IEEE Control Systems Letters paper presented at
2024 IEEE 63rd Conference on Decision and Control (CDC)
December 16-19, 2024. MiCo, Milan, Italy

A New Approach to the Energy-to-Peak Performance Analysis of Continuous-time Markov Jump Linear Systems

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***Abstract*— The energy-to-peak (L_2–L_∞) performance of continuous-time Markov jump linear systems (MJLS) is studied from a new perspective: by considering an output norm which is different from the one treated in the existing MJLS literature, we devise upper and lower bounds to the worst-case energy-to-peak gain (L_2–L_∞ induced norm). The lower bound can be efficiently computed by solving an ordinary differential equation, or, if the jump process is irreducible, a Lyapunov (algebraic) equation. The upper bound is obtained from the controllability Gramian of the MJLS (a set of coupled Lyapunov equations). As a by-product of this result, we are also able to show the consistence of the corresponding L_2–L_∞ system norm vis-à-vis the H$_2$ norm of MJLS: it is proven that the H_2 norm is an upper bound to the worst-case energy-to-peak gain, a feature which was not proven in other works devoted to the MJLS case. In the case without jumps, it is shown that both bounds coincide with the actual L_2–L_∞ system norm.**

I. Introduction

The incorporation of stochastic jump phenomena into systems and control theory has motivated a significant part of the scientific research which has been conducted over the last 50 years or so. The safety critical requirements of modern technologies such as robotics, aeronautics, electronics and distributed computing, to cite a few areas, have pushed the boundaries in all possible directions, giving rise to a mature and rich theory. Among the various existing approaches, we can cite hybrid systems, switched systems, and Markov jump systems as themes of recurring interest [1]–[5].

Notwithstanding the formidable progresses witnessed by the research community in recent times, many seemingly simple extensions have somewhat resisted the test of time, and, despite all the efforts for their solution, still remain to be solved. Such is the case, for instance, of the performance analysis of Markov jump linear systems in the energy-to-peak (L_2-L_∞) sense. Some of the classical references dealing with worst-case performance in the input-output sense (induced system norms) include, for instance, [6]–[8]. In the MJLS case, we can cite [9]–[12] as a sample of other references which dealt with L_2-L_∞ performance, chiefly with respect to the design of L_2-L_∞ filters, and with no examination of the inherent conservatism involved in the performance analysis. There are also several works devoted to the discrete-time MJLS case, such as [13]–[15]. This scenario, however, falls outside the scope of the present note (which is the continuous-time setup).

The interest in the L_2-L_∞ metrics typically comes from situations in which large *instantaneous* excursions are undesirable (which is intrinsically different, for instance, from the H_∞ setup, in which performance is measured from the smoother perspective of L_2-L_2, or energy-to-energy, gain). In practice, the L_2-L_∞ perspective is useful in those situations where the outputs must not exceed some predefined threshold, at the penalty of adverse phenomena such as saturations, damages to electromechanical components, ketoacidosis, or reaching an economic default, for instance. From an analyst's perspective, the computation of L_2-L_∞ system norms helps us to know beforehand whether such adversities can occur, in a worst-case scenario. (Of course, in cases where the exact norm cannot be computed precisely, the next best thing is to obtain some bounds for it, or, in control/filtering design problems, to ensure that a guaranteed cost is attained – even if at the expense of some conservatism.)

A. Our approach and contributions

Every one of the aforementioned papers devoted to MJLS considered the output norm (with E standing for expectation)

$$y \ \mapsto \ \sup_{t\geq 0} \sqrt{E\big[\|y(t)\|^2\big]}. \tag{1}$$

In this note, however, we shall consider the output norm $\|y\|_\infty = \sup_{t\geq 0} \max_{i=1,\dots,p} E[|y_i(t)|]$, where $y_i(t)$ is the ith output at time t. Following the nomenclature of [6], in the deterministic case (i.e., the scenario without jumps) we are thus considering an $L^{\infty,\infty}$ norm for the output, whereas (1) reduces to the $L^{\infty,2}$ norm. Therefore, in a sense, we are generalizing the classical setup towards a different direction than the existing literature. The profits of this choice of norm are the following contributions:

- We consider an output norm which is not only consistent with the L_∞ norm of deterministic signals, but is also consistent with the H_2 norm of MJLS: it is shown that the H_2 norm is an upper bound to the worst-case energy-to-peak gain. The lack of such a result for setups involving the norm (1) leaves open the possibility that the L_2-L_∞ cost considered by [9]–[12] has no connection with the H_2 norm, as shown in Section IV.
- The possibility that there is a gap between the L_2-L_∞ norm and its upper bound motivates us to derive a lower bound as well. It is shown that the computation of this lower bound is tantamount to the solution of an ordinary differential equation, or, if the Markov jump process is

This work was supported by Fundação Carlos Chagas Filho de Amparo à Pesquisa do Estado do Rio de Janeiro (FAPERJ) under the JCNE grant E-26/201.465/2022, and by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), grant 316397/2023-7.

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Joint submission with L-CSS (doi.org/10.1109/LCSYS.2024.3406492).

irreducible, of a Lyapunov (algebraic) equation. To the best of the author's knowledge, the analysis of the lower bound is new and has no counterpart in the literature that considers the output norm (1). Nevertheless, it is also a (conservative) lower bound for the L_2-to-L_∞ norm with respect to (1). In the case without jumps, the lower and upper bounds possess the good feature of coinciding with the actual L_2-L_∞ norm (Section III-A).

