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Design of Sparse Control with Minimax Concave Penalty

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***Abstract*— In this paper, we propose a novel computational method for sparse control, also known as maximum hands-off control, using the minimax concave penalty. The sparse control problem is formulated as an L^0-optimal control problem, which is known to be hard to solve. To overcome this difficulty, we propose using the minimax concave penalty as a surrogate for the L^0 norm. We demonstrate the equivalence between the original and proposed control problems without relying on the normality assumption, which is typically required when approximating the L^0 norm with the L^1 norm. Furthermore, we present an effective numerical algorithm for the proposed optimal control based on the Alternating Direction Method of Multipliers (ADMM). A design example is shown to illustrate the effectiveness of the proposed method.**

I. Introduction

Sparse control [1], [2], [3], [4], also known as maximum hands-off control, is a type of control that has the minimum support length. In other words, sparse control deactivates actuation for as long as possible. Such a control can reduce not only consumption of energy but also generation of harmful exhaust gases in a vehicle, for example. Due to these benefits, sparse control is often referred to as *green control* [5]. Theoretical results on sparse control have been actively reported for various systems, including stochastic control systems [6], infinite-dimensional systems [7], discrete-time linear systems [8], [9], [10], and nonlinear systems [11], [12]. Additionally, applications in diverse fields have been proposed, such as thermally activated building systems (TABS) [13], mobility networks [14], quadrotors [15], spacecrafts [16], [17], [18], and robotics [19].

Mathematically, the design of continuous-time sparse control is formulated as an optimal control problem with L^0 norm minimization, which is challenging due to its nonconvex and discontinuous nature. A common approach to circumvent this difficulty is to approximate the L^0 norm by the L^1 norm, simplifying the control problem to the classical minimum-fuel control problem [20]. Under the assumption of normality of the L^1 optimal control problem, the two optimal control problems are proved to be equivalent [1], [4]. Specifically, for linear time-invariant systems, sparse control computation can be efficiently addressed as a convex optimization problem after time-discretization. However, if the system matrix A is nonsingular, the normality assumption may not hold for certain initial states, which precludes the use of the L^1 optimization. Indeed, a gap between the optimal solutions of L^0 and L^1 optimal control have been observed in some systems [3].

The objective of this study is to address this problem using a nonconvex function known as the *minimax concave penalty* (MCP) [21], [22], [23], which has been used for finite-dimensional optimization problems in signal processing. Namely, we approximate the L^0 norm in the sparse optimal control problem with the minimax concave penalty. This paper first establishes the equivalence between the newly formulated control problem and the original L^0 optimal control, notably *without the normality assumption*. Subsequently, we show an advantage of employing the minimax concave penalty over other non-convex functions such as the smoothly clipped absolute deviation (SCAD) and the log-sum penalty (LSP). in numerical computations. It has been shown in [24] that the nonconvex optimal control problem can be directly formulated as difference-of-convex (DC) programming after time discretization for non-convex penalty functions such as MCP, SCAD, and LSP. However, we show that the problem can also be equivalently transformed into an optimization problem that can be more effectively solved by the alternating direction method of multipliers (ADMM) [25], [26] compared to the DC programming. Actually, the computational time of the proposed algorithm is comparable with the convex optimization algorithm based on L^1-norm regularization as shown in the numerical example in Section V.

The organization of this paper is as follows: Section II formulates the sparse optimal control problem. Section III proposes adopting the minimax concave penalty to solve the control problem, and show the equivalence between the original and proposed control problems. Section IV shows a numerical computation for the proposed control, which can be reduced to an optimization problem that can be effectively solved by ADMM. Section V illustrates an example of sparse control to show the effectiveness of the proposed method. Finally, Section VI makes concluding remarks.

II. Sparse Control Problem

Consider the following state equation.

