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PI boundary consensus of networked hyperbolic systems with application to multi-lane traffic synchronization

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Abstract— **This paper studies the proportional-integral (PI) boundary feedback control problem for consensus of hyperbolic multi-agent systems (MASs), and provides an application to the synchronization of a multi-lane road traffic flow system. Firstly, we propose a PI boundary consensus protocol for the hyperbolic MASs of conservation laws in the presence of unknown constant input disturbances. Secondly, we present the consensus analysis under undirected communication topologies by employing the Lyapunov approach, obtaining sufficient conditions w.r.t. the PI boundary control matrices and Laplacian matrices for ensuring the asymptotic consensus. We further integrate the spectral decomposition technique with Lyapunov approach to derive the sufficient conditions related to Laplacian eigenvalues, which are more tractable, under the assumption that the undirected graph is connected. Finally, we provide an application to the synchronization of a multi-lane road traffic flow system described by the Aw-Rascle equation, and give numerical simulation results to demonstrate the effectiveness of the PI boundary consensus protocol.**

I. INTRODUCTION

Hyperbolic Partial Differential Equations (PDEs) are a class of significant mathematical models that are widely used to describe physical processes, such as gas flow in pipe networks [1], water flow in networks of open channels [2] and vehicle traffic flow [3]. In the last few decades, a great deal of importance is attached to the study of multi-agent systems with hyperbolic PDEs, in which the dynamics of each agent is described by hyperbolic PDEs and the agents interact through their boundaries. The controllability and observability for a networked linear hyperbolic PDE system with coupled boundary conditions were analyzed in [4], and a stabilizing controller was also designed.

Consensus means that all agents reach an agreement with a previously unknown value or trajectory, which has been widely applied in connected vehicles [5], intelligent transportation systems [6] and etc. However, extensive studies on consensus merely focused on MASs with Ordinary Differential Equations (ODEs). With the consideration of spatio-temporal characteristics, it is of greater practical and theoretical significance to study the consensus of multi-agent systems with PDEs. For example, the motion synchronization control of two-manipulator flexible beam systems was discussed in [7], and the cooperative attitude tracking problem of multiple flexible spacecrafts modelled by PDEs was investigated in [8].

In recent years, consensus of networked PDE MASs has attracted increasing attention. Qiu et al. investigated the adaptive output feedback consensus problem for parabolic PDE agents under undirected networks in [9], and they made extensions to switching topologies in [10]. Lu et al. [11] considered the consensus problem of networked hyperbolic systems based on event-triggered boundary feedback control. Chen et al. [12] studied the bipartite consensus problem for a network of PDEs. More recently, Liu et al. [13] studied the average consensus control of MASs with event-triggered mechanism based on PDE.

On the other hand, proportional-integral (PI) control is one of the most common control algorithms to regulate the output or process variable of a system to approach a set-point, since its advantage is to cancel forced oscillations and attenuate load disturbances. Zhang et al. [14] investigated the PI boundary feedback control for the linear hyperbolic system of balance laws. Subsequently, they further considered the PI boundary stabilization of nonlinear hyperbolic systems of balance laws for the H^2 -norm in [15]. To the best of authors' knowledge, PI boundary control for the consensus of hyperbolic MASs has not been studied yet.

In this paper, we consider the PI boundary consensus problem for multi-agent systems with hyperbolic PDEs. Firstly, we propose a PI boundary consensus protocol for a networked multi-agent system subject to unknown constant input disturbances, in which each agent is modeled by a hyperbolic system of two linear conservation laws. Secondly, we perform consensus analysis for the networked system under undirected topologies by using Lyapunov approach, in which we give sufficient conditions with respect to the PI boundary control matrices and Laplacian eigenvalues. In addition, we present an application to the synchronization of a multi-lane road traffic flow system based on Aw-Rascle Equations.

This paper is organized as follows. In section II, we display the problem formulation. The consensus analysis under undirected communication topologies is provided in Section III. Section IV provides an application to multi-lane traffic synchronization, and section V concludes the paper.

Notations: For a function $f(x) = (f_1(x), f_2(x))^{\mathrm{T}} \in L^2((0,L);\mathbb{R}^2)$, denote its L^2-norm as $\|f(x)\|_{L^2((0,L);\mathbb{R}^2)} \triangleq$

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$\sqrt{\int_0^L \left(f_1^2(x)+f_2^2(x)\right)dx}$. Given a real matrix A, A^{T} denotes the transpose of A, A^{-1} denotes the inverse of A, $A<(\leq)\ 0$ denotes A is a negative definite (semi-definite) matrix, and $\lambda_{\max}(A)$ denotes the largest real part of eigenvalues of A. For a diagonal real matrix $\Lambda=\mathrm{diag}\{\lambda_1,...,\lambda_N\}$, $|\Lambda|=\mathrm{diag}\{|\lambda_1|,...,|\lambda_N|\}$. For a partitioned symmetric matrix, the symbol $\star$ stands for the symmetric blocks. $\mathbf{1}=(1,1,...,1)^{\mathrm{T}}$ and $\mathbf{0}=(0,0,...,0)^{\mathrm{T}}$ are column vectors with appropriate dimensions if no confusion arises.

II. Problem Formulation

Consider a group of agents governed by hyperbolic systems of two linear conservation laws in Riemann coordinates:

$$\begin{cases} \partial_t\xi_i^1(x,t)+\gamma_1\partial_x\xi_i^1(x,t)=0 \\ \partial_t\xi_i^2(x,t)-\gamma_2\partial_x\xi_i^2(x,t)=0 \end{cases}, \quad i=1,...,N,$$

where $\xi_i^j(x,t):\ [0,L]\times\mathbb{R}_+\to\mathbb{R}$ with $j=1,2$, and $\gamma_1>0>-\gamma_2$. Define $\xi_i=(\xi_i^1,\xi_i^2)^{\mathrm{T}}$ as the state for agent i, and then we have

$$\partial_t\xi_i(x,t)+\Gamma\partial_x\xi_i(x,t)=0, \quad i=1,...,N, \tag{1}$$

where $\Gamma=\mathrm{diag}\{\gamma_1,-\gamma_2\}$.