- The algebraic properties of the proposed upper bound for the L_2-L_∞ norm make it possible for us to derive a linear matrix inequalities-based procedure for the synthesis of state-feedback controllers, in the spirit of [7] (Section III-B).

This paper is organized as follows. The bare essentials of notation and preliminaries are enclosed in Section II. Some auxiliary results of germane importance are provided in Section II-A. Section III features the main results (Theorem 11 and Corollaries 13, 14, and 15), and Section IV provides a numerical example which compares our results with a recent work from the literature. Finally, some concluding remarks are included in Section V.

II. Preliminaries

Throughout the paper, $\mathfrak{e}_i$ will stand for a column vector whose entries are all equal to zero, except for the ith entry, which is one. The Euclidean norm of $x \in \mathbb{R}^n$ is $\|x\| = \sqrt{x'x} = \mathrm{tr}(xx')^{1/2}$, and $|x_i| = |\mathfrak{e}_i'x|$ is the modulus of the ith component of x. If M is a positive semidefinite matrix ($M \geq 0$), then we define $d_{\max}(M)$ as its largest diagonal entry. The Kronecker delta is δ_{ij}, and 1_{A} is the indicator random variable (equal to one if the event A occurs and equal to zero otherwise). The Kronecker product and sum are denoted $\otimes, \oplus$, and $\mathrm{diag}(X_1, \ldots, X_N)$ stands for the block-diagonal matrix whose diagonal blocks are $X_1, \ldots, X_N$.

Consider a complete stochastic basis $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}, P)$, carrying a continuous-time Markov process $\theta = \{\theta(t);\ t \geq 0\}$ with state space $\mathcal{S} = \{1, \ldots, N\}$, such that, for $i, j \in \mathcal{S}$:

$$P\big\{\theta(t+h) = j \,|\, \theta(t) = i\big\} = \delta_{ij} + \lambda_{ij}h + o(h), \tag{2a}$$

where $o(h)$ means $h \to 0 \Rightarrow o(h)/h \to 0$, and the transition rate matrix $\Lambda = [\lambda_{ij}] \in \mathbb{R}^{N\times N}$ satisfies the usual constraints

$$\sum_{j=1}^{N} \lambda_{ij} \equiv 0, \quad \text{and} \quad i \neq j \implies \lambda_{ij} \geq 0. \tag{2b}$$

Our interest is with the performance of the MJLS

$$\mathcal{G} : \begin{cases} \dot{x}(t) = A_{\theta(t)}x(t) + B_{\theta(t)}v(t), \quad t \geq 0 \\ y(t) = C_{\theta(t)}x(t) \end{cases} \tag{3}$$

with $A_{\theta(t)} \in \mathbb{R}^{n\times n}, B_{\theta(t)} \in \mathbb{R}^{n\times m}$, and $C_{\theta(t)} \in \mathbb{R}^{p\times n}$. System (3) is intimately related to the state transition matrix $\Phi(t,\tau)$, which is the (unique) solution of the random differential equation

$$\frac{\partial}{\partial t}\Phi(t,\tau) = A_{\theta(t)}\Phi(t,\tau), \quad \Phi(\tau,\tau) = I, \quad t \geq \tau. \tag{4}$$

Definition 1: System (3) is said to be mean square stable whenever, for all $x(0)$ satisfying $E[\|x(0)\|^2] < \infty$, we have $v \equiv 0 \implies \lim_{t\to\infty} E\big[\|x(t)\|^2\big] = 0$.

The input norm that will be considered in the paper is

$$\|v\|_2 = \left(\int_0^\infty E\big[\|v(t)\|^2\big]dt\right)^{1/2}, \tag{5a}$$

whereas the output norm is, by definition,

$$\|y\|_\infty = \sup_{t\geq 0} \max_{j=1,\ldots,p} E\big[|y_j(t)|\big], \quad y_j(t) = \mathfrak{e}_j'y(t), \tag{5b}$$

so our interest will be with what is commonly known as the energy-to-peak performance (L_2-to-L_∞ induced norm):

Definition 2: If system (3) is mean square stable, we define its worst-case energy-to-peak gain as

$$\|\mathcal{G}\| = \sup_v \big\{\|y\|_\infty;\ v \in \mathcal{V},\ x(0) = 0\big\} \tag{6}$$

where $\mathcal{V}$ stands for the set of all $\mathbb{R}^m$-valued, adapted, input processes with unit energy; i.e., $v \in \mathcal{V}$ if and only if:

(i) $v(t) \in \mathbb{R}^m$ is $\mathfrak{F}_t$-measurable for all $t \geq 0$;
(ii) $\|v\|_2 = 1$.

In Section III, we will show that the mean square stability of (3) allows us to obtain upper bounds for $\|\mathcal{G}\|$ in terms of the H_2 norm:

Definition 3: The H_2 norm of system (3) is defined as

$$\|\mathcal{G}\|_{H_2} = \left(\sum_{i=1}^{m}\int_0^\infty E\big[\|y^i(t)\|^2\big]dt\right)^{1/2}, \tag{7}$$

where y^i is the output of (3) driven by a unit impulse applied to the ith input channel, and $x(0) = 0$.