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + \boldsymbol{b}u(t),\ t \in [0,T],\ \boldsymbol{x}(0) = \boldsymbol{\xi} \in \mathbb{R}^d, \quad (1)$$

where $A \in \mathbb{R}^{d\times d}$ and $\boldsymbol{b} \in \mathbb{R}^d$ are fixed, and $\boldsymbol{x}(t) \in \mathbb{R}^d$ indicates the state and $u(t) \in \mathbb{R}$ the control at time t. T represents the terminal time, and $\boldsymbol{\xi}$ is the initial state. For this linear system, we consider the optimal control problem of finding the control $u(t)$ that minimizes the following cost

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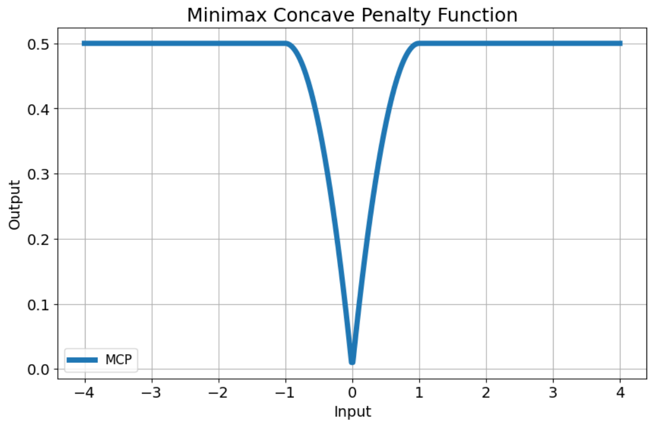

Fig. 1. Minimax concave penalty function with $\alpha = 1, \beta = 1$

function:

$$J(u) = \|\boldsymbol{x}(T)\|_2^2 + \lambda \|u\|_0, \tag{2}$$

where $\lambda > 0$, and $\|u\|_0$ indicates the L^0 norm (the length of the support) of function u. We also consider the following magnitude constraint on the control:

$$|u(t)| \leq 1, \quad \forall t \in [0, T]. \tag{3}$$

This problem is hard to solve due to the nonconvexity and discontinuity of the L^0 norm. In the next section, we adopt the minimax concave penalty as a surrogate for the L^0 norm for efficiently solving the control problem.

The optimal control problem is the Bolza type with the cost function including the ℓ^2 norm of the final state $\boldsymbol{x}(T)$. This is different from the original formulation of maximum hands-off control [1] in which the final state constraint $\boldsymbol{x}(T) = \boldsymbol{0}$ is used and the cost function does not include $\boldsymbol{x}(T)$, namely the Lagrange type. The method and theory in the following sections are still valid when we consider the original formulation.

III. Sparse Control via Minimax Concave Penalty

Here we introduce the minimax concave penalty function $\psi_{\mathrm{MC}}(u; \alpha, \beta) : \mathbb{R} \mapsto \mathbb{R}_+$ with hyperparameters $\alpha > 0$ and $\beta > 0$ defined by

$$\psi_{\mathrm{MC}}(u; \alpha, \beta) \triangleq \begin{cases} \alpha|u| - \dfrac{1}{2\beta}u^2, & |u| \leq \alpha\beta, \\ \dfrac{1}{2}\beta\alpha^2, & |u| > \alpha\beta. \end{cases} \tag{4}$$

Figure 1 shows the curve of the minimax concave penalty function with $\alpha = 1$ and $\beta = 1$. From this figure, we can see that the function is non-convex, but sharply pointed at the origin. This property is known to induce sparsity more effectively than the commonly used ℓ^1 norm.

We note that for a multi-input system with control input $\boldsymbol{u}(t) \in \mathbb{R}^m$, we define the vector minimax concave (VMC) penalty function [22] defined by

$$\psi_{\mathrm{VMC}}(\boldsymbol{u}; \alpha, \beta) \triangleq \sum_{i=1}^{n} \psi_{\mathrm{MC}}(u_i; \alpha, \beta).$$

For simplicity, we focus on single-input systems, but the results shown below are the same for multi-input systems.