Let $\xi_{i,in}(t)=(\xi_i^1(0,t),\xi_i^2(L,t))^{\mathrm{T}}$, and $\xi_{i,out}(t)=(\xi_i^1(L,t),\xi_i^2(0,t))^{\mathrm{T}}$ for all $i=1,...,N$. For each hyperbolic agent i, the boundary condition is given as

$$\xi_{i,in}(t)=A\xi_{i,out}(t)+u_i(t)+\theta_i, \tag{2}$$

where $A\in\mathbb{R}^{2\times 2}$ denotes the internal coupling matrix, and $u_i(t)\in\mathbb{R}^2$ denotes the boundary control of hyperbolic agent i. $\theta_i\in\mathbb{R}^2$ is an additive outside unknown disturbance which corrupts system (1) on the boundaries of input. We assume $u_i(t)$ follows the following proportional-integral (PI) boundary consensus protocol:

$$\begin{aligned} u_i(t)=&K_P\sum_{j=1}^N a_{ij}(\xi_{j,out}(t)-\xi_{i,out}(t)) \\ &+K_I\sum_{j=1}^N a_{ij}\int_0^t\left(\xi_{j,out}(\tau)-\xi_{i,out}(\tau)\right)d\tau, \end{aligned} \tag{3}$$

where $K_P,K_I\in\mathbb{R}^{2\times 2}$ are gain matrices, $a_{ij}>0$ while hyperbolic agent i receives the information of hyperbolic agent j, and $a_{ij}=0$ otherwise.

The communication topology among the group of hyperbolic agents is represented by a time-invariant undirected graph $\mathcal{G}=\{\mathcal{V},\mathcal{E},\mathcal{A}\}$, consisting of a node set $\mathcal{V}=\{1,2,...,N\}$, a time-invariant edge set $\mathcal{E}\subseteq\{(i,j):i,j\in\mathcal{V}\}$, and a corresponding time-invariant adjacency matrix $\mathcal{A}=[a_{ij}]$ with non-negative entries $a_{ij}\geq 0$ for all i,j. Iff $(j,i)\in\mathcal{E}$, $(i,j)\in\mathcal{E}$, which means the information is exchanged between node i and node j, and thus $a_{ij}=a_{ji}>0$. An undirected graph is connected if there is a path between any two nodes, which is a sequence of consecutive edges. Self-loops are not considered in this paper. The Laplacian matrix $\mathcal{L}=[l_{ij}]$ is defined as $l_{ij}=-a_{ij}$ for $i\neq j$ and $l_{ii}=\sum_{j\neq i}a_{ij}$ for all i.

Definition 1: System (1)-(2) with PI boundary consensus protocol (3) is said to achieve consensus asymptotically if

$$\lim_{t\to+\infty}\|\xi_i-\xi_j\|_{L^2((0,L);\mathbb{R}^2)}=0,\ \forall i,j=1,...,N. \tag{4}$$

In the next section, we aim to derive sufficient conditions with respect to internal coupling matrix A, gain matrices K_P,K_I and Laplacian matrix $\mathcal{L}$ for ensuring the asymptotic consensus of system (1)-(2) under PI boundary consensus protocol (3).

III. Main Results

The boundary condition (2) for each hyperbolic system i under the boundary consensus protocol (3) can be expressed as:

$$\begin{aligned} \xi_{i,in}(t)=&A\xi_{i,out}(t)+K_P\sum_{j=1}^N a_{ij}(\xi_{j,out}(t)-\xi_{i,out}(t)) \\ &+K_I\sum_{j=1}^N a_{ij}\int_0^t\left(\xi_{j,out}(\tau)-\xi_{i,out}(\tau)\right)d\tau+\theta_i. \end{aligned} \tag{5}$$

Let $w=(w_1,w_2,...,w_N)^{\mathrm{T}}$ with $w_i=1/N, i=1,2,...,N$. Let $\xi^*=\sum_{i=1}^N w_i\xi_i$, and then following (1), we have

$$\partial_t\xi^*+\Gamma\partial_x\xi^*=0. \tag{6}$$

Let $\theta^*=\sum_{i=1}^N w_i\theta_i$, and the boundary condition (5) w.r.t. ξ^* can be rewritten as

$$\xi^*_{in}=A\xi^*_{out}+\theta^*. \tag{7}$$

Define $e_i=\xi_i-\xi^*$, $\tilde{\theta}_i=\theta_i-\theta^*$ for all $i=1,...,N$. Then following (1), (5) and (6)-(7), we have

$$\partial_t e_i+\Gamma\partial_x e_i=0, \tag{8}$$

$$\begin{aligned} e_{i,in}(t)=&Ae_{i,out}(t)+K_P\sum_{j=1}^N a_{ij}(e_{j,out}(t)-e_{i,out}(t)) \\ &+K_I\sum_{j=1}^N a_{ij}\int_0^t(e_{j,out}(\tau)-e_{i,out}(\tau))d\tau+\tilde{\theta}_i \end{aligned} \tag{9}$$

for all $i=1,...,N$.

Define $e=(e_1^{\mathrm{T}},e_2^{\mathrm{T}},...,e_N^{\mathrm{T}})^{\mathrm{T}}$, $e_{in}=(e_{1,in}^{\mathrm{T}},e_{2,in}^{\mathrm{T}},...,e_{N,in}^{\mathrm{T}})^{\mathrm{T}}$, $e_{out}=(e_{1,out}^{\mathrm{T}},e_{2,out}^{\mathrm{T}},...,e_{N,out}^{\mathrm{T}})^{\mathrm{T}}$, and $\tilde{\theta}=(\tilde{\theta}_1^{\mathrm{T}},\tilde{\theta}_2^{\mathrm{T}},...,\tilde{\theta}_N^{\mathrm{T}})^{\mathrm{T}}$. Then system (8)-(9) can be rewritten into the following compact form:

$$\partial_t e+(I_N\otimes\Gamma)\partial_x e=0, \tag{10}$$

$$e_{in}(t)=(I_N\otimes A-\mathcal{L}\otimes K_P)e_{out}(t)-\mathcal{L}\otimes K_I\int_0^t e_{out}(\tau)d\tau+\tilde{\theta}. \tag{11}$$

Then we can obtain that the asymptotic consensus of system (1) under boundary condition (5) is equivalent to the asymptotic stability of error system (10)-(11).

