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Mean Field Game for Strategic Bidding of Energy Consumers in Congested Distribution Networks

Amirreza Silani, Simon H. Tindemans

Abstract— **The proliferation of batteries, photovoltaic cells and Electric Vehicles (EVs) in electric power networks can result in network congestion. A redispatch market that allows the Distribution System Operators (DSOs) to relieve congested networks by asking the energy consumers to adjust their scheduled consumption is an alternative to upgrading network capacity. However, energy consumers can strategically increase their bids on the day-ahead market in anticipation of payouts from the redispatch market. This behaviour, which is called increase-decrease gaming, can aggravate congestion and allow the energy consumers to extract windfall profits from the DSO. In this paper, we model the increase-decrease game for large populations of energy consumers in power networks using a mean field game approach. The agents (energy consumers) maximize their individual welfare on the day-ahead market with anticipation of the redispatch market, coupled via the electricity price. We show that there exists a Nash equilibrium for this game and use an algorithm that converges to the Nash equilibrium for the infinite population case.**

I. INTRODUCTION

Nowadays, the number of distributed energy resources such as batteries, photovoltaic cells and Electric Vehicles (EVs) connected to the power network is increasing. The advantage of these resources is the decarbonization of the energy system. However, they bring new challenges for power networks that is network congestion. One way to address this problem is using Local Flexibility Markets (LFMs) [1], [2]. Typically, LFM proposals consider energy consumption schedules after closure of the day-ahead market. If the consumption schedules of flexible loads and inflexible loads result in congestion, then the Distribution System Operator (DSO) asks the energy consumers to redispatch their consumption schedules in the LFM, which is also called the redispatch market [3]. The energy consumers who reduce their consumptions on the redispatch market will be paid by the DSO. However, the energy consumers can anticipate the redipatch market and bid strategically on the day-ahead market to maximize their individual welfare. This is called the increase-decrease game [4]. The increase-decrease game can aggravate congestion and allow the energy consumers to extract windfall profits from the DSO. Modeling and analyzing the increase-decrease game in the electrical energy markets are recently attracting much research attention (see [3]–[6]). In [3], different types of market failures are introduced and demonstrated in a toy model. In [4], it is shown that combining a regional with a locational market leads to opportunities that rational firms exploit and the increase-decrease game is possible even in absence of market power. A two-stage game is proposed in [5] to analyze imperfect competition of producers in zonal power markets with a real-time and a day-ahead market. A profit decomposition method to evaluate the impacts of bidding strategies on the payoffs and an index to quantify market power are presented in [6]. Although, these papers investigate the increase-decrease game in different electrical energy markets, to the best of our knowledge, there exists no work in the literature providing a suitable model and Nash equilibrium analysis for the increase-decrease game applied to distribution network congestion. Moreover, in [7]–[9], the mean field game theory is used to solve the constrained charging control of large populations of EVs. Thus, the mean field game theory can also be exploited for modeling and analyzing the increase-decrease game. The case, explored in this paper, considers a large population of energy consumers that do not have market power (individually).

A. Silani and S.H. Tindemans are with the Department of Electrical Sustainable Energy, Delft University of Technology, The Netherlands ({a.silani, s.h.tindemans}@tudelft.nl)

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In this paper, we model the increase-decrease game for large populations of energy consumers using a mean field game approach. The energy consumers maximize their individual welfare on the day-ahead market with anticipation of the redispatch market, coupled via the electricity price. Then, we show that there exists a Nash equilibrium for this game. Finally, we use an iterative algorithm that converges to the Nash equilibrium for the infinite population case. The contributions of the paper can then be listed as follows: (i) the increase-decrease game for large populations of energy consumers in power networks is modeled using a mean field game approach; (ii) the existence and the uniqueness of the Nash equilibrium for this game are theoretically proved; (iii) an iterative algorithm that converges to the Nash equilibrium for the infinite population case is used.

