phi(q) + \lfloor \eta \rceil^{\alpha} \\ \eta \end{pmatrix}. \tag{31}$$

Then, the system (28) can be rewritten as

$$\begin{pmatrix} \dot{\sigma} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} -2A(q) \frac{\partial \lfloor \eta \rceil^{\alpha}}{\partial \eta} & 0_m \\ -2A(q) & -\left(\frac{\partial \lfloor \eta \rceil^{\alpha}}{\partial \eta} \right)^{-1} A(q) \end{pmatrix} \begin{pmatrix} \nabla_{\sigma} H_{\rm smc}^{\xi} \\ \nabla_{\eta} H_{\rm smc}^{\xi} \end{pmatrix},$$

$$H_{\rm smc}^{\xi}(\sigma, \eta) = \frac{1}{2} \|\eta\|^2 + \beta \|\sigma\|_1. \tag{32}$$

The symbol $(\cdot)^{\xi}$ represents $(\cdot)$ expressed in the ξ coordinates. The dynamics of σ and η can be expressed as

$$\dot{\sigma}_i = -2\alpha\beta a_i(q) |\eta_i|^{\alpha-1} \operatorname{sgn} \sigma_i, \tag{33}$$

$$\dot{\eta}_i = -2\beta a_i(q) \operatorname{sgn} \sigma_i - \frac{a_i(q)}{\alpha} \lfloor \eta_i \rceil^{2-\alpha}. \tag{34}$$

Here, let us consider the case that $\eta_i = 0$ and $\sigma_i \neq 0$. In this case, it follows from (33) and (34) that $\dot{\eta}_i \leq -2\beta\varepsilon$ when $\sigma_i > 0$ and $\dot{\eta}_i \geq 2\beta\varepsilon$ when $\sigma_i < 0$. It means that there exists a vicinity $|\eta_i| < \delta$ such that $\dot{\eta}_i \leq -2\beta\varepsilon$ holds with $\sigma_i > 0$ and $\dot{\eta}_i \geq 2\beta\varepsilon$ holds with $\sigma_i < 0$. Thus, $\eta_i = 0$ is not an attractor and the state passes through the vicinity $|\eta_i| < \delta$ in a finite time. On the other hand, it follows outside the vicinity that

$$\dot{\sigma}_i \leq -2\alpha\beta\varepsilon\delta^{\alpha-1},\ \sigma_i > 0, \tag{35}$$

$$\dot{\sigma}_i \geq \ \ 2\alpha\beta\varepsilon\delta^{\alpha-1},\ \sigma_i < 0, \tag{36}$$

which implies the sliding variable σ_i becomes zero in a finite time. This argument holds for all i, so $\sigma = 0$ achieves in a finite time.

In the last part of the proof, let us prove that the state variable converges to zero in a finite time with the Lyapunov function $H_{\rm smc}$. The time derivative of $H_{\rm smc}$ can be derived as

$$\dot{H}_{\rm smc} = \dot{H}_{\rm smc}^{\xi} = -\sum_{i=1}^{m} 2a_i(q) \left(\sqrt{\alpha}\beta |\eta_i|^{\frac{\alpha-1}{2}} \operatorname{sgn} \sigma_i + \frac{\lfloor \eta_i \rceil^{\frac{3-\alpha}{2}}}{2\sqrt{\alpha}} \right)^2 - \frac{1}{2\alpha} \sum_{i=1}^{m} a_i(q) |\eta_i|^{3-\alpha} \leq 0.$$

This equation shows that $H_{\rm smc}$ works as a Lyapunov function of the closed-loop system (28). Moreover, considering the equivalent control system in the sliding mode where $\dot{\sigma} =$

$\sigma = 0$ holds, we obtain

$$\dot{H}^{\xi}_{\rm smc} = -\frac{1}{\alpha}\sum_{i=1}^{m} a_i(q)|\eta_i|^{3-\alpha} \le -\frac{\varepsilon}{\alpha}\|\eta\|^{3-\alpha}_{3-\alpha}$$

By using the inequality of the general p-norm, it follows from $\|\eta\|_2 \le \|\eta\|_{3-\alpha}$ and $H^{\xi}_{\rm smc} = \|\eta\|^2/2$ in the sliding mode that

$$\dot{H}^{\xi}_{\rm smc} \le -\frac{\varepsilon}{\alpha}\|\eta\|^{3-\alpha}_{2} = -\frac{2^{\frac{3-\alpha}{2}}\varepsilon}{\alpha}(H^{\xi}_{\rm smc})^{\frac{3-\alpha}{2}}. \tag{37}$$