Lemma 1: The linear hyperbolic MAS (1)-(2) with PI boundary consensus protocol (3) achieves consensus asymptotically, if K_I is reversible and there exist diagonal matrix

$P_1 \in \mathbb{R}^{2\times 2}$, symmetric matrix $P_2 \in \mathbb{R}^{2\times 2}$, matrix $P_3 \in \mathbb{R}^{2\times 2}$, and a real constant $\mu > 0$ such that the following matrix inequalities

$$P = \begin{bmatrix} P_1 & P_3 \\ \star & P_2 \end{bmatrix} > 0, \tag{12}$$

$$\Omega(x) = \begin{bmatrix} \Omega_{11}(x) & \Omega_{12}(x) & \Omega_{13}(x) \\ \star & \Omega_{22} & \Omega_{23} \\ \star & \star & \Omega_{33} \end{bmatrix} < 0, \tag{13}$$

hold for all $x \in [0, L]$, where

$$\begin{aligned}
\Omega_{11}(x) &= -\mu(I_N \otimes |\Gamma|P_1(x)) \\
\Omega_{12}(x) &= -\mu(I_N \otimes |\Gamma|P_3(x)) \\
\Omega_{13}(x) &= -\mathcal{L} \otimes P_3(x) \\
\Omega_{22} &= \frac{1}{L}[e^{\mu L} I_N \otimes K_I^{\mathrm{T}}|\Gamma|P_1 K_I + e^{\mu L} I_N \otimes K_I^{\mathrm{T}}|\Gamma|P_3 \\
&\quad + e^{\mu L} I_N \otimes P_3^{\mathrm{T}}|\Gamma|K_I] \\
\Omega_{23} &= \frac{1}{L}[e^{\mu L} I_N \otimes K_I^{\mathrm{T}}|\Gamma|P_1 A - e^{\mu L}\mathcal{L} \otimes K_I^{\mathrm{T}}|\Gamma|P_1 K_P \\
&\quad + e^{\mu L} I_N \otimes P_3^{\mathrm{T}}|\Gamma|A - e^{\mu L}\mathcal{L} \otimes P_3^{\mathrm{T}}|\Gamma|K_P \\
&\quad - I_N \otimes |\Gamma|P_3] - \mathcal{L} \otimes P_2 \\
\Omega_{33} &= \frac{1}{L}[e^{\mu L} I_N \otimes A^{\mathrm{T}}|\Gamma|P_1 A - e^{\mu L}\mathcal{L}^{\mathrm{T}} \otimes K_P^{\mathrm{T}}|\Gamma|P_1 A \\
&\quad - e^{\mu L}\mathcal{L} \otimes A^{\mathrm{T}}|\Gamma|P_1 K_P + e^{\mu L}\mathcal{L}^2 \otimes K_P^{\mathrm{T}}|\Gamma|P_1 K_P \\
&\quad - I_N \otimes |\Gamma|P_1]
\end{aligned}$$

with $P_1(x) = P_1 \mathrm{diag}\{e^{\mu(L-x)}, e^{\mu x}\}$, $P_3(x) = P_3 \mathrm{diag}\{e^{\mu(L-x)}, e^{\mu x}\}$.

Proof: We construct the Lyapunov function candidate in the following form as

$$V(t) = \int_0^L \begin{bmatrix} e \\ \eta \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} I_N \otimes P_1(x) & I_N \otimes P_3(x) \\ \star & I_N \otimes P_2 \end{bmatrix} \begin{bmatrix} e \\ \eta \end{bmatrix} dx, \tag{14}$$

where

$$\eta(t) = -\mathcal{L} \otimes \int_0^t e_{out}(\tau)d\tau + I_N \otimes K_I^{-1}\tilde{\theta}. \tag{15}$$

The first step of the proof is to compute the time-derivative of V along the solution (e, η). It yields:

$$\begin{aligned}
\dot{V}(t) &= \int_0^L [2e_t^{\mathrm{T}}(I_N \otimes P_1(x))e + e_t^{\mathrm{T}}(I_N \otimes P_3(x))\eta \\
&\quad + e^{\mathrm{T}}(I_N \otimes P_3(x))\dot{\eta} + \dot{\eta}^{\mathrm{T}}(I_N \otimes P_3^{\mathrm{T}}(x))e \\
&\quad + \eta^{\mathrm{T}}(I_N \otimes P_3^{\mathrm{T}}(x))e_t]dx + 2L\dot{\eta}^{\mathrm{T}}(I_N \otimes P_2)\eta \\
&= \dot{V}_1 + \dot{V}_2 + \dot{V}_3
\end{aligned}$$

with

$$\begin{aligned}
\dot{V}_1 &\triangleq 2\int_0^L e_t^{\mathrm{T}}(I_N \otimes P_1(x))edx, \\
\dot{V}_2 &\triangleq \int_0^L [e_t^{\mathrm{T}}(I_N \otimes P_3(x))\eta + e^{\mathrm{T}}(I_N \otimes P_3(x))\dot{\eta} \\
&\quad + \dot{\eta}^{\mathrm{T}}(I_N \otimes P_3^{\mathrm{T}}(x))e + \eta^{\mathrm{T}}(I_N \otimes P_3^{\mathrm{T}}(x))e_t]dx, \\
\dot{V}_3 &\triangleq 2L\dot{\eta}^{\mathrm{T}}(I_N \otimes P_2)\eta.
\end{aligned}$$