It should be clear from Definitions 2 and 3 that there is no loss of generality in the following assumption, which will be considered in the remainder of this paper:

Assumption 4: System (3) is mean square stable, and $x(0) = 0 \in \mathbb{R}^n$.

Remark 5: Mean square stability (MSS) is the most extensively studied stability notion for (3), and some criteria for checking it include the Hurwitz property of the matrix $\Lambda' \otimes I + \mathrm{diag}(A_1 \oplus A_1, \ldots, A_N \oplus A_N)$, as well as the solvability of linear matrix inequalities or algebraic (Lyapunov) equations. For a comprehensive account on the subject, including examples which show that MSS neither implies nor is implied by the asymptotic stability of each individual operating mode, see [5, Chapter 3].

A. Auxiliary Results

A corollary of [5, Theorems 3.25 and 5.4] will be useful:

Corollary 6: If (3) is mean square stable, then there is a unique solution to the coupled Lyapunov-like equations

$$A_iQ_i + Q_iA_i' + \sum_{j=1}^{N}\lambda_{ji}Q_j + \nu_iB_iB_i' = 0, \quad i \in \mathcal{S}, \tag{8}$$

where $\nu_i \equiv P\{\theta(0) = i\}$. In addition, we have $\|\mathcal{G}\|_{H_2}^2 = \mathrm{tr}(\sum_{i=1}^{N} C_iQ_iC_i') < \infty$.

Remark 7: In [5, Section 5.3], the solution of (8) is called the controllability Gramian of (3). As shown in [5, Proposition 3.20], one elementary way for computing it analytically is via vectorization [16]. Later on, we shall devise upper bounds for $\|\mathcal{G}\|$ in terms of the matrix $\sum_{i=1}^{N} C_i Q_i C_i'$, in the spirit of [6, Theorem 1].

Our main results will rely heavily on the performance analysis of the multiple-input single-output (MISO) systems $\mathcal{G}_1, \ldots, \mathcal{G}_p$ defined as

$$\mathcal{G}_j : \begin{cases} \dot{x}(t) = A_{\theta(t)} x(t) + B_{\theta(t)} v(t) \\ y_j(t) = \mathfrak{e}_j' C_{\theta(t)} x(t). \end{cases} \tag{9}$$

Lemma 8: Under Assumption 4, we have

$$\|\mathcal{G}_j\| \leq \|\mathcal{G}\| \qquad \forall\, j \in \{1, \ldots, p\}. \tag{10}$$

Proof: We have for all $j \in \{1, \ldots, p\}$ and $t \geq 0$ that $E|y_j(t)| \leq \max_{i=1,\ldots,p} E|y_i(t)|$, so

$$\|y_j\|_\infty = \sup_{t\geq 0} E\big[|y_j(t)|\big] \leq \sup_{t\geq 0} \max_{i=1,\ldots,p} E\big[|y_i(t)|\big] = \|y\|_\infty,$$

and the result follows by taking the supremum of both sides with respect to $v \in \mathcal{V}$. ■

The remainder of this subsection comprises two technical results which will also be necessary for our further development. For their proofs, see the Appendix.

Lemma 9: We have, for small $h > 0$ and all $s \geq 0$, that

$$E(1_{\{\theta(s+h)=i\}} - 1_{\{\theta(s)=i\}} \,|\, \mathfrak{F}_s) = \lambda_{\theta(s)i} h + o(h). \tag{11}$$

Lemma 10: We have that

$$E\big(C_{\theta(t)} \Phi(t,\tau) \,|\, \mathfrak{F}_\tau\big) = \bar{C} e^{F(t-\tau)} (\mathfrak{e}_{\theta(\tau)} \otimes I_n), \tag{12}$$

where

$$\bar{C} = [C_1 \ \cdots \ C_N], \quad F = \Lambda' \otimes I + \operatorname{diag}(A_1, \ldots, A_N). \tag{13}$$

III. Main Results

The main result of this paper, stated next, relates the L_2-L_∞ and H_2 norms of system (3), as well as the cost

$$|\mathcal{G}|_t = \sqrt{d_{\max}\left(\int_0^t \bar{C} e^{F(t-\tau)} \Pi(\tau) e^{F'(t-\tau)} \bar{C}' d\tau\right)} \tag{14}$$

with $\bar{C}, F$ as in (13), and

$$\Pi(\tau) = \begin{bmatrix} P\{\theta(\tau)=1\} B_1 B_1' & & 0 \\ & \ddots & \\ 0 & & P\{\theta(\tau)=N\} B_N B_N' \end{bmatrix}. \tag{15}$$

Theorem 11: Under Assumption 4, the worst-case energy-to-peak gain of system (3) is bounded as follows:

$$\sup_{t\geq 0} |\mathcal{G}|_t \leq \|\mathcal{G}\| \leq \sqrt{d_{\max}\left(\sum_{i=1}^{N} C_i Q_i C_i'\right)}, \tag{16}$$

with $|\mathcal{G}|_t$ as in (14), and Q as in Corollary 6. In particular, this yields the more conservative bound $\|\mathcal{G}\| \leq \|\mathcal{G}\|_{H_2}$.