We fix the hyperparameters α and β, and we reformulate the original sparse control problem in Section II with the following function as a surrogate for the L^0 norm:

$$\psi(u) = \frac{\psi_{\mathrm{MC}}(u; \alpha, \beta)}{\psi_{\mathrm{MC}}(1; \alpha, \beta)}. \tag{5}$$

Note that this function has the following properties, which are used to show the equivalence theorem below:

(i) $\psi(0) = 0$,
(ii) $\psi(1) = \psi(-1) = 1$,
(iii) $\psi(u) \leq u^0$ for any $u \in [-1, 1]$,

where $u^0 = 1$ if $u \neq 0$ and $0^0 = 0$. Then the proposed optimal control problem with the minimax concave penalty is described as follows:

Problem 1: Given $\boldsymbol{\xi} \in \mathbb{R}^d$, $T > 0$, $\lambda > 0$, $\alpha > 0$, and $\beta > 0$, find a control u over $[0, T]$ that solves

$$\begin{aligned} \underset{u}{\text{minimize}} \quad & \|\boldsymbol{x}(T)\|_2^2 + \lambda \int_0^T \psi(u(t))dt \\ \text{subject to} \quad & \dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + \boldsymbol{b}u(t), \quad t \in [0, T], \\ & \boldsymbol{x}(0) = \boldsymbol{\xi}, \\ & |u(t)| \leq 1, \quad t \in [0, T]. \end{aligned}$$

Throughout the paper, we denote by J_{MC} the cost function in Problem 1.

First, we show that the optimal control has the following property called the *bang-off-bang property*:

Lemma 1 (bang-off-bang property): Suppose that there exists at least one optimal solution to Problem 1. Then, any optimal solution to Problem 1 takes values belonging to the set $\{0, \pm 1\}$ almost everywhere.

Proof: Let us take any solution u^* to Problem 1. It follows from Theorem 22.2 and Corollary 22.3 in [27] that there exists a function p that satisfies

$$u^*(t) \in \underset{u \in [-1,1]}{\arg\max} \left(p(t)^\top \boldsymbol{b}u - \lambda\psi(u)\right) \tag{6}$$

almost everywhere. For any $t \in [0, T]$, the function $\phi(u) \triangleq p(t)^\top \boldsymbol{b}u - \lambda\psi(u)$ is convex and is not constant over $[0, 1]$ from the definition of ψ. Hence, we have

$$\underset{u \in [0,1]}{\arg\max}\ \phi(u) \subset \{0, 1\}. \tag{7}$$

Similarly, we have

$$\underset{u \in [-1,0]}{\arg\max}\ \phi(u) \subset \{-1, 0\}. \tag{8}$$

The result immediately follows from (6), (7), and (8). ∎

By using the bang-off-bang property, we have the following equivalence theorem between the original and the proposed optimal controls.

Theorem 1 (equivalence): Suppose that there exists at least one optimal solution to Problem 1. Let us denote by $\mathcal{U}_0^*$ and $\mathcal{U}_{\mathrm{MC}}^*$ the sets of all optimal solutions to the sparse optimal control problem in Section II and Problem 1, respectively. Then, we have $\mathcal{U}_0^* = \mathcal{U}_{\mathrm{MC}}^*$.

Proof: Let us take any $u_{\rm MC}^* \in \mathcal{U}_{\rm MC}^*$. From Lemma 1, the control satisfies $u_{\rm MC}^*(t) \in \{0, \pm 1\}$ almost everywhere, and let us put

$$E_0 \triangleq \{t \in [0,T] : u_{\rm MC}^*(t) = 0\},$$
$$E_1 \triangleq \{t \in [0,T] : |u_{\rm MC}^*(t)| = 1\}.$$

Note that the sets E_0 and E_1 are disjoint and $\mu_{\rm L}(E_0) + \mu_{\rm L}(E_1) = T$, where $\mu_{\rm L}$ denotes the Lebesgue measure. Then, we have

$$\begin{aligned}\int_0^T \psi(u_{\rm MC}^*(t))dt &= \sum_{i=0}^{1} \int_{E_i} \psi(u_{\rm MC}^*(t))dt \\ &= \int_{E_1} \psi(u_{\rm MC}^*(t))dt \\ &= \int_{E_1} \psi(1)dt = \mu_{\rm L}(E_1) = \|u_{\rm MC}^*\|_0.\end{aligned} \tag{9}$$