II. MODELING AND PROBLEM FORMULATION

In this section, we describe the role of energy consumers and DSO on the day-ahead and redispatch market. Then, we model the increase-decrease game for large populations of energy consumers using a mean field game approach.

A. Problem formulation

We consider the problem of energy consumption for a population of n energy consumers (agents) with flexible loads. We consider the set of agents $\mathcal{N} := \{1, 2, \dots, n\}$.

For notational simplicity, we consider the problem for one time slot, but the model is trivially applied to multiple (non-coupled) time slots. For each $i \in \mathcal{N}$, let E_i^{d} be the consumption schedule on the day-ahead market and E_i^{r} be the anticipated consumption reduction on the redispatch market (when congestion occurs) for its flexible loads, u_i be the utility (*i.e.*, the agents would like to use energy) and $E_i^{\max}$ is the maximum consumption.

The DSO investigates the schedules on the day-ahead market and asks the agents to reduce their consumptions on the redispatch market if a congestion problem is detected. The agents who reduce their consumptions on the redispatch market will be paid by the DSO.

Let E^{inflx} be the total inflexible load and cap be the total capacity of generation. Also, let $f^{\mathrm{d}}(\cdot)$ and $f^{\mathrm{r}}(\cdot)$ be the electricity supply price functions on the day-ahead and redispatch market, respectively. We note that in many electricity markets, the pay-as-cleared pricing rule is used [4]. Now, we consider the following assumption on the supply price functions.

Assumption 1: **(Pay-as-cleared pricing rule).** The pay-as-cleared pricing rule is used on the day-ahead and redispatch market. The day-ahead and redispatch supply price functions $f^{\mathrm{d}}(\cdot)$ and $f^{\mathrm{r}}(\cdot)$ are continuous and given by $f^{\mathrm{d}}\big(\frac{1}{cap}(\sum_{i\in\mathcal{N}} E_i^{\mathrm{d}} + E^{\mathrm{inflx}})\big) = f^{\mathrm{d}}\big(\frac{1}{c^n}(\frac{1}{n}\sum_{i\in\mathcal{N}} E_i^{\mathrm{d}} + d^n)\big)$ and $f^{\mathrm{r}}\big(\frac{1}{cap}(\sum_{i\in\mathcal{N}} E_i^{\mathrm{d}} - \sum_{i\in\mathcal{N}} E_i^{\mathrm{r}} + E^{\mathrm{inflx}})\big) = f^{\mathrm{r}}\big(\frac{1}{c^n}(\frac{1}{n}\sum_{i\in\mathcal{N}} E_i^{\mathrm{d}} - \frac{1}{n}\sum_{i\in\mathcal{N}} E_i^{\mathrm{r}} + d^n)\big)$, where $c^n = \frac{cap}{n}$ and $d^n = \frac{E^{\mathrm{inflx}}}{n}$. Moreover, there exist positive values of $\bar{c} := \lim_{n\to\infty} c^n$ and $\bar{d} := \lim_{n\to\infty} d^n$.

Assumption 1 implicitly defines supply functions for a market-based clearing mechanism, which is used in [7]–[9].

B. Increase-decrease game

The agents can anticipate the redispatch market with some degree of accuracy (e.g. using the historical data of load) and bid strategically on the day-ahead market to maximize their individual welfare. This is called the increase-decrease game. Indeed, the agents can anticipate when the congestion may occur in the network. Then, they modify their schedules on the day-ahead market such that they will be paid by the DSO to reduce it on the redispatch market.