Proof: The triangle and Cauchy-Schwarz inequalities imply, for all $t \geq 0$, $j \in \{1, \ldots, p\}$ and $v \in \mathcal{V}$, the following:

$$\begin{aligned} E|y_j(t)| &= E\left|\int_0^t \mathfrak{e}_j' C_{\theta(t)} \Phi(t,s) B_{\theta(s)} v(s) ds\right| \\ &\leq \int_0^t E\big[|\mathfrak{e}_j' C_{\theta(t)} \Phi(t,s) B_{\theta(s)} v(s)|\big] ds \\ &\leq \int_0^t E\big[\|\mathfrak{e}_j' C_{\theta(t)} \Phi(t,s) B_{\theta(s)}\| \|v(s)\|\big] ds \\ &\leq \left(\int_0^t E\big[\|\mathfrak{e}_j' C_{\theta(t)} \Phi(t,s) B_{\theta(s)}\|^2\big] ds\right)^{1/2} \|v\|_2, \end{aligned}$$

so $\|\mathcal{G}\|$ is upper bounded by the quantity

$$\sup_{t\geq 0} \max_{j=1,\ldots,p} \left(\int_0^t E\big[\|\mathfrak{e}_j' C_{\theta(t)} \Phi(t,s) B_{\theta(s)}\|^2\big] ds\right)^{1/2}. \tag{17}$$

For later use, let $y_j^i(\tau) = \mathfrak{e}_j' C_{\theta(\tau)} \Phi(\tau, 0) B_{\theta(0)} \mathfrak{e}_i$ stand for the output response of (9) with respect to an impulsive input such as the one considered in Definition 3. The tower rule, combined with time homogeneity, yields

$$\begin{aligned} &\int_0^t E\big[\|\mathfrak{e}_j' C_{\theta(t)} \Phi(t,s) B_{\theta(s)}\|^2\big] ds \\ &= \int_0^t E\Big[E\Big(\|\mathfrak{e}_j' C_{\theta(t)} \Phi(t,s) B_{\theta(s)}\|^2 \,\Big|\, \mathcal{F}_s\Big)\Big] ds \\ &= \int_0^t E\Big[E\Big(\|\mathfrak{e}_j' C_{\theta(t-s)} \Phi(t-s,0) B_{\theta(0)}\|^2 \,\Big|\, \mathcal{F}_0\Big)\Big] ds \\ &= \sum_{i=1}^{N} \int_0^t E\big[|\mathfrak{e}_j' C_{\theta(\tau)} \Phi(\tau,0) B_{\theta(0)} \mathfrak{e}_i|^2\big] d\tau \\ &= \sum_{i=1}^{N} \int_0^t E\big[y_j^i(\tau)^2\big] d\tau \leq \sum_{i=1}^{N} \int_0^\infty E\big[y_j^i(\tau)^2\big] d\tau = \|\mathcal{G}_j\|_{H_2}^2 \end{aligned}$$

with $\leq$ becoming an equality when $t \to \infty$. We have therefore proven that $\|\mathcal{G}\| \leq \max_{j=1,\ldots,p} (\|\mathcal{G}_j\|_{H_2}^2)^{1/2} = \max_{j=1,\ldots,p} \|\mathcal{G}_j\|_{H_2}$, so, from Corollary 6, we promptly get

$$\|\mathcal{G}\|^2 \leq \max_{j=1,\ldots,p} \sum_{i=1}^{N} \mathfrak{e}_j' C_i Q_i C_i' \mathfrak{e}_j = d_{\max}\left(\sum_{i=1}^{N} C_i Q_i C_i'\right),$$

yielding the second $\leq$ in (16). Using also $d_{\max}(\cdot) \leq \operatorname{tr}(\cdot)$, we have from Corollary 6 that $\|\mathcal{G}\| \leq \|\mathcal{G}\|_{H_2}$.

In order to prove the first inequality in (16), we shall analyze $\|\mathcal{G}_j\|$ for all $j \in \{1, \ldots, p\}$, and then use Lemma 8. Using the already proven part of (16), we know that $\|\mathcal{G}_j\| \leq \|\mathcal{G}\|_{H_2}$ is finite, so we can interchange the order of suprema in the following:

$$\begin{aligned} \|\mathcal{G}_j\| &= \sup_{v\in\mathcal{V}} \sup_{t\geq 0} E\big[|y_j(t)|\big] \\ &= \sup_{t\geq 0} \sup_{v\in\mathcal{V}} E\big[|y_j(t)|\big] \geq \sup_{t\geq 0} \sup_{v\in\mathcal{V}} E\big[y_j(t)\big], \end{aligned} \tag{18}$$

where $\geq$ is because of the monotonicity of expectation. Using now the tower rule, we get

$$E\big[y(t)\big] = \int_0^t E\Big(C_{\theta(t)} \Phi(t,\tau) B_{\theta(\tau)} v(\tau)\Big) d\tau$$

$$= \int_0^t E\Big(E\big(C_{\theta(t)}\Phi(t,\tau)\,\big|\,\mathfrak{F}_\tau\big)B_{\theta(\tau)}v(\tau)\Big)d\tau$$
$$= \int_0^\infty E\big(U(t,\tau)'v(\tau)\big)d\tau, \tag{19}$$

where, from Lemma 10:

$$U(t,\tau)' := \begin{cases} \bar{C}e^{F(t-\tau)}(\mathfrak{e}_{\theta(\tau)} \otimes B_{\theta(\tau)}), & \tau \in [0,t], \\ 0, & \text{otherwise.} \end{cases} \tag{20}$$