Note also that we have

$$\int_0^T \psi(u(t))dt \leq \|u\|_0 \tag{10}$$

for any $u \in \mathcal{U}$, where $\mathcal{U} \triangleq \{u : |u(t)| \leq 1 \text{ on } [0,T]\}$, since $\psi(u(t)) \leq u(t)^0$ for any t. Hence, for any $u \in \mathcal{U}$, we have

$$J(u_{\rm MC}^*) = J_{\rm MC}(u_{\rm MC}^*) \leq J_{\rm MC}(u) \leq J(u),$$

where the first relation follows from (9), the second relation follows from the optimality of $u_{\rm MC}^* \in \mathcal{U}_{\rm MC}^*$, and the third relation follows from (10). This implies $u_{\rm MC}^* \in \mathcal{U}_0^*$. Hence, the set $\mathcal{U}_0^*$ is not empty, and $\mathcal{U}_{\rm MC}^* \subset \mathcal{U}_0^*$.

We next take any $u_0^* \in \mathcal{U}_0^*$. For any $u \in \mathcal{U}$, we have

$$J_{\rm MC}(u_0^*) \leq J(u_0^*) = J(u_{\rm MC}^*) = J_{\rm MC}(u_{\rm MC}^*) \leq J_{\rm MC}(u),$$

where the first relation follows from (10), the second relation follows from $\mathcal{U}_{\rm MC}^* \subset \mathcal{U}_0^*$, the third relation follows from (9), and the fourth relation follows from the optimality of $u_{\rm MC}^* \in \mathcal{U}_{\rm MC}^*$. This implies $u_0^* \in \mathcal{U}_{\rm MC}^*$, and hence $\mathcal{U}_0^* \subset \mathcal{U}_{\rm MC}^*$. This gives the result. ■

From this theorem, we can focus on solving Problem 1 for sparse optimal control. In the next section, we show that Problem 1 can be reduced to an optimization problem that can be numerically solved after time discretization.

IV. Numerical Solution via ADMM

Here we show a numerical computation method for solving Problem 1.

A. Time discretization

First, we discretize Problem 1 by time discretization. For this, we split the time interval $[0,T]$ into n subintervals. Let the width of time discretization be $h > 0$, and let $T \triangleq nh$. Then, we assume the zero-order hold assumption [28] on the control $u(t)$. Namely, the control $u(t)$ is constant in each subinterval, that is,

$$u(t) \triangleq u(jh) \triangleq u_{\rm d}[j],\ t \in [jh, (j+1)h),$$
$$j = 0, 1, \ldots, n-1.$$

Then, the state equation (1) can be described as the following discrete-time state equation:

$$\boldsymbol{x}_{\rm d}[j+1] = A_{\rm d}\boldsymbol{x}_{\rm d}[j] + \boldsymbol{b}_{\rm d}u_{\rm d}[j],$$
$$j = 0, 1, \ldots, n-1,\ \boldsymbol{x}_{\rm d}[0] = \boldsymbol{\xi},$$

where $\boldsymbol{x}_{\rm d}[j]$ represents the discrete-time state defined by $\boldsymbol{x}_{\rm d}[j] \triangleq \boldsymbol{x}(jh)$, and

$$A_{\rm d} \triangleq e^{Ah},\ \boldsymbol{b}_{\rm d} \triangleq \int_0^h e^{At}\boldsymbol{b}dt.$$

The terminal state $\boldsymbol{x}(T)$ is given as

$$\boldsymbol{x}(T) = \boldsymbol{x}_{\rm d}[n] = \Phi\boldsymbol{u} - \boldsymbol{\zeta},$$

where $\boldsymbol{\zeta} \triangleq -A_{\rm d}^n\boldsymbol{\xi} \in \mathbb{R}^d$ and

$$\Phi \triangleq \begin{bmatrix} A_{\rm d}^{n-1}\boldsymbol{b}_{\rm d} & A_{\rm d}^{n-2}\boldsymbol{b}_{\rm d} & \cdots & \boldsymbol{b}_{\rm d} \end{bmatrix} \in \mathbb{R}^{d\times n},$$
$$\boldsymbol{u} \triangleq \begin{bmatrix} u_{\rm d}[0] & u_{\rm d}[1] & \ldots & u_{\rm d}[n-1] \end{bmatrix}^\top \in \mathbb{R}^n.$$