Let $\bar{E}^{\mathrm{d},n} = \frac{1}{n}\sum_{i\in\mathcal{N}} E_i^{\mathrm{d}}$, $\bar{E}^{\mathrm{r},n} = \frac{1}{n}\sum_{i\in\mathcal{N}} E_i^{\mathrm{r}}$ be the mean values of the consumption schedules on the day-ahead market and the anticipated consumption reduction on the redispatch market, respectively. Each agent $i \in \mathcal{N}$ maximizes its individual welfare (utility minus day-ahead costs, plus anticipated redispatch revenue) $\mathcal{J}_i(E_i^{\mathrm{d}}, E_i^{\mathrm{r}}; E_{-i}^{\mathrm{d}}, E_{-i}^{\mathrm{r}})$ by solving the optimization problem

$$\begin{aligned} &\max_{E_i^{\mathrm{d}}, E_i^{\mathrm{r}}} \mathcal{J}_i(E_i^{\mathrm{d}}, E_i^{\mathrm{r}}; E_{-i}^{\mathrm{d}}, E_{-i}^{\mathrm{r}}) \\ &\text{s.t. } 0 \le E_i^{\mathrm{d}} \le E_i^{\max},\ 0 \le E_i^{\mathrm{r}} \le E_i^{\mathrm{d}}, \end{aligned} \tag{1}$$

where $\mathcal{J}_i(E_i^{\mathrm{d}}, E_i^{\mathrm{r}}; E_{-i}^{\mathrm{d}}, E_{-i}^{\mathrm{r}}) = \big(u_i - f^{\mathrm{d}}\big(\frac{1}{c^n}(\bar{E}^{\mathrm{d},n} + d^n)\big)\big)E_i^{\mathrm{d}} + \big(f^{\mathrm{r}}\big(\frac{1}{c^n}(\bar{E}^{\mathrm{d},n} - \bar{E}^{\mathrm{r},n} + d^n)\big) - u_i\big)E_i^{\mathrm{r}}$ and $E_{-i}^{\mathrm{d}} := \big\{E_j^{\mathrm{d}} | j \in \mathcal{N} - \{i\}\big\}$, $E_{-i}^{\mathrm{r}} := \big\{E_j^{\mathrm{r}} | j \in \mathcal{N} - \{i\}\big\}$ are the strategies of all other agents.

We note that the individual welfare of i-th agent depends on the strategies $E_i^{\mathrm{d}}, E_i^{\mathrm{r}}$ of the agent i and the strategies $E_{-i}^{\mathrm{d}}, E_{-i}^{\mathrm{r}}$ of all the other agents. Then, a set of agents' strategies is stable if each agent cannot make more welfare by changing its own strategies when the strategies of other agents are fixed. Following [8, Definition 3.1], this concept is called Nash equilibrium and is defined as follows.

Definition 1: **(Nash equilibrium).** Let the set of admissible strategies Ξ_i for each agent $i \in \mathcal{N}$ be defined as $\Xi_i := \{(E_i^{\mathrm{d}}, E_i^{\mathrm{r}}) |\ 0 \le E_i^{\mathrm{d}} \le E_i^{\max},\ 0 \le E_i^{\mathrm{r}} \le E_i^{\mathrm{d}}\}$. Then, a set of strategies $\{E_i^{\mathrm{d}*}, E_i^{\mathrm{r}*}\}_{i\in\mathcal{N}}$ is a Nash equilibrium if for all $(E_i^{\mathrm{d}}, E_i^{\mathrm{r}}) \in \Xi_i$ and $i \in \mathcal{N}$

$$\mathcal{J}_i(E_i^{\mathrm{d}*}, E_i^{\mathrm{r}*}; E_{-i}^{\mathrm{d}*}, E_{-i}^{\mathrm{r}*}) \ge \mathcal{J}_i(E_i^{\mathrm{d}}, E_i^{\mathrm{r}}; E_{-i}^{\mathrm{d}*}, E_{-i}^{\mathrm{r}*}). \tag{2}$$