This equation shows that the Hamiltonian function $H^{\xi}_{\rm smc}$ converges to zero in a finite time in the sliding mode. Therefore, together with the finite time convergence of the sliding variable σ, the state variable of the closed-loop system converges to zero in a finite time. This completes the proof. □

This theorem shows the unification of passivity-based control and terminal sliding mode control. It allows us to construct an energy-based Lyapunov function that works both inside and outside the sliding manifold. The constants α and β are free parameters. The sliding variable σ converges faster if we take a lager value of β, and the state variable converges faster in the sliding mode if we take a lager value of α.

In addition, we can certainly construct a sliding surface inspired by (9) in the previous frameworks [12], [13] when the parameters ϕ and ψ are selected as

$$\phi(q) = \lfloor q \rceil^{\frac{1}{\alpha}},\ \psi(\eta) = \eta,\ \alpha > 1. \tag{38}$$

In this case, however, the input u diverges as the position q approaches to the origin because there is a singular term $\partial\phi(q)/\partial q$ in the feedback input (27). On the other hand, the proposed KPES method discussed in Lemma 1 allows us to select nonlinear function depending on η and then achieve nonsingular sliding mode control with Lyapunov stability.

The robustness of the proposed controller can be analyzed by using the Hamiltonian function $H^{\xi}_{\rm smc}$ in the similar way to [13], but the details are omitted due to page limitation.

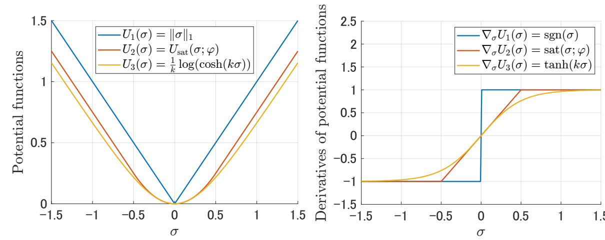
Remark 2: In practical applications of sliding mode control, the signum function $\mathrm{sgn}\,(\cdot)$ is replaced by a continuous function such as the saturation function $\mathrm{sat}\,(\cdot)$ to alleviate the chattering oscillation. Here, the function $\mathrm{sat}\,(\cdot)$ is defined by

$$\mathrm{sat}\,(x;\varphi) = \begin{cases} \mathrm{sgn}\,(x), & |x| > \varphi, \\ x/\varphi, & |x| \le \varphi, \end{cases} \tag{39}$$

where $\varphi > 0$ is a tuning parameter. Such a continuous approximation of input works well but asymptotic stability of the origin of the closed-loop system is not always ensured in the conventional frameworks of nonsingular terminal sliding mode control. See [11], [10], for details. The proposed method, on the other hand, can reduce chattering by selecting a potential function with a smooth bottom. Figure 2 shows the relation between potential function and the input. For example, the saturation function $\mathrm{sat}(\cdot)$ is realized by selecting

$$U_{\rm sat}(\sigma;\varphi) = \sum_{i=1}^{m} U_i(\sigma_i;\varphi),\ U_i(\sigma_i;\varphi) = \begin{cases} |\sigma_i| - \frac{\varphi}{2}, & |\sigma_i| > \varphi \\ \frac{1}{2\varphi}\sigma_i^2, & |\sigma_i| \le \varphi \end{cases}$$

as $U(\sigma)$. In the case, asymptotic stability of the closed-loop system is still ensured by Lemma 1 because a smooth function is selected as $U(\sigma)$. Therefore, it is possible to adjust the discontinuity of the control input without losing Lyapunov stability.



(a) Poential functions (b) Derivatives

Fig. 2. The relation between the potential function $U(\sigma)$ and input u. Figure (a) shows the potential function candidates and Figure (b) shows their derivatives which are used in the control input u.