From this, the first term of the cost function (2) is represented as $\|\boldsymbol{x}(T)\|_2^2 = \|\Phi\boldsymbol{u} - \boldsymbol{\zeta}\|_2^2$. The second term of (2) is also transformed as

$$\begin{aligned}\lambda \int_0^T \psi\big(u(t)\big)dt &= \frac{\lambda}{\psi_{\rm MC}\big(1;\alpha,\beta\big)} \int_0^T \psi_{\rm MC}\big(u(t);\alpha,\beta\big)dt \\ &= \frac{\lambda h}{\psi_{\rm MC}\big(1;\alpha,\beta\big)} \sum_{j=0}^{n-1} \psi_{\rm MC}\big(u_{\rm d}[j];\alpha,\beta\big) \\ &= \tilde{\lambda}\|\boldsymbol{u}\|_{\rm MC}\end{aligned}$$

where

$$\tilde{\lambda} \triangleq \frac{\lambda h}{\psi_{\rm MC}\big(1;\alpha,\beta\big)},\quad \|\boldsymbol{u}\|_{\rm MC} \triangleq \sum_{j=0}^{n-1} \psi_{\rm MC}\big(u_{\rm d}[j];\alpha,\beta\big)$$

Finally, the constraint (3) on the control input is described as $\|\boldsymbol{u}\|_\infty \leq 1$.

Consequently, the optimal control problem (Problem 1) is transformed into

$$\begin{aligned}&\underset{\boldsymbol{u}\in\mathbb{R}^n}{\text{minimize}} && \|\Phi\boldsymbol{u} - \boldsymbol{\zeta}\|_2^2 + \tilde{\lambda}\|\boldsymbol{u}\|_{\rm MC} \\ &\text{subject to} && \|\boldsymbol{u}\|_\infty \leq 1.\end{aligned} \tag{11}$$

Although the cost function in (11) is nonconvex, the optimization problem can be efficiently solved by the alternating direction method of multipliers (ADMM) as shown in the next subsection.

B. ADMM algorithm

Here we derive the ADMM-based algorithm for the optimization problem in (11).

First, let us define the constraint set C by

$$C \triangleq \{\boldsymbol{u} \in \mathbb{R}^n : \|\boldsymbol{u}\|_\infty \leq 1\}, \tag{12}$$

and its indicator function I_C by

$$I_C(\boldsymbol{u}) \triangleq \begin{cases} 0, & \boldsymbol{u} \in C, \\ +\infty, & \boldsymbol{u} \notin C. \end{cases} \tag{13}$$

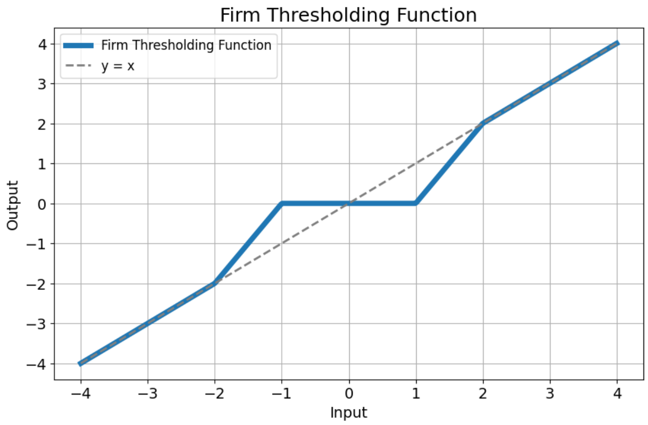

Fig. 2. Firm thresholding function with $c_1 = 1, c_2 = 2$

Then, the optimization problem (11) is equivalently described by

$$\underset{\boldsymbol{u}\in\mathbb{R}^n}{\text{minimize}} \ \ \|\Phi\boldsymbol{u}-\boldsymbol{\zeta}\|_2^2+\tilde{\lambda}\|\boldsymbol{u}\|_{\text{MC}}+I_C(\boldsymbol{u}).$$