If we assume that $\bar{E}^{\mathrm{d},n}$ and $\bar{E}^{\mathrm{r},n}$ are fixed references, then (1) leads to a linear program in E_i^{d} and E_i^{r}, whose optimal solution is discontinuous in these fixed references. Therefore, in analogy with [7], [8], we consider the following optimization problem for each agent $i \in \mathcal{N}$:

$$\begin{aligned} &\max_{E_i^{\mathrm{d}}, E_i^{\mathrm{r}}} \mathcal{J}_{\sigma i}(E_i^{\mathrm{d}}, E_i^{\mathrm{r}}; E_{-i}^{\mathrm{d}}, E_{-i}^{\mathrm{r}}) \\ &\text{s.t. } 0 \le E_i^{\mathrm{d}} \le E_i^{\max},\ 0 \le E_i^{\mathrm{r}} \le E_i^{\mathrm{d}}, \end{aligned} \tag{3}$$

where $\mathcal{J}_{\sigma i}(E_i^{\mathrm{d}}, E_i^{\mathrm{r}}; E_{-i}^{\mathrm{d}}, E_{-i}^{\mathrm{r}}) = \big(u_i - f^{\mathrm{d}}\big(\frac{1}{c^n}(\bar{E}^{\mathrm{d},n} + d^n)\big)\big)E_i^{\mathrm{d}} + \big(f^{\mathrm{r}}\big(\frac{1}{c^n}(\bar{E}^{\mathrm{d},n} - \bar{E}^{\mathrm{r},n} + d^n)\big) - u_i\big)E_i^{\mathrm{r}} - \sigma\big((E_i^{\mathrm{d}} - \bar{E}^{\mathrm{d},n})^2 + (E_i^{\mathrm{r}} - \bar{E}^{\mathrm{r},n})^2\big)$, with $\sigma \in \mathbb{R}_{\ge 0}$. The additional term in (3) regularizes the problem and can in practice be made arbitrarily small by choosing σ sufficiently small.

The solution to (3) depends on $\bar{E}^{\mathrm{d},n}$ and $\bar{E}^{\mathrm{r},n}$, *i.e.*, the population averages. Thus, the optimization problems (3) are coupled and finding the optimal solution is complex. To address this issue, we consider the fixed references z^{d} and z^{r} instead of $\bar{E}^{\mathrm{d},n}$ and $\bar{E}^{\mathrm{r},n}$, respectively. Then, each agent $i \in \mathcal{N}$ solves the following optimization problem:

$$\begin{aligned} &\max_{E_i^{\mathrm{d}}, E_i^{\mathrm{r}}} \mathcal{J}_{\sigma i}(E_i^{\mathrm{d}}, E_i^{\mathrm{r}}; z^{\mathrm{d}}, z^{\mathrm{r}}) \\ &\text{s.t. } 0 \le E_i^{\mathrm{d}} \le E_i^{\max},\ 0 \le E_i^{\mathrm{r}} \le E_i^{\mathrm{d}}, \end{aligned} \tag{4}$$

where $\mathcal{J}_{\sigma i}(E_i^{\mathrm{d}}, E_i^{\mathrm{r}}; z^{\mathrm{d}}, z^{\mathrm{r}}) = \big(u_i - f^{\mathrm{d}}\big(\frac{1}{c^n}(z^{\mathrm{d}} + d^n)\big)\big)E_i^{\mathrm{d}} + \big(f^{\mathrm{r}}\big(\frac{1}{c^n}(z^{\mathrm{d}} - z^{\mathrm{r}} + d^n)\big) - u_i\big)E_i^{\mathrm{r}} - \sigma\big((E_i^{\mathrm{d}} - z^{\mathrm{d}})^2 + (E_i^{\mathrm{r}} - z^{\mathrm{r}})^2\big)$.