Then, using the error system (10) and integration by parts for $\dot{V}_1$ and $\dot{V}_2$, substituting the PI boundary condition (11) into $\dot{V}_1$ and $\dot{V}_2$, and letting $\tilde{K}_P = I_N \otimes A - \mathcal{L} \otimes K_P$ and $\tilde{K}_I = I_N \otimes K_I$, we have

$$\begin{aligned}
\dot{V}_1 &= 2\int_0^L [-(I_N \otimes \Gamma)e_x]^{\mathrm{T}}(I_N \otimes P_1(x))edx \\
&= -e_{out}^{\mathrm{T}}(I_N \otimes |\Gamma|P_1)e_{out} + [\tilde{K}_P e_{out} + \tilde{K}_I\eta]^{\mathrm{T}}(e^{\mu L} I_N \otimes \\
&\quad |\Gamma|P_1)[\tilde{K}_P e_{out} + \tilde{K}_I\eta] - \mu\int_0^L e^{\mathrm{T}}(I_N \otimes |\Gamma|P_1(x))edx \\
&= \int_0^L \begin{bmatrix} e \\ \eta \\ e_{out} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -\mu(I_N \otimes |\Gamma|P_1(x)) & 0 & 0 \\ \star & \frac{\tilde{K}_I^{\mathrm{T}}(e^{\mu L} I_N \otimes |\Gamma|P_1)\tilde{K}_I}{L} & \frac{\tilde{K}_I^{\mathrm{T}}(e^{\mu L} I_N \otimes |\Gamma|P_1)\tilde{K}_P}{L} \\ \star & \star & \frac{\tilde{K}_P^{\mathrm{T}}(e^{\mu L} I_N \otimes |\Gamma|P_1)\tilde{K}_P - I_N \otimes |\Gamma|P_1}{L} \end{bmatrix} \begin{bmatrix} e \\ \eta \\ e_{out} \end{bmatrix} dx,
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
\dot{V}_2 &= \int_0^L \big[[-(I_N \otimes \Gamma)e_x]^{\mathrm{T}}(I_N \otimes P_3(x))\eta + \eta^{\mathrm{T}}(I_N \otimes P_3^{\mathrm{T}}(x)) \\
&\quad [-(I_N \otimes \Gamma)e_x]\big]dx + \int_0^L [e^{\mathrm{T}}(I_N \otimes P_3(x))(-\mathcal{L} \otimes I)e_{out} \\
&\quad + e_{out}^{\mathrm{T}}(-\mathcal{L} \otimes I)(I_N \otimes P_3^{\mathrm{T}}(x))e]dx \\
&= -e_{out}^{\mathrm{T}}(I_N \otimes |\Gamma|P_3)\eta - \eta^{\mathrm{T}}(I_N \otimes P_3^{\mathrm{T}}|\Gamma|)e_{out} \\
&\quad + [\tilde{K}_P e_{out} + \tilde{K}_I\eta]^{\mathrm{T}}(e^{\mu L} I_N \otimes |\Gamma|P_3)\eta \\
&\quad + \eta^{\mathrm{T}}(e^{\mu L} I_N \otimes P_3^{\mathrm{T}}|\Gamma|)[\tilde{K}_P e_{out} + \tilde{K}_I\eta] \\
&\quad - \mu\int_0^L \left[e^{\mathrm{T}}(I_N \otimes |\Gamma|P_3(x))\eta + \eta^{\mathrm{T}}(I_N \otimes P_3^{\mathrm{T}}(x)|\Gamma|)e\right] dx \\
&\quad - \int_0^L \left[e^{\mathrm{T}}(\mathcal{L} \otimes P_3(x))e_{out} + e_{out}^{\mathrm{T}}(\mathcal{L} \otimes P_3^{\mathrm{T}}(x))e\right] dx \\
&= \int_0^L \begin{bmatrix} e \\ \eta \\ e_{out} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 0 & -\mu(I_N \otimes |\Gamma|P_3(x)) & -\mathcal{L} \otimes P_3(x) \\ \star & \frac{\tilde{K}_I^{\mathrm{T}}(e^{\mu L} I_N \otimes |\Gamma|P_3) + (e^{\mu L} I_N \otimes P_3^{\mathrm{T}}|\Gamma|)\tilde{K}_I}{L} & \frac{(e^{\mu L} I_N \otimes P_3^{\mathrm{T}}|\Gamma|)\tilde{K}_P - I_N \otimes |\Gamma|P_3}{L} \\ \star & \star & 0 \end{bmatrix} \begin{bmatrix} e \\ \eta \\ e_{out} \end{bmatrix} dx.
\end{aligned} \tag{17}$$

Moreover, the time-derivative of V_3 can be rewritten as

$$\dot{V}_3 = \int_0^L \begin{bmatrix} e \\ \eta \\ e_{out} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 0 & 0 & 0 \\ \star & 0 & -\mathcal{L} \otimes P_2 \\ \star & \star & 0 \end{bmatrix} \begin{bmatrix} e \\ \eta \\ e_{out} \end{bmatrix} dx. \tag{18}$$

Finally, by combining (16)-(18), and substituting $\tilde{K}_P$, $\tilde{K}_I$, we have the time-derivative of V as

$$\dot{V} = \int_0^L \begin{bmatrix} e \\ \eta \\ e_{out} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \Omega_{11}(x) & \Omega_{12}(x) & \Omega_{13}(x) \\ \star & \Omega_{22} & \Omega_{23} \\ \star & \star & \Omega_{33} \end{bmatrix} \begin{bmatrix} e \\ \eta \\ e_{out} \end{bmatrix} dx. \tag{19}$$