Defining new variables $\boldsymbol{z}_0 \in \mathbb{R}^d, \boldsymbol{z}_1, \boldsymbol{z}_2 \in \mathbb{R}^n$ by

$$\boldsymbol{z}_0 = \Phi\boldsymbol{u},\ \ \boldsymbol{z}_1 = \boldsymbol{z}_2 = \boldsymbol{u},$$

we obtain the following optimization problem suitable for the alternating direction method of multipliers (ADMM) [25]:

$$\begin{aligned} &\underset{\boldsymbol{u}\in\mathbb{R}^n,\boldsymbol{z}\in\mathbb{R}^m}{\text{minimize}} && \|\boldsymbol{z}_0-\boldsymbol{\zeta}\|_2^2+\tilde{\lambda}\|\boldsymbol{z}_1\|_{\text{MC}}+I_C(\boldsymbol{z}_2)\\ &\text{subject to} && \boldsymbol{z}=\Psi\boldsymbol{u}, \end{aligned} \tag{14}$$

where $m \triangleq d + 2n$, and

$$\boldsymbol{z} \triangleq \begin{bmatrix}\boldsymbol{z}_0\\ \boldsymbol{z}_1\\ \boldsymbol{z}_2\end{bmatrix} \in \mathbb{R}^m,\ \ \Psi \triangleq \begin{bmatrix}\Phi\\ I\\ I\end{bmatrix} \in \mathbb{R}^{m\times n}.$$

Although the function $\tilde{\lambda}\|\boldsymbol{z}_1\|_{\text{MC}}$ is nonconvex, it has the following useful property [23], [22]:

Lemma 2: Let $\boldsymbol{y} \in \mathbb{R}^n$ and $\gamma > 0$. Define $f : \mathbb{R}^n \mapsto \mathbb{R}$ by

$$f(\boldsymbol{x}) = \frac{1}{2}\|\boldsymbol{y}-\boldsymbol{x}\|_2^2+\gamma\tilde{\lambda}\|\boldsymbol{x}\|_{\text{MC}}.$$

If $\beta \geq \gamma\tilde{\lambda}$ holds, then f is convex. Moreover, the minimizer of f is given by

$$\boldsymbol{x}^* = \text{firm}(\boldsymbol{y};\alpha\gamma\tilde{\lambda},\alpha\beta),$$

where $\alpha > 0$ and $\beta > 0$ are the hyperparemeters in the minimax concave penalty function in (4), and $\text{firm}(\boldsymbol{y};c_1,c_2)$ is the *firm thresholding function* (see Figure 2) defined by

$$[\text{firm}(\boldsymbol{y};c_1,c_2)]_i \triangleq \begin{cases} 0, & |y_i| \leq c_1,\\ \text{sgn}(y_i)\frac{c_2(|y_i|-c_1)}{c_2-c_1}, & c_1 < |y_i| \leq c_2,\\ y_i, & |y_i| > c_2, \end{cases}$$

for $i = 1, 2, \ldots, n$ with $c_2 > c_1 > 0$.

Algorithm 1 ADMM algorithm for solving optimization problem (14)