Now, suppose that the population size of the agents is large. Then, we can assume that the population size is infinity to approximate the optimization problem (3). Indeed, $\bar{E}^{\mathrm{d}} = \lim_{n\to\infty} \bar{E}^{\mathrm{d},n}$ and $\bar{E}^{\mathrm{r}} = \lim_{n\to\infty} \bar{E}^{\mathrm{r},n}$ exist and are finite and for the infinite population case, the optimization problem (3) becomes

$$\begin{aligned} &\max_{E_i^{\mathrm{d}}, E_i^{\mathrm{r}}} \mathcal{J}_{\sigma i}^{\infty}(E_i^{\mathrm{d}}, E_i^{\mathrm{r}}; E_{-i}^{\mathrm{d}}, E_{-i}^{\mathrm{r}}) \\ &\text{s.t. } 0 \le E_i^{\mathrm{d}} \le E_i^{\max},\ 0 \le E_i^{\mathrm{r}} \le E_i^{\mathrm{d}}, \end{aligned} \tag{5}$$

where $\mathcal{J}_{\sigma i}^{\infty}(E_i^{\mathrm{d}}, E_i^{\mathrm{r}}; E_{-i}^{\mathrm{d}}, E_{-i}^{\mathrm{r}}) = \big(u_i - f^{\mathrm{d}}\big(\frac{1}{\bar{c}}(\bar{E}^{\mathrm{d}} + \bar{d})\big)\big)E_i^{\mathrm{d}} + \big(f^{\mathrm{r}}\big(\frac{1}{\bar{c}}(\bar{E}^{\mathrm{d}} - \bar{E}^{\mathrm{r}} + \bar{d})\big) - u_i\big)E_i^{\mathrm{r}} - \sigma\big((E_i^{\mathrm{d}} - \bar{E}^{\mathrm{d}})^2 + (E_i^{\mathrm{r}} - \bar{E}^{\mathrm{r}})^2\big)$. Similar to (4), the welfare function $\mathcal{J}_{\sigma i}^{\infty}(E_i^{\mathrm{d}}, E_i^{\mathrm{r}}; z^{\mathrm{d}}, z^{\mathrm{r}})$ and the optimization problem for the infinite population case with fixed references z^{d} and z^{r} can be defined.

In the following assumption, we consider the restrictive conditions for σ based on the derivatives of the day-ahead and redispatch supply price functions.

Assumption 2: **(Restrictive conditions for σ).** The functions $f^{\mathrm{d}}(\cdot)$ and $f^{\mathrm{r}}(\cdot)$ are strictly increasing and continuously differentiable. For the infinite population case, there exists $\gamma_2 > \gamma_1 > 0$ such that $\frac{1}{2\bar{c}} \max_{x^{\mathrm{d}}\in[\underline{x}^{\mathrm{d}},\bar{x}^{\mathrm{d}}]} \frac{\partial f^{\mathrm{d}}(x^{\mathrm{d}})}{\partial x^{\mathrm{d}}} \leq \sigma \leq \frac{\gamma_1}{\bar{c}} \min_{x^{\mathrm{d}}\in[\underline{x}^{\mathrm{d}},\bar{x}^{\mathrm{d}}]} \frac{\partial f^{\mathrm{d}}(x^{\mathrm{d}})}{\partial x^{\mathrm{d}}}$, $\frac{1}{2\bar{c}} \max_{x^{\mathrm{r}}\in[\underline{x}^{\mathrm{r}},\bar{x}^{\mathrm{r}}]} \frac{\partial f^{\mathrm{r}}(x^{\mathrm{r}})}{\partial x^{\mathrm{r}}} \leq \sigma \leq \frac{\gamma_2}{\bar{c}} \min_{x^{\mathrm{r}}\in[\underline{x}^{\mathrm{r}},\bar{x}^{\mathrm{r}}]} \frac{\partial f^{\mathrm{r}}(x^{\mathrm{r}})}{\partial x^{\mathrm{r}}}$ and $\frac{1}{2} < \max(\gamma_2, \frac{\gamma_1\gamma_2}{\gamma_2-\gamma_1}) < 1$, where $\underline{x}^{\mathrm{y}}$ and $\bar{x}^{\mathrm{y}}$ are the minimum and maximum possible x^{y}, with $\mathrm{y} = \mathrm{d}, \mathrm{r}$ subject to the constraints of (4). For the finite population case, these inequalities hold with $\bar{E}^{\mathrm{d},n}$, $\bar{E}^{\mathrm{r},n}$, d^n and c^n.