Due to the condition (13) that $\Omega(x) < 0$, there exists a constant $v = -\frac{\lambda_{\max}(\Omega(x))}{\lambda_{\max}(P)} > 0$ for all $x \in [0, L]$, such that

$$\dot{V} \leq \lambda_{\max}(\Omega(x)) \int_0^L \begin{bmatrix} e \\ \eta \\ e_{out} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} e \\ \eta \\ e_{out} \end{bmatrix} dx \\ \leq -vV. \quad (20)$$

Further, we can obtain that $V(t) \leq \exp(-vt)V(0)$, and therefore $\lim_{t\to\infty} V(t) = 0$. Through the definition of $V(t)$, $\lim_{t\to\infty} V(t) = 0$ implies $\lim_{t\to\infty} \|e(x,t)\|_{L^2} = 0$, which indicates that system (10)-(11) is asymptotically stable in L^2-norm. ■

To ensure the consensus of system (1) under boundary condition (5), the sufficient condition is closely related to the Laplacian matrix $\mathcal{L}$. Following the condition (13) given in Lemma 1, we need to judge the negative definiteness of a $6N$-dimensional matrix, and thus it is difficult to check when N is large. We will employ the spectral decomposition technique to derive a sufficient condition in relation to the Laplacian eigenvalues in place of the Laplacian matrix $\mathcal{L}$, which will make the condition more tractable.

Before giving the main result of this paper, an assumption is given as follows.

Assumption 1: The communication topology $\mathcal{G}$ is undirected and connected.

Let $\lambda_i,\ i = 1, ..., N$, and $0 = \lambda_1 < \lambda_2 \leq ... \leq \lambda_N$ denote the eigenvalues of $\mathcal{L}$. Since $\mathcal{G}$ is undirected, $w = \mathbf{1}_N/N$ is the non-negative left eigenvector of $\mathcal{L}$ associated with eigenvalue 0.

Theorem 1: The linear hyperbolic MAS (1)-(2) with PI boundary consensus protocol (3) achieves consensus asymptotically, if Assumption 1 holds, K_I is reversible and there exist diagonal matrix $P_1 \in \mathbb{R}^{2\times2}$, symmetric matrix $P_2 \in \mathbb{R}^{2\times2}$, matrix $P_3 \in \mathbb{R}^{2\times2}$, and a real constant $\mu > 0$ such that the following matrix inequalities

$$P = \begin{bmatrix} P_1 & P_3 \\ \star & P_2 \end{bmatrix} > 0,$$

$$\Pi_i(x) = \begin{bmatrix} \Pi_{11}(x) & \Pi_{12}(x) & \Pi_{i,13}(x) \\ \star & \Pi_{22} & \Pi_{i,23} \\ \star & \star & \Pi_{i,33} \end{bmatrix} < 0, \quad (21)$$

hold for all $x \in [0, L]$ and all $i = 2, ..., N$, where

$$\Pi_{11}(x) = -\mu|\Gamma|P_1(x),$$
$$\Pi_{12}(x) = -\mu|\Gamma|P_3(x),$$
$$\Pi_{i,13}(x) = -\lambda_i P_3(x),$$
$$\Pi_{22} = \frac{1}{L}(e^{\mu L}K_I^{\mathrm{T}}|\Gamma|P_1K_I + e^{\mu L}K_I^{\mathrm{T}}|\Gamma|P_3 + e^{\mu L}P_3^{\mathrm{T}}|\Gamma|K_I),$$
$$\Pi_{i,23} = \frac{1}{L}[e^{\mu L}K_I^{\mathrm{T}}|\Gamma|P_1A + e^{\mu L}P_3^{\mathrm{T}}|\Gamma|A - |\Gamma|P_3 \\ - \lambda_i(e^{\mu L}K_I^{\mathrm{T}}|\Gamma|P_1K_P + e^{\mu L}P_3^{\mathrm{T}}|\Gamma|K_P + LP_2)],$$
$$\Pi_{i,33} = \frac{1}{L}[e^{\mu L}A^{\mathrm{T}}|\Gamma|P_1A - |\Gamma|P_1 - \lambda_i(e^{\mu L}K_P^{\mathrm{T}}|\Gamma|P_1A \\ + e^{\mu L}A^{\mathrm{T}}|\Gamma|P_1K_P) + \lambda_i^2 e^{\mu L}K_P^{\mathrm{T}}|\Gamma|P_1K_P].$$

Proof: Choose $\psi_i \in \mathbb{R}^N$ such that $\psi_i^{\mathrm{T}}\mathcal{L} = \lambda_i\psi_i^{\mathrm{T}}$, and define an orthogonal matrix $\Psi = (\mathbf{1}_N/\sqrt{N}, \psi_2, ..., \psi_N)$ to transform $\mathcal{L}$ into a diagonal form

$$\Lambda \triangleq \mathrm{diag}\{0, \lambda_2, ..., \lambda_N\} = \Psi^{\mathrm{T}}\mathcal{L}\Psi. \quad (22)$$

Let $\tilde{e} = (\Psi \otimes I_2)^{\mathrm{T}}e$, $\tilde{\eta} = (\Psi \otimes I_2)^{\mathrm{T}}\eta$, $\tilde{e}_{out} = (\Psi \otimes I_2)^{\mathrm{T}}e_{out}$, where $\tilde{e}_1 = 1/\sqrt{N}\sum_{i=1}^N e_i = 0$, $\tilde{\eta}_1 = 1/\sqrt{N}\sum_{i=1}^N \eta_i = 0$, $\tilde{e}_{1,out} = 1/\sqrt{N}\sum_{i=1}^N e_{i,out} = 0$. According to (22), we have

$$e^{\mathrm{T}}\Omega_{11}(x)e = \tilde{e}^{\mathrm{T}}(\Psi^{\mathrm{T}} \otimes I_2)[-2\mu(I_N \otimes |\Gamma|P_1(x))](\Psi \otimes I_2)\tilde{e} \\ = \tilde{e}^{\mathrm{T}}[-2\mu(\Psi^{\mathrm{T}}I_N\Psi \otimes |\Gamma|P_1(x))]\tilde{e} \\ = \sum_{i=2}^N \tilde{e}_i^{\mathrm{T}}[-2\mu|\Gamma|P_1(x)]\tilde{e}_i, \quad (23)$$