Considering all $v \in \mathcal{V}$, we therefore have from the Cauchy-Schwarz inequality that

$$E\big[y_j(t)\big] \leq \left(\int_0^t E\big(\mathfrak{e}_j' U(t,\tau)'U(t,\tau)\mathfrak{e}_j\big)d\tau\right)^{1/2}, \tag{21}$$

with equality holding for the following $v \in \mathcal{V}$:

$$v(\tau) = \left(\int_0^t E\big(\mathfrak{e}_j' U(t,\tau)'U(t,\tau)\mathfrak{e}_j\big)d\tau\right)^{-1/2} U(t,\tau)\mathfrak{e}_j.$$

Putting together Lemma 8 and (18), we therefore obtain for all $t \geq 0$ that

$$\|\mathcal{G}\|^2 \geq \max_{1\leq j\leq p}\left\{\int_0^t E\big(\mathfrak{e}_j' U(t,\tau)'U(t,\tau)\mathfrak{e}_j\big)d\tau\right\}$$
$$= d_{\max}\left(\int_0^t E\big(U(t,\tau)'U(t,\tau)\big)d\tau\right),$$

which completes the proof, because

$$E\big(U(t,\tau)'U(t,\tau)\big)$$
$$= \bar{C}e^{F(t-\tau)}E\big[(\mathfrak{e}_{\theta(\tau)}\mathfrak{e}_{\theta(\tau)}') \otimes (B_{\theta(\tau)}B_{\theta(\tau)}')\big]e^{F'(t-\tau)}\bar{C}'$$
$$= \bar{C}e^{F(t-\tau)}\Pi(\tau)e^{F'(t-\tau)}\bar{C}'.$$

■

Remark 12: It is presently unknown whether the inequality $\|\mathcal{G}\| \leq \|\mathcal{G}\|_{H_2}$ (which is also valid in the case without jumps treated in [6]) would still hold if the output norm (1) were considered instead of (5b).

The upper bound in (16) can be computed via well known results in the literature, such as linear matrix inequalities or coupled Lyapunov equations, as indicated in Corollary 6 (see [5, Chapter 5] and [17] for details). As for the lower bound, we propose computationally amenable procedures in the sequel:

Corollary 13: Under the same conditions of Theorem 11, we have for all $t \geq 0$ that

$$|\mathcal{G}|_t = \sqrt{d_{\max}\left(\bar{C}Z(t)\bar{C}'\right)}, \tag{22}$$

where $Z(t) \in \mathbb{R}^{Nn\times Nn}$ satisfies the following ODE:

$$\dot{Z}(t) = FZ(t) + Z(t)F' + \Pi(t), \quad Z(0) = 0. \tag{23}$$

If, in addition, the jump process θ is irreducible, with invariant probabilities $\pi_1, \ldots, \pi_N$, then we also have that

$$\|\mathcal{G}\| \geq \sqrt{d_{\max}\big(\bar{C}Z\bar{C}'\big)}, \tag{24}$$

with $Z \in \mathbb{R}^{Nn\times Nn}$ denoting the (unique) solution of the Lyapunov equation

$$FZ + ZF' + \Pi = 0, \tag{25}$$

with $\Pi = \operatorname{diag}(\pi_1 B_1 B_1', \ldots, \pi_N B_N B_N')$.

Proof: Letting $Z(t) = \int_0^t e^{F(t-\tau)}\Pi(\tau)e^{F'(t-\tau)}d\tau$, we have, for small h, that

$$Z(t+h) = e^{Fh}\left(Z(t) + \int_t^{t+h} e^{F(t-\tau)}\Pi(\tau)e^{F'(t-\tau)}d\tau\right)e^{F'h}$$
$$\approx Z(t) + h\big(FZ(t) + Z(t)F' + \Pi(t)\big) + o(h),$$

yielding (23).

Before we directly address the irreducible case, notice that the uniqueness of Z as in (25) stems from the Hurwitz property of F (a consequence of mean square stability, as shown in [5, Proposition 3.13 and Theorem 3.15]). Let then vec and vec^{-1} stand for the usual vectorization operation and its inverse [16]. We can decompose $Z(t) = Z + \text{vec}^{-1}[\zeta(t)]$, with Z as in (25) and

$$\dot{\zeta}(t) = (F \oplus F)\zeta(t) + w(t), \qquad \zeta(0) = 0,$$

where $w(t) \equiv \text{vec}[\Pi(t) - \Pi]$ is bounded and vanishes to zero then $t \to \infty$. Invoking [18, Fact 16.21.20], we therefore conclude that $t \to \infty \Rightarrow \zeta(t) \to 0 \Rightarrow Z(t) \to Z$, so the fact that (22) holds for all t yields (24), when $t \to \infty$. ■

A. The case without jumps

This section considers the case where $N = 1$, so that $(A_i, B_i, C_i) \equiv (A, B, C)$ and $\lambda_{ij} = 0$ in (2)–(3). In this case (already studied in [6, Theorem 1d]), we have the following:

Corollary 14: In the case without jumps, there is no conservatism in (16) and (24):

$$\sup_{t\geq 0}|\mathcal{G}|_t = \|\mathcal{G}\| = \sqrt{d_{\max}(CZC')}. \tag{26}$$

If, in addition, the output of (3) is a scalar process ($p = 1$), then we also have $\|\mathcal{G}\| = \|\mathcal{G}\|_{H_2}$ in (16).