III. Mean field game approach

In this section, we investigate the existence and uniqueness of the Nash equilibrium for the increase-decrease game modeled usings a mean field game approach. Then, we use an iterative algorithm converging to the Nash equilibrium.

Let $\bar{E}^{\mathrm{d}*} = \lim_{n\to\infty} \bar{E}^{\mathrm{d},n*}$ and $\bar{E}^{\mathrm{r}*} = \lim_{n\to\infty} \bar{E}^{\mathrm{r},n*}$, with $\bar{E}^{\mathrm{d},n*} = \frac{1}{n}\sum_{i\in\mathcal{N}} E_i^{\mathrm{d}*}$, $\bar{E}^{\mathrm{r},n*} = \frac{1}{n}\sum_{i\in\mathcal{N}} E_i^{\mathrm{r}*}$ and $z = (z^{\mathrm{d}}, z^{\mathrm{r}})$. Also, consider the definition of the Nash equilibrium given in Definition 1 and the welfare function given in (5) for each agent $i \in \mathcal{N}$. Then, in the following theorem, we show for the infinite population case the conditions that govern a Nash equilibrium.

Theorem 1: **(Conditions for a Nash equilibrium).** Let Assumption 1 hold. Then, a set of strategies $\{E_i^{\mathrm{d}*}, E_i^{\mathrm{r}*}\}_{i\in\mathcal{N}}$ is a Nash equilibrium for the infinite population case if

a. $E_i^{\mathrm{d}*}(z)$ and $E_i^{\mathrm{r}*}(z)$ solve the optimization problem (4) for the infinite population case and with fixed z^{d} and z^{r} for all $i \in \mathcal{N}$.

b. $z^{\mathrm{d}} = \bar{E}^{\mathrm{d}*}$ and $z^{\mathrm{r}} = \bar{E}^{\mathrm{r}*}$, implying that z^{d} and z^{r} can be obtained by averaging the optimal strategies of all agents.

Proof: Consider the set of strategies $\{E_i^{\mathrm{d}*}, E_i^{\mathrm{r}*}\}_{i\in\mathcal{N}}$ solving the optimization (4) for the infinite population case (*i.e.*, with $\mathcal{J}_{\sigma i}^{\infty}(E_i^{\mathrm{d}}, E_i^{\mathrm{r}}; z^{\mathrm{d}}, z^{\mathrm{r}})$) and with fixed z^*, where $z^* = (z^{\mathrm{d}*}, z^{\mathrm{r}*}) = (\bar{E}^{\mathrm{d}*}, \bar{E}^{\mathrm{r}*})$. For the infinite population case, *i.e.*, when $n \to \infty$, each agent's strategies E_i^{d} and E_i^{r} has ignorable effect on the population averages $\bar{E}^{\mathrm{d}}$ and $\bar{E}^{\mathrm{r}}$. Thus, for all $i \in \mathcal{N}$, we have $\lim_{n\to\infty} \frac{1}{n}\big(E_i^{\mathrm{d}} + \sum_{j\in\mathcal{N}-\{i\}} E_j^{\mathrm{d}*}\big) = \lim_{n\to\infty} \frac{1}{n}\sum_{j\in\mathcal{N}} E_j^{\mathrm{d}*} = \bar{E}^{\mathrm{d}*} = z^{\mathrm{d}*}$ and $\lim_{n\to\infty} \frac{1}{n}\big(E_i^{\mathrm{r}} + \sum_{j\in\mathcal{N}-\{i\}} E_j^{\mathrm{r}*}\big) = \lim_{n\to\infty} \frac{1}{n}\sum_{j\in\mathcal{N}} E_j^{\mathrm{r}*} = \bar{E}^{\mathrm{r}*} = z^{\mathrm{r}*}$. Then, for each agent $i \in \mathcal{N}$, we have $\mathcal{J}_{\sigma i}^{\infty}(E_i^{\mathrm{d}}, E_i^{\mathrm{r}}; E_{-i}^{\mathrm{d}*}, E_{-i}^{\mathrm{r}*}) = \big(u_i - f^{\mathrm{d}}\big(\frac{1}{\bar{c}}(\bar{E}^{\mathrm{d}*} + \bar{d})\big)\big)E_i^{\mathrm{d}} + \big(f^{\mathrm{r}}\big(\frac{1}{\bar{c}}(\bar{E}^{\mathrm{d}*} - \bar{E}^{\mathrm{r}*} + \bar{d})\big) - u_i\big)E_i^{\mathrm{r}} - \sigma\big((E_i^{\mathrm{d}} - \bar{E}^{\mathrm{d}*})^2 + (E_i^{\mathrm{r}} - \bar{E}^{\mathrm{r}*})^2\big) = \mathcal{J}_{\sigma i}^{\infty}(E_i^{\mathrm{d}}, E_i^{\mathrm{r}}; z^{\mathrm{d}*}, z^{\mathrm{r}*})$. Hence, each $E_i^{\mathrm{d}*}$ and $E_i^{\mathrm{r}*}$ maximize $\mathcal{J}_{\sigma i}^{\infty}(E_i^{\mathrm{d}}, E_i^{\mathrm{r}}; z^{\mathrm{d}*}, z^{\mathrm{r}*})$ and also $\mathcal{J}_{\sigma i}^{\infty}(E_i^{\mathrm{d}}, E_i^{\mathrm{r}}; E_{-i}^{\mathrm{d}*}, E_{-i}^{\mathrm{r}*})$ with respect to the constraints of (5). Thus, following Definition 1, $\{E_i^{\mathrm{d}*}, E_i^{\mathrm{r}*}\}_{i\in\mathcal{N}}$ is a Nash equilibrium for the infinite population case. ■