$$e^{\mathrm{T}}\Omega_{12}(x)\eta = \sum_{i=2}^N \tilde{e}_i^{\mathrm{T}}[-\mu|\Gamma|P_3(x)]\tilde{\eta}_i, \quad (24)$$

$$e^{\mathrm{T}}\Omega_{13}(x)e_{out} = \sum_{i=2}^N \tilde{e}_i^{\mathrm{T}}[-\lambda_iP_3(x)]\tilde{e}_{i,out}, \quad (25)$$

$$\eta^{\mathrm{T}}\Omega_{22}\eta = \sum_{i=2}^N \eta_i^{\mathrm{T}}[\frac{1}{L}(e^{\mu L}K_I^{\mathrm{T}}|\Gamma|P_1K_I + e^{\mu L}K_I^{\mathrm{T}}|\Gamma|P_3 \\ + e^{\mu L}P_3^{\mathrm{T}}|\Gamma|K_I)]\tilde{\eta}_i, \quad (26)$$

$$\eta^{\mathrm{T}}\Omega_{23}e_{out} = \tilde{\eta}^{\mathrm{T}}[\frac{1}{L}(e^{\mu L}\Psi^{\mathrm{T}}I_N\Psi \otimes K_I^{\mathrm{T}}|\Gamma|P_1A \\ - e^{\mu L}\Psi^{\mathrm{T}}\mathcal{L}\Psi \otimes K_I^{\mathrm{T}}|\Gamma|P_1K_P + e^{\mu L}\Psi^{\mathrm{T}}I_N\Psi \otimes P_3^{\mathrm{T}}|\Gamma|A \\ - e^{\mu L}\Psi^{\mathrm{T}}\mathcal{L}\Psi \otimes P_3^{\mathrm{T}}|\Gamma|K_P - \Psi^{\mathrm{T}}I_N\Psi \otimes |\Gamma|P_3) \\ - \Psi^{\mathrm{T}}\mathcal{L}\Psi \otimes P_2]\tilde{e}_{out} \\ = \tilde{\eta}^{\mathrm{T}}[\frac{1}{L}(e^{\mu L}I_N \otimes K_I^{\mathrm{T}}|\Gamma|P_1A - e^{\mu L}\Lambda \otimes K_I^{\mathrm{T}}|\Gamma|P_1K_P \\ + e^{\mu L}I_N \otimes P_3^{\mathrm{T}}|\Gamma|A - e^{\mu L}\Lambda \otimes P_3^{\mathrm{T}}|\Gamma|K_P - I_N \otimes |\Gamma|P_3) \\ - \Lambda \otimes P_2]\tilde{e}_{out} \\ = \sum_{i=2}^N \tilde{\eta}_i^{\mathrm{T}}[\frac{1}{L}(e^{\mu L}K_I^{\mathrm{T}}|\Gamma|P_1A - e^{\mu L}\lambda_iK_I^{\mathrm{T}}|\Gamma|P_1K_P + e^{\mu L} \\ P_3^{\mathrm{T}}|\Gamma|A - e^{\mu L}\lambda_iP_3^{\mathrm{T}}|\Gamma|K_P - |\Gamma|P_3) - \lambda_iP_2]\tilde{e}_{i,out}, \quad (27)$$

$$e_{out}^{\mathrm{T}}\Omega_{33}e_{out} = \tilde{e}_{out}^{\mathrm{T}}[\frac{1}{L}(e^{\mu L}\Psi^{\mathrm{T}}I_N\Psi \otimes A^{\mathrm{T}}|\Gamma|P_1A \\ - e^{\mu L}\Psi^{\mathrm{T}}\mathcal{L}^{\mathrm{T}}\Psi \otimes K_P^{\mathrm{T}}|\Gamma|P_1A - e^{\mu L}\Psi^{\mathrm{T}}\mathcal{L}\Psi \otimes A^{\mathrm{T}}|\Gamma|P_1K_P \\ + e^{\mu L}\Psi^{\mathrm{T}}\mathcal{L}^{\mathrm{T}}\mathcal{L}\Psi \otimes K_P^{\mathrm{T}}|\Gamma|P_1K_P - \Psi^{\mathrm{T}}I_N\Psi \otimes |\Gamma|P_1)]\tilde{e}_{out} \\ = \sum_{i=2}^N \tilde{e}_{i,out}^{\mathrm{T}}[\frac{1}{L}(e^{\mu L}A^{\mathrm{T}}|\Gamma|P_1A - e^{\mu L}\lambda_iK_P^{\mathrm{T}}|\Gamma|P_1A \\ - e^{\mu L}\lambda_iA^{\mathrm{T}}|\Gamma|P_1K_P + e^{\mu L}\lambda_i^2K_P^{\mathrm{T}}|\Gamma|P_1K_P - |\Gamma|P_1)]\tilde{e}_{i,out}. \quad (28)$$

Then $\dot{V}(t)$ in (19) can be rewritten into

$$\dot{V}(t) = \sum_{i=2}^N \int_0^L \begin{bmatrix} \tilde{e}_i \\ \tilde{\eta}_i \\ \tilde{e}_{i,out} \end{bmatrix}^{\mathrm{T}} \cdot \begin{bmatrix} \Pi_{11}(x) & \Pi_{12}(x) & \Pi_{i,13}(x) \\ \star & \Pi_{22} & \Pi_{i,23} \\ \star & \star & \Pi_{i,33} \end{bmatrix} \begin{bmatrix} \tilde{e}_i \\ \tilde{\eta}_i \\ \tilde{e}_{i,out} \end{bmatrix} dx. \quad (29)$$

Following the similar arguments in the proof of Lemma 1, we prove that system (1) with boundary condition (5) achieves consensus asymptotically, since (21) holds for all $x \in [0, L]$ and all $i = 2, ..., N$. ■

Remark 1: Since undirected graphs are assumed in this paper, the final agreed is equal to the averaged state ξ^* following dynamics (6)-(7) with the initial state $\xi^*(x, 0) = \sum_{i=1}^{N} \xi_i(x, 0)/N$.