IV. Numerical Example

This section demonstrates the effectiveness of the proposed method through numerical simulations. Let us consider a mechanical port-Hamiltonian system in the form (1) with $m = 2$. The system parameters are given as follows:

$$M(q) = \begin{pmatrix} 1 & 1 + \frac{1}{2}\cos(q_1+q_2) \\ 1 + \frac{1}{2}\cos(q_1+q_2) & 2 + \cos(q_1+q_2) \end{pmatrix},$$
$$D_0(q,p) = \mathrm{diag}\,(0.2, 0.2).$$

For this system, we select $T(q)$ satisfying (2) as

$$T(q) = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} \frac{2}{\sqrt{2-\cos(q_1+q_2)}} & 0 \\ 0 & \frac{1}{\sqrt{2+\cos(q_1+q_2)}} \end{pmatrix}.$$

The control input is designed by (27) with the following free parameters

$$\phi(q) = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} q, \quad \alpha = 1.2, \quad \beta = 2.0.$$

Note that this choice satisfies the condition (17). The control objective is to stabilize at the origin under the initial condition $x(0) = (q(0)^\top, p(0)^\top)^\top = (-2, 3, 1, 0)^\top$.

Figures 3–7 show the results of the numerical simulations. Figures 3 and 4 show that the sliding variable becomes zero in a finite time ($t \approx 0.7$) and the state variable convergence to zero. Figures 5 illustrates the response of the 2-norm of the state $\|\chi\| = \|(q^\top, \eta^\top)^\top\|$ in the logarithmic scale. As the figure is plotted in the scale, the straight line means the exponential decay. The result of $\|\chi\|$ is not plotted by a straight line, so it decreases faster than exponential decay in the sliding mode ($t > 0.7$), which implies finite time convergence of the state variable. The solid (blue and red) lines in Fig. 6 shows the history of the input u. When the state reaches one of the sliding surfaces ($t \approx 0.2$), the chattering occurs in the input, which can be reduced by selecting a smooth potential function as mentioned in Remark 2. In addition, the input does not diverges when η_1 and/or η_2 becomes zero, which shows that there is no singular problem in the input. Despite the existence of chattering, the Hamiltonian function decreases monotonically shown by the solid blue line in Fig. 7. This shows that the Hamiltonian function works as a Lyapunov function both inside and outside the sliding surface.

The dotted (yellow and green) lines in Figs. 6 and 7 also show the simulation results where $\operatorname{sgn}\sigma$ in Eq. (27) is replaced with $\operatorname{sat}(\sigma; 0.5)$. The chattering in the input is removed due to a continuous approximation of input while Hamiltonian function still decreases monotonically. Therefore, Lyapunov stability of the closed-loop system can be ensured even if we use a continuous input, which is the main advantage of the proposed controller. These results exhibit the effectiveness of the proposed method.

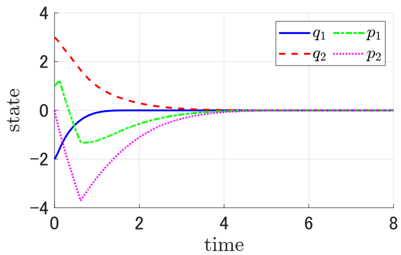


Fig. 3. Responses of the state x with discontinuous input.

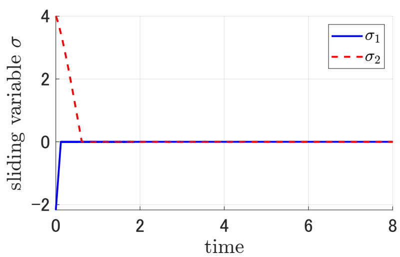


Fig. 4. Responses of the sliding variable σ with discontinuous input.

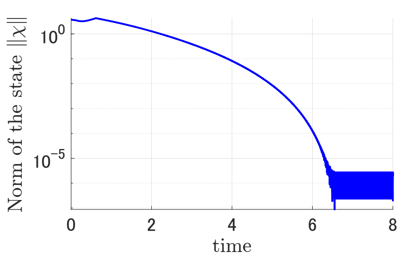


Fig. 5. Response of the 2-norm of the state $\|\chi\|$ with discontinuous input.

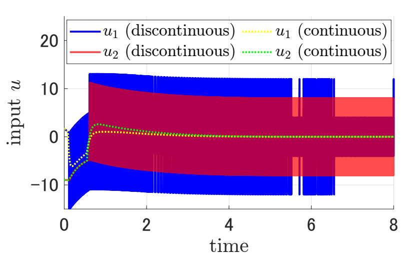


Fig. 6. Responses of the discontinuous and continuous inputs u.