Proof: In the case without jumps, mean square stability becomes equivalent to the asymptotic stability of $A = F$, which is then a Hurwitz matrix by hypothesis. This implies, in (14), the following (when $t \to \infty$):

$$|\mathcal{G}|_t \to \sqrt{d_{\max}\left(\int_0^\infty Ce^{As}BB'e^{A's}C'ds\right)} = \sqrt{d_{\max}(CZC')},$$

where Z, just as in (25), satisfies $AZ + ZA' + BB' = 0$, i.e., it is the controllability Gramian of the pair (A, B). The final statement for $p = 1$ is because of the obvious fact that $d_{\max}(\cdot) = \text{tr}(\cdot)$ for scalar arguments (see [6] as well). ■

B. L_2-L_∞ control

Consider now the following controlled version of (3):

$$\mathcal{G}_K : \begin{cases} \dot{x}(t) = \big(A_{\theta(t)} + G_{\theta(t)}K_{\theta(t)}\big)x(t) + B_{\theta(t)}v(t) \\ y(t) = \big(C_{\theta(t)} + H_{\theta(t)}K_{\theta(t)}\big)x(t) \end{cases} \tag{27}$$

along with the problem of designing controller gains $K_1, \ldots, K_N \in \mathbb{R}^{q\times n}$ such that (27) is stabilized in the mean square sense, and $\|\mathcal{G}_K\|$ is less than a prespecified value γ.

The lack of an exact characterization of the L_2-L_∞ norm (6) makes the direct optimization of this criterion rather impossible, as of now. An indirect alternative, based on

Theorem 11 and expressed through linear matrix inequalities, is to work with the upper bound in (16), in the spirit of [7]:

Corollary 15: Given $\gamma > 0$, suppose there are matrices $P_1, R_1, S_1, \ldots, P_N, R_N, S_N$ such that

$$\sum_{i=1}^{N} \begin{bmatrix} C_i & H_i \end{bmatrix} \begin{bmatrix} P_i & R_i' \\ R_i & S_i \end{bmatrix} \begin{bmatrix} C_i' \\ H_i' \end{bmatrix} < \gamma^2 I, \tag{28a}$$

coupled with

$$\mathrm{Her}(A_i P_i + G_i R_i) + \sum_{j=1}^{N} \lambda_{ji} P_j + \nu_i B_i B_i' < 0, \tag{28b}$$

$$\begin{bmatrix} P_i & R_i' \\ R_i & S_i \end{bmatrix} > 0, \qquad i \in \mathcal{S}, \tag{28c}$$

where $\mathrm{Her}(\cdot) \equiv (\cdot) + (\cdot)'$ and $\nu_i \equiv P\{\theta(0) = i\}$, are satisfied. In this case, the controller gains $K_i \equiv R_i P_i^{-1}$ render system (27) mean square stable, with $\|\mathcal{G}_K\| < \gamma$.

Proof: Substituting $R_i \equiv K_i P_i$ in (28b), we get $\mathrm{Her}[(A_i + G_i K_i) P_i] + \sum_{j=1}^{N} \lambda_{ji} P_j + \nu_i B_i B_i' < 0$. Using then [5, Theorem 3.25], we get mean square stability of (27), along with the guarantee that there are $Q_i < P_i$, $i \in \mathcal{S}$, such that $\mathrm{Her}[(A_i + G_i K_i) Q_i] + \sum_{j=1}^{N} \lambda_{ji} Q_j + \nu_i B_i B_i' \equiv 0$. Considering now (28a), we obtain, after using Schur's complements in (28c), that

$$\begin{aligned} \gamma^2 I &> \sum_{i=1}^{N} \begin{bmatrix} C_i & H_i \end{bmatrix} \begin{bmatrix} P_i & P_i K_i' \\ K_i P_i & R_i P_i^{-1} R_i' \end{bmatrix} \begin{bmatrix} C_i' \\ H_i' \end{bmatrix} \\ &\geq \sum_{i=1}^{N} (C_i + H_i K_i) Q_i (C_i + H_i K_i)', \end{aligned} \tag{29}$$

so $d_{\max}[\sum_{i=1}^{N} (C_i + H_i K_i) Q_i (C_i + H_i K_i)'] < \gamma^2$, and the desired result follows from Theorem 11. ■

IV. Numerical Example

In the numerical example of [12, Section VI], the authors considered various observational scenarios involving the linearized longitudinal dynamics of an unmanned aerial vehicle. We will consider here just the scenario of cluster observations, and that the filter order is equal to three; our particular interest in this scenario stems from the fact that the filter parameters (A_{f1}, B_{f1} and so forth) were fully described therein. These parameters correspond to a MISO (multiple-input single-output) system, whose Markov jump process is irreducible, with stationary probabilities $\pi_1 \approx 0.5224, \pi_2 \approx 0.2985, \pi_3 \approx 0.1791$.