IV. Application to Multi-lane Traffic Synchronization

In this section, we present an application to a multi-lane traffic synchronization problem. Consider a traffic flow system on a multi-lane road that comprises N lanes and operates under boundary consensus control.

A. Multi-lane Traffic Flow Model

We use the Aw-Rascle Equations to illustrate each lane of the multi-lane road traffic flow system:

$$\begin{cases} \partial_t \rho_i + \partial_x(\rho_i v_i) = 0 \\ \partial_t v_i + (v_i - a\rho_i)\partial_x v_i = 0 \end{cases}, \quad i = 1, ..., N. \tag{30}$$

$\rho_i(x, t)$ and $v_i(x, t)$ are the vehicle density and average speed respectively for the i-th lane at position $x \in [0, L]$ and time $t \in [0, +\infty)$ with L representing the length of the road. The term $a\rho_i$ represents the traffic pressure with $a = v_f/\rho_m$, where v_f denotes the free speed and ρ_m denotes the maximum density.

The aim of the multi-lane traffic synchronization is to guide the traffic flow states of all lanes to an agreement at any position on the road, i.e. $\lim_{t\to\infty} (\rho_i(x, t) - \rho_j(x, t)) = 0$ and $\lim_{t\to\infty} (v_i(x, t) - v_j(x, t)) = 0$ for all $x \in [0, L]$ and all $i \neq j$, so that we can avoid lane-changing behaviors in order to improve the road safety.

Define (ρ^*, v^*) as a steady state, and $\tilde{\rho}_i = \rho_i - \rho^*$, $\tilde{v}_i = v_i - v^*$ as the deviation state. Similarly, let $w_i = v_i + a\rho_i$, denote $w^* = v^* + a\rho^*$, and let $\tilde{w}_i = w_i - w^*$. Then system (30) could be linearized under the Riemann coordinate as

$$\partial_t \xi_i + \Gamma \partial_x \xi_i = 0, \quad i = 1, ..., N, \tag{31}$$

where $\xi_i = (\tilde{w}_i, \tilde{v}_i)^{\mathrm{T}}$, and $\Gamma = \mathrm{diag}\{v^*, v^* - a\rho^*\}$. ρ^* and v^* are pre-set constants satisfying $v^* - a\rho^* < 0$, which illustrates the speed of vehicle is shifted from downstream to upstream and the steady traffic state is in congestion mode.

B. PI Boundary Consensus Control

By using the coil or video technologies, we assume that we can measure the average speed $v_i(0, t)$ at the upstream boundary and the vehicle density $\rho_i(L, t)$ at the downstream boundary. Then, they can be transmitted to neighboring lanes through the vehicle-to-vehicle (V2V) and vehicle-to-infrastructure (V2I) communication. Suppose the multi-lane road traffic flow consists of connected autonomous vehicles, and we can autonomously adjust their velocities and headways. Therefore, it is reasonable to assume that $v_i(L, t)$ at the downstream boundary and $\rho_i(0, t)$ at the upstream boundary are the control variables. Then, the boundary consensus control law is as follows

$$\begin{cases} v_i(L,t) = k_1\tilde{v}_i(0,t) + k_2\hat{v}_i(0,t) + k_2\int_0^t \hat{v}_i(0,\tau)d\tau + v^* \\ \rho_i(0,t) = k_3\tilde{v}_i(0,t) + k_4\hat{v}_i(0,t) + k_5\int_0^t \hat{v}_i(0,\tau)d\tau \\ \quad +k_6\big(\tilde{\rho}_i(L,t) + \frac{k_2}{k_1}\hat{\rho}_i(L,t) + \frac{k_2}{k_1}\int_0^t \hat{\rho}_i(L,\tau)d\tau\big) \\ \quad +\frac{k_2 k_6}{a}\big(\check{v}_i(0,t) + \int_0^t \check{v}_i(0,\tau)d\tau\big) + \delta_i + \rho^* \end{cases} \tag{32}$$

for $i = 1, ..., N$, where $k_1, k_2, k_3, k_4, k_5, k_6$ are feedback gains, δ_i represents an unknown constant input disturbance at the upstream boundary,

$$\hat{v}_i(0,t) = \sum_{j=1}^{N} a_{ij}(v_j(0,t) - v_i(0,t)),$$

$$\hat{\rho}_i(L,t) = \sum_{j=1}^{N} a_{ij}(\rho_j(L,t) - \rho_i(L,t)),$$

$$\check{v}_i(0,t) = \sum_{j=1}^{N} a_{ij}(\hat{v}_j(0,t) + \int_0^t \hat{v}_j(0,\tau)d\tau - \hat{v}_i(0,t) - \int_0^t \hat{v}_i(0,\tau)d\tau),$$

$a_{ij} = 1$ if the i-th lane receives the information from the j-th lane, and $a_{ij} = 0$ otherwise.