Initial values: $\boldsymbol{z}[0], \boldsymbol{v}[0] = [\boldsymbol{v}_0[0]^\top, \boldsymbol{v}_1[0]^\top, \boldsymbol{v}_2[0]^\top]^\top$
Positive constants: $\alpha > 0,\ \beta > 0,\ \gamma > 0,\ \tilde{\lambda} > 0$
Maximum iteration number: $\texttt{MAX\_ITER} > 0$
Initial iteration number: $k := 0$
while $k < \texttt{MAX\_ITER}$ **do**
$\quad \boldsymbol{u}[k+1] := (\Psi^\top\Psi)^{-1}\Psi^\top(\boldsymbol{z}[k]-\boldsymbol{v}[k])$
$\quad \boldsymbol{z}[k+1] := \begin{bmatrix} \frac{1}{2\gamma+1}\big(2\gamma\boldsymbol{\zeta}+(\Phi\boldsymbol{u}[k+1]+\boldsymbol{v}_0[k])\big)\\ \text{firm}\big(\boldsymbol{u}[k+1]+\boldsymbol{v}_1[k];\alpha\gamma\tilde{\lambda},\alpha\beta\big)\\ \Pi_C(\boldsymbol{u}[k+1]+\boldsymbol{v}_2[k]) \end{bmatrix}$
$\quad \boldsymbol{v}[k+1] := \boldsymbol{v}[k]+\Psi\boldsymbol{u}[k+1]-\boldsymbol{z}[k+1]$
$\quad k := k+1$
end while
return $\boldsymbol{u}[k], \boldsymbol{z}[k], \boldsymbol{v}[k]$

Now, the ADMM-based algorithm for the optimization problem (14) is given in Algorithm 1. In this algorithm, Π_C is the projection operator onto the set C in (12) defined by

$$\Pi_C(\boldsymbol{v}) = \big[\text{sat}(v_1), \text{sat}(v_2), \ldots, \text{sat}(v_n)\big]^\top$$

where $\text{sat}(v_i) \triangleq \text{sgn}(v_i)\min(1,|v_i|)$, and v_i is the i-th element of vector $\boldsymbol{v}$. Although the optimization is nonconvex, the convergence of this algorithm is guaranteed [22].

V. Numerical Example

We here consider the following double integrator:

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\boldsymbol{x}(t) + \begin{bmatrix}0\\ 1\end{bmatrix}u(t),\ \ \boldsymbol{x}(0) = \begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix}.$$

It is reported in [3] that with the initial state $(\xi_1, \xi_2) = (10, -3)$ and the terminal time $T = 5$, the L^1-optimal control problem is not normal and hence we cannot adopt the L^1-norm approximation to obtain sparse control for this system. On the other hand, our method does not require the normality assumption on the optimal control problem, and hence we can obtain a sparse control signal for this system.

For time discretization, we set the sampling time $h = 0.05$, and the number of subintervals $n = 100$. We solve the sparse control problem (Problem 1) with hyperparameters $\alpha = 1$ and $\beta = 1$ for the minimax concave panalty in (4). We change the regularization parameter $\tilde{\lambda}$ in the range $[0.0001, 4]$. We compare the conventional L^1 optimization that minimizes

$$\|\boldsymbol{x}(T)\|_2^2 + \lambda\int_0^T |u(t)|dt.$$

Figure 3 shows the ℓ^2 norm of the terminal state, i.e., $\|\boldsymbol{x}(T)\|_2$ for each value of the regularization parameter $\tilde{\lambda}$ after time discretization. It can be observed that the control performance measured by $\|\boldsymbol{x}(T)\|_2$ by the proposed control is always better than or equal to that by the L^1 optimization for any regularization parameters.

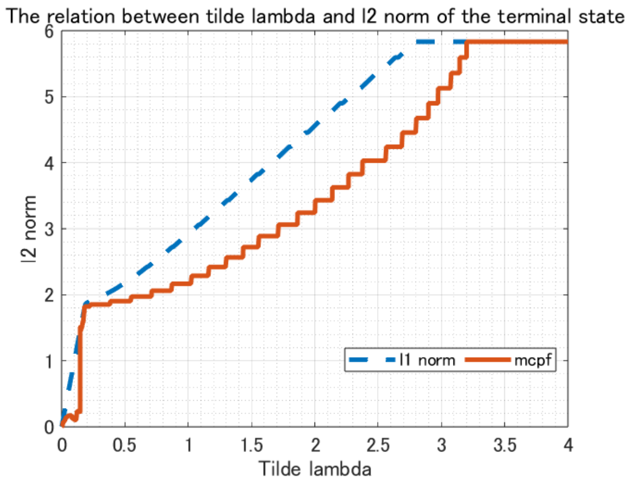

Fig. 3. The relation between $\tilde{\lambda}$ and $\|\boldsymbol{x}(T)\|_2$ by the proposed method (solid line) and the L^1 optimization (dashed line).