Our purpose here is to show, in this particular example, how tight the bounds in Theorem 11 and Corollary 13 are, and to compare them with the worst-case gain analysis of [12, Proposition 2] (which, as it will be shown in Section V, is yet another upper bound to $\|\mathcal{G}\|$).

Figure 1 features a plot of the right-hand side of (22), which was obtained through the numerical solution of the ODE (23) until practical convergence ($\|\dot{Z}(120)\| < 10^{-3}$, in this example), and compares it with three other bounds (which are static, in the sense that they do not involve integration). The figure evidences that, in this example:

- There is no loss in replacing this procedure (solving an ODE) with the solution of the equation (25); the latter yields the same result with less computational effort.
- The difference between the lower and upper bounds in (16) is as tight as 10%: the H_2 norm is 288.1911, and $(\bar{C} Z \bar{C}')^{1/2}$ is 261.5105.
- The upper bound from [12, Proposition 2] is around 423.7447, which is about 62% larger than $(\bar{C} Z \bar{C}')^{1/2}$. Even though this discrepancy is a natural consequence of the fact that [12] considers a different output norm, it also clearly illustrates that [12, Proposition 2] is not, in general, a tight upper bound to the performance considered here.

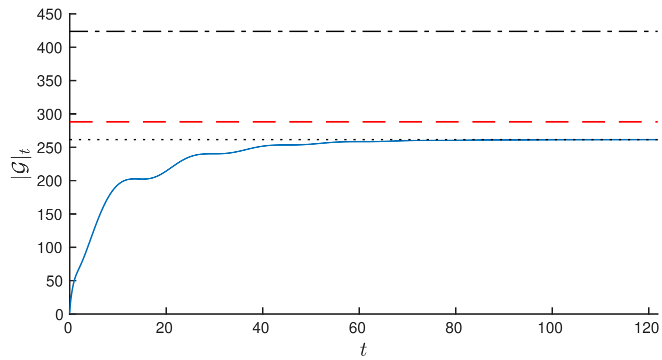

Fig. 1. Comparison between the lower bound $|\mathcal{G}|_t = (\bar{C} Z(t) \bar{C}')^{1/2}$ in (22), obtained by numerically solving the ODE (23) (blue curve), and various other static bounds. Dotted, black: $(\bar{C} Z \bar{C}')^{1/2} \approx 261.5105$ as in (24), obtained by solving the algebraic equation (25). Dashed, red: H_2 norm, approximately 288.1911 (upper bound in (16)). Dot-dash, black: minimum γ in [12, Proposition 2], around 423.7447.

V. Concluding Remarks

1) Conservatism of the H_2 norm in the MISO case: As shown in the MISO example of the preceding section (Figure 1), the H_2 norm can be strictly larger than $\sup_t |\mathcal{G}|_t$ in (14). This leaves open a question: dissimilarly to the linear time-invariant case studied in [6], [7], is there a gap between the energy-to-peak performance and the H_2 norm, i.e., is it possible that $\|\mathcal{G}\| < \|\mathcal{G}\|_{H_2}$ in the MISO case with jumps?

An examination of the proof of Theorem 11 up until (17) suggests that, in order to "close the gap" between these two norms, we should be able to produce a disturbance $v \in \mathcal{V}$ for which the expression which precedes (17) becomes equal to $E|y_j(t)|$. Dissimilarly to the LTI case, however, choosing $v(s) \propto B'_{\theta(s)} \Phi(t,s)' C'_{\theta(t)} \mathfrak{e}_j$ is not a viable option, because this disturbance is not adapted to the filtration $\{\mathfrak{F}_s\}$. A deeper study of these matters is worthy of future investigation.

2) Relationship with the output norm (1)*:* As mentioned in Section I, references [9]–[12] addressed the energy-to-peak performance with respect to the output norm (1) (denoted $\|y\|_{\infty,2}$ in the sequel), yielding a different induced norm than (6). Using Jensen's inequality, we get that

$$\left(E\left[|y_j(t)|\right]\right)^2 \leq E\left[|y_j(t)|^2\right] \leq E\left[\|y(t)\|^2\right] \tag{30}$$

for all $t \geq 0$ and $j \in \{1, \ldots, p\}$, which yields $\|y\|_\infty = \sup_{t\geq 0} \max_{j=1,\ldots,p} E[|y_j(t)|] \leq \sup_{t\geq 0}\{E[\|y(t)\|^2]\}^{1/2} = \|y\|_{2,\infty}$ for all $v \in \mathcal{V}$. Therefore:

- The lower bound in (16) is also a lower bound to the energy-to-peak performance induced by (1).
- Upper bounds for the energy-to-peak performance induced by (1), such as [12, Proposition 2] (or the performance itself), are also upper bounds to the performance (6) considered here. As shown in Figure 1, however, this can yield quite conservative estimates of (6).