Then the boundary condition of the system under the consensus control is

$$\begin{aligned} \xi_{i,in} =& A\xi_{i,out} + K_P \sum_{j=1}^{N} a_{ij}(\xi_{j,out} - \xi_{i,out}) \\ &+ K_I \sum_{j=1}^{N} a_{ij} \int_0^t (\xi_{j,out}(s) - \xi_{i,out}(s))\, ds + \theta_i, \end{aligned}$$

where

$$A = \begin{bmatrix} k_6 & ak_3 - k_1k_6 + 1 \\ 0 & k_1 \end{bmatrix}, K_P = \begin{bmatrix} \frac{k_2k_6}{k_1} & ak_4 - k_1k_2k_6 - k_2k_6 \\ 0 & k_2 \end{bmatrix},$$

$$K_I = \begin{bmatrix} \frac{k_2k_6}{k_1} & ak_5 - k_1k_2k_6 - k_2k_6 \\ 0 & k_2 \end{bmatrix}, \theta_i = \begin{bmatrix} a\delta_i & 0 \end{bmatrix}^{\mathrm{T}}.$$

C. Simulation

In our simulation, the multi-lane road traffic flow system (30) consists of three lanes. Set related traffic and road parameters as $L = 1\ km, \rho^* = 80\ veh./km, v^* = 30\ km/h, \rho_m = 160\ veh./km$ and $v_f = 120\ km/h$. The upstream boundary input disturbance $\delta_{1,2,3} = -1, 0, 1$. The initial states of the three-lane road traffic flow system are as follows:

$$\begin{cases} \rho_1(x,0) &= 100 + 5\cos(\pi x) \\ v_1(x,0) &= 25 - 5\sin(\pi x) \\ \rho_2(x,0) &= 80 + 8\cos(0.6\pi x) \\ v_2(x,0) &= 30 + 8\sin(0.6\pi x) \\ \rho_3(x,0) &= 60 + 3\cos(0.8\pi x) \\ v_3(x,0) &= 35 - 3\sin(0.8\pi x) \end{cases}, \quad x \in [0, L].$$

Set $k_1 = 1, k_2 = 0.3, k_3 = -2.8, k_4 = k_5 = -\frac{14}{15}$, and $k_6 = -1$. Thus, we have

$$A = \begin{bmatrix} -1 & -0.1 \\ 0 & 1 \end{bmatrix}, K_P = K_I = \begin{bmatrix} -0.3 & -0.1 \\ 0 & 0.3 \end{bmatrix}.$$

Considering the undirected communication topology for the three-lane road traffic flow system as shown in Fig. 1,

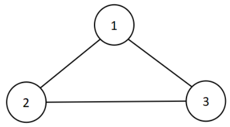

Fig. 1. Communication topology of the three-lane road traffic flow system.

we have $\mathcal{L} = [2\ -1\ -1; -1\ 2\ -1; -1\ -1\ 2]$, and hence $\lambda_2(\mathcal{L}) = \lambda_3(\mathcal{L}) = 3$.

Solving the inequality conditions (12) and (21), we obtain $P_1 = [5.2743\ \ 0; 0\ \ 2.6871]$, $P_2 = [22.3531\ -10.6835; -10.6835\ \ 25.5264]$, and $P_3 = [3.3108\ 4.1817; 4.1817\ 0.6132]$. Hence the sufficient conditions for ensuring the asymptotic consensus given in Theorem 1 are satisfied.

As shown in Fig. 2, we can see the spatio-temporal evolution of state deviations between any two lanes. In more specific words, as shown in Fig. 2(a), for all $x \in [0, L]$, $\rho_1 - \rho_2$ and $\rho_1 - \rho_3$ converge to 0. As shown in Fig. 2(b), for all $x \in [0, L]$, $v_1 - v_2$ and $v_1 - v_3$ converge to 0. We can conclude that the multi-lane traffic synchronization is achieved. Especially, in Fig. 3, we can see the spatial trajectories of ρ_1 and v_1 at $t = 0.6\ hr$, demonstrating that the final agreed traffic density and speed vary with x instead of being stabilized to (ρ^*, v^*).

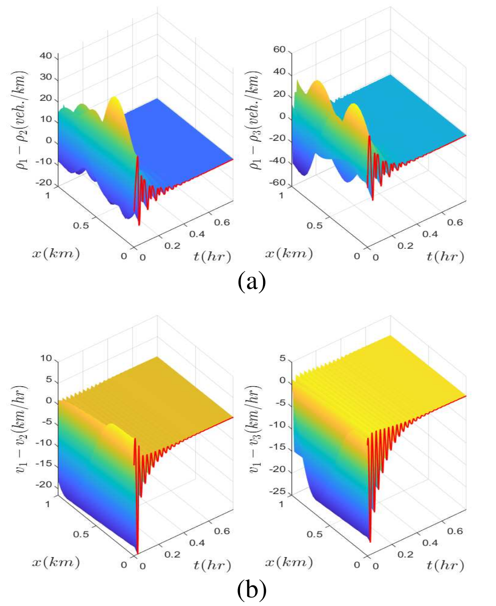

Fig. 2. The spatio-temporal evolution of state deviations between any two lanes. (a) $\rho_1 - \rho_2$ and $\rho_1 - \rho_3$; (b) $v_1 - v_2$ and $v_1 - v_3$.

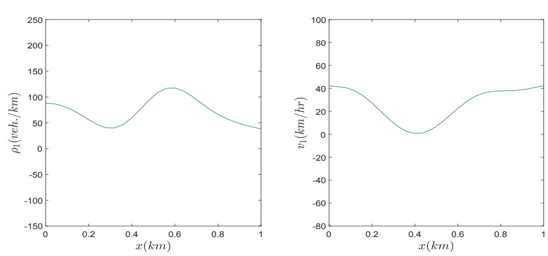

Fig. 3. The spatial trajectory of the final state of lane 1.

V. Conclusion

This paper has proposed a PI boundary consensus protocol for hyperbolic multi-agent systems with input disturbances, and presented an asymptotic consensus analysis under undirected communication topologies. We have employed the Lyapunov approach to prove that the proposed PI boundary consensus protocol could drive the hyperbolic multi-agent systems reach asymptotical consensus, in which the sufficient conditions w.r.t. the PI boundary control matrices and Laplacian matrices are derived. Then, we integrate the spectral decomposition technique with Lyapunov approach to derive sufficient conditions related to Laplacian eigenvalues, which are more tractable. Furthermore, we have provided an application to the synchronization of a multi-lane road traffic flow system, and simulation results have been provided to verify the theoretical results.

Our future research may include exploring sufficient consensus conditions with less number of LMIs and extending the results to the directed communication topology case.

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