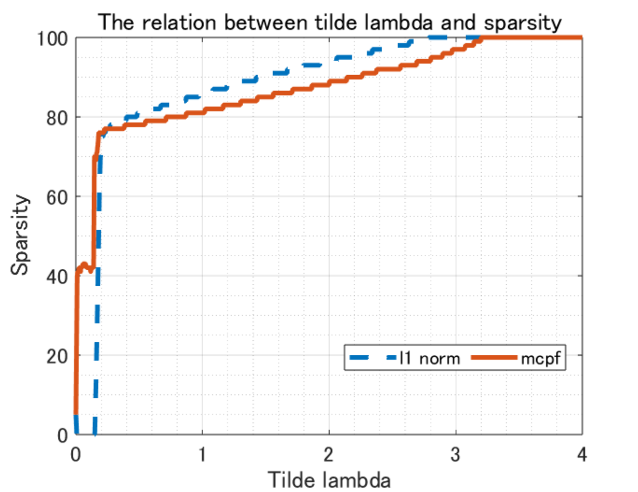

Fig. 4. The relation between $\tilde{\lambda}$ and the sparsity by the proposed method (solid line) and the L^1 optimization (dashed line).

Figure 4 shows the number of subintervals on which the control is zero, which is a measure of sparsity, against the regularization parameter. We can see that for small values of the regularization parameter $\tilde{\lambda}$, the proposed method gives sparse control while the L^1-based control exhibits no sparsity.

To clearly illustrate the advantage of the proposed method, we choose regularization parameters of the two control methods such that $\|\boldsymbol{x}(T)\|_2 = 0.087$. Figure 5 shows the control signals by the proposed and the L^1-based methods. The sparsity of the proposed method is 41 while that of the L^1-based control is 0, that is, no sparse control. Moreover, we show the computational time by the proposed method is almost the same as that by the L^1-based method. The comparison is summarized in Table I. This result well illustrates the effectiveness of the proposed method. The MATLAB codes in this example can be downloaded at [29].

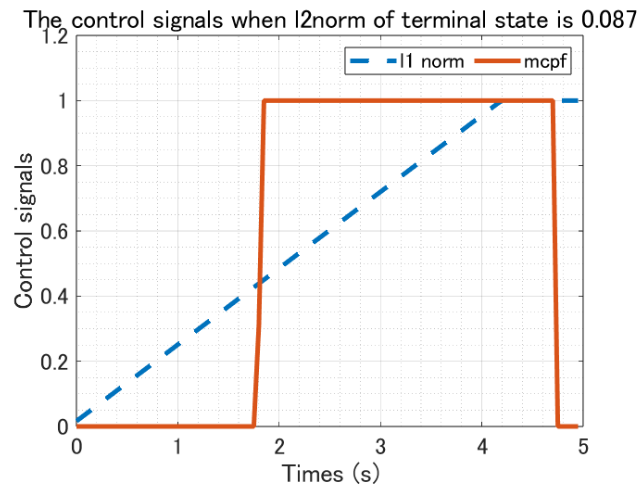

Fig. 5. The control signals that achieve $\|\boldsymbol{x}(T)\|_2 = 0.087$: proposed (solid line) and L^1-optimal (dashed line)

method	sparsity	computatinal time (s)
proposed method	41	0.1086
L^1-based method	0	0.1153

TABLE I

COMPARISON OF SPARSITY AND COMPUTATIONAL TIME

VI. Conclusion

In this paper, we have proposed a sparse control method using the minimax concave penalty. This method is particularly advantageous for sparse control problems that are non-normal where L^1-based approaches may fail. To illustrate this, we provided a numerical example that demonstrates the effectiveness of our method in addressing such non-normal problems. Future work will focus on constructing an equivalence theorem for discrete-time systems, and determining the optimal choice of the hyperparameters α and β in the minimax concave penalty.

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