APPENDIX

Proof of Lemma 9: We indeed have that

$$
\begin{aligned}
&E\big(1_{\{\theta(s+h)=i\}} \,|\, \mathfrak{F}_s\big) = E\big(1_{\{\theta(s+h)=i\}} \,|\, \theta(s)\big) \\
&= \sum_{j=1}^{N} P\{\theta(s+h) = i \mid \theta(s) = j\} 1_{\{\theta(s)=j\}} \\
&= \big(1 + \lambda_{ii} h + o(h)\big) 1_{\{\theta(s)=i\}} + \sum_{j\neq i} \big(\lambda_{ji} h + o(h)\big) 1_{\{\theta(s)=j\}} \\
&= 1_{\{\theta(s)=i\}} + \lambda_{\theta(s)i} h + o(h),
\end{aligned}
$$

concluding the proof. ■

Proof of Lemma 10: Let, for each $i, j \in \mathcal{S}$ and $s \geq 0$:

$$
H_i(s, j) = E\big(\Phi(s, 0) 1_{\{\theta(s)=i\}} \,\big|\, \theta(0) = j\big) \in \mathbb{R}^{n\times n}. \quad (31)
$$

Bearing in mind that the evolution of Φ in (4) is entirely described in terms of θ, and that the system's coefficients are homogeneous, we have that

$$
\begin{aligned}
&H_i\big(t - \tau, \theta(\tau)\big) = \sum_{j=1}^{N} H_i(t - \tau, j) 1_{\{\theta(\tau)=j\}} \\
&= \sum_{j=1}^{N} E\big(\Phi(t - \tau, 0) 1_{\{\theta(t-\tau)=i\}} \mid \theta(0) = j\big) 1_{\{\theta(\tau)=j\}} \\
&= \sum_{j=1}^{N} E\big(\Phi(t, \tau) 1_{\{\theta(t)=i\}} \mid \theta(\tau) = j\big) 1_{\{\theta(\tau)=j\}} \\
&= E\big(\Phi(t, \tau) 1_{\{\theta(t)=i\}} \mid \mathfrak{F}_\tau\big) \\
&\overset{*}{=} 1_{\{\theta(\tau)=i\}} I + \int_\tau^t E\big(\Psi_i(s) \,\big|\, \mathfrak{F}_\tau\big) ds, \qquad (32)
\end{aligned}
$$

where $\overset{*}{=}$ is from Dynkin's formula [5, Section 4.3], and

$$
\Psi_i(s) = \lim_{h\to 0} \frac{E\big(\Phi(s + h, \tau) 1_{\{\theta(s+h)=i\}} \,|\, \mathfrak{F}_s\big) - \Phi(s, \tau) 1_{\{\theta(s)=i\}}}{h}.
$$

We have now, for small $h > 0$, that

$$
\begin{aligned}
&E\big(\Phi(s + h, \tau) 1_{\{\theta(s+h)=i\}} \,|\, \mathfrak{F}_s\big) - \Phi(s, \tau) 1_{\{\theta(s)=i\}} \\
&= E\big[\Phi(s + h, \tau)\big(1_{\{\theta(s+h)=i\}} - 1_{\{\theta(s)=i\}}\big) \,|\, \mathfrak{F}_s\big] \\
&\quad + E\big[\big(\Phi(s + h, \tau) - \Phi(s, \tau)\big) 1_{\{\theta(s)=i\}} \,|\, \mathfrak{F}_s\big] \\
&\approx (I + hA_{\theta(s)})\Phi(s, \tau) E\big[1_{\{\theta(s+h)=i\}} - 1_{\{\theta(s)=i\}} \,|\, \mathfrak{F}_s\big] \\
&\quad + hA_i \Phi(s, \tau) 1_{\{\theta(s)=i\}} \\
&\approx (I + hA_{\theta(s)})\Phi(s, \tau)\lambda_{\theta(s)i} h + hA_{\theta(s)} \Phi(s, \tau) 1_{\{\theta(s)=i\}},
\end{aligned}
$$

thanks to Lemma 9. This allows us to write $\Psi_i(s) = A_i \Phi(s, \tau) 1_{\{\theta(s)=i\}} + \lambda_{\theta(s)i} \Phi(s, \tau)$, so

$$
\begin{aligned}
&H_i\big(t - \tau, \theta(\tau)\big) - 1_{\{\theta(\tau)=i\}} I \\
&\equiv \int_\tau^t \left(A_i H_i(s - \tau, \theta(\tau)) + \sum_{k=1}^{N} \lambda_{ki} H_k(s - \tau, \theta(\tau)) \right) ds,
\end{aligned}
$$

which, letting $H(s, j) = \big[H_1(s, j)' \ \cdots \ H_N(s, j)'\big]'$, is just

$$
H\big(t - \tau, \theta(\tau)\big) - (\mathfrak{e}_{\theta(\tau)} \otimes I) = \int_\tau^t F H\big(s - \tau, \theta(\tau)\big) ds, \quad (33)
$$

from which we can conclude the proof:

$$
\begin{aligned}
E\big(C_{\theta(t)} \Phi(t, \tau) \,|\, \mathfrak{F}_\tau\big) &= \sum_{i=1}^{N} C_i E\big(\Phi(t, \tau) 1_{\{\theta(t)=i\}} \,|\, \mathfrak{F}_\tau\big) \\
&= \bar{C} H\big(t - \tau, \theta(\tau)\big) = \bar{C} e^{F(t-\tau)} (\mathfrak{e}_{\theta(\tau)} \otimes I). \qquad (34)
\end{aligned}
$$

■

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