V. Conclusion

This work proposes a new passivity-based controller that achieves nonsingular terminal sliding mode control with Lyapunov stability. To construct the controller, we extend KPES so that a special class of artificial potential functions can be chosen. This result gives a unified approach that

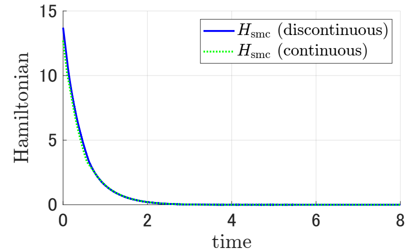


Fig. 7. Responses of the Hamiltonian function $H_{\rm smc}$ with discontinuous and continuous inputs.

merges passivity-based control and sliding mode control, and makes it possible to adjust the controller from discontinuous to continuous without losing Lyapunov stability. The numerical example demonstrates the effectiveness of the proposed method.

References

[1] A. van der Schaft, *L2-Gain and Passivity Techniques in Nonlinear Control*. Springer International Publishing, 2017.

[2] H. Rodríguez and R. Ortega, "Stabilization of electromechanical systems via interconnection and damping assignment," *International Journal of Robust and Nonlinear Control*, vol. 13, no. 12, pp. 1095–1111, 2003.

[3] K. Fujimoto, S. Sakai, and T. Sugie, "Passivity based control of a class of Hamiltonian systems with nonholonomic constraints," *Automatica*, vol. 48, no. 12, pp. 3054–3063, 2012.

[4] K. Fujimoto and T. Sugie, "Canonical transformation and stabilization of generalized Hamiltonian systems," *IEEE Control Systems Letters*, vol. 42, no. 3, pp. 217–227, 2001.

[5] R. Ortega, A. van der Schaft, B. Maschke, and G. Escobar, "Interconnection and damping assignment passivity-based control of port-controlled Hamiltonian systems," *Automatica*, vol. 38, no. 4, pp. 585–596, 2002.

[6] J.-J. E. Slotine and W. Li, *Applied Nonlinear Control*. Prentice-Hall International Editions, 1991.

[7] A. Ferrara, G. P. Incremona, and M. Cucuzzella, *Advanced and Optimization Based Sliding Mode Control: Theory and Applications*. Society for Industrial and Applied Mathematics, 2019.

[8] S. T. Venkataraman and S. Gulati, "Control of nonlinear systems using terminal sliding modes," in *1992 American Control Conference*. IEEE, 1992.

[9] Y. Wu, X. Yu, and Z. Man, "Terminal sliding mode control design for uncertain dynamic systems," *IEEE Control Systems Letters*, vol. 34, no. 5, pp. 281–287, 1998.

[10] Y. Feng, X. Yu, and F. Han, "On nonsingular terminal sliding-mode control of nonlinear systems," *Automatica*, vol. 49, no. 6, pp. 1715–1722, 2013.

[11] Y. Feng, X. Yu, and Z. Man, "Non-singular terminal sliding mode control of rigid manipulators," *Automatica*, vol. 38, no. 12, pp. 2159–2167, 2002.

[12] K. Fujimoto, N. Sakata, I. Maruta, and J. Ferguson, "A passivity based sliding mode controller for simple port-Hamiltonian systems," *IEEE Control Systems Letters*, vol. 5, no. 3, pp. 839–844, 2021.

[13] N. Sakata, K. Fujimoto, and I. Maruta, "Passivity-based sliding mode control for mechanical port-hamiltonian systems," *IEEE Transactions on Automatic Control*, vol. 69, no. 8, pp. 5605–5612, 2024.

[14] J. Ferguson, A. Donaire, and R. H. Middleton, "Kinetic-potential energy shaping for mechanical systems with applications to tracking," *IEEE Control Systems Letters*, vol. 3, no. 4, pp. 960–965, 2019.

[15] A. Venkatraman, R. Ortega, I. Sarras, and A. van der Schaft, "Speed observation and position feedback stabilization of partially linearizable mechanical systems," *IEEE Transactions on Automatic Control*, vol. 55, no. 5, pp. 1059–1074, 2010.

[16] J. G. Romero, R. Ortega, and I. Sarras, "A globally exponentially stable tracking controller for mechanical systems using position feedback," *IEEE Transactions on Automatic Control*, vol. 60, no. 3, pp. 818–823, 2015.

[17] A. Levant and B. Shustin, "Quasi-continuous MIMO sliding-mode control," *IEEE Transactions on Automatic Control*, vol. 63, no. 9, pp. 3068–3074, 2018.