According to the mean field game theory, each agent does not consider the individual strategies of other agents to maximize its welfare, but each agent is affected by the mass average trajectories $\bar{E}^{\mathrm{d}}$ and $\bar{E}^{\mathrm{r}}$ [8]. We note that Theorem 1 provides the conditions that govern a Nash equilibrium and is different from the Nash equilibrium definition. Let the saturation function $\mathrm{sat}_{[c_1,c_2]}(\cdot)$ be defined as $\mathrm{sat}_{[c_1,c_2]}(s) :=$ c_1 if $s \leq c_1$, c_2 if $s \geq c_2$, s otherwise. Then, in the following lemma, we obtain the unique solution to the problem (4).

Lemma 1: **(Optimal solution to** (4)**).** Let Assumption 1 hold and consider the optimization problem (4). Then, its optimal solution is unique and given by

$$\begin{aligned} E_i^{\mathrm{d}*} = g^{\mathrm{d}}(B^*, z) =& \mathrm{sat}_{[0,E_i^{\max}]}\big(\tfrac{1}{2\sigma}\big(u_i - f^{\mathrm{d}}\big(\tfrac{1}{c^n}(z^{\mathrm{d}} + d^n)\big) \\ &+ 2\sigma z^{\mathrm{d}} + B^*(z)\big)\big) \\ E_i^{\mathrm{r}*} = g^{\mathrm{r}}(B^*, z) =& \mathrm{sat}_{[0,E_i^{\max}]}\big(\tfrac{1}{2\sigma}\big(f^{\mathrm{r}}\big(\tfrac{1}{c^n}(z^{\mathrm{d}} - z^{\mathrm{r}} + d^n)\big) \\ &- u_i + 2\sigma z^{\mathrm{r}} - B^*(z)\big)\big), \end{aligned} \tag{6}$$

where $B^*(z) = \min_{B\in\mathbb{R}_{\geq 0}}\{B : g^{\mathrm{r}}(B, z) \leq g^{\mathrm{d}}(B, z)\}$.

Proof: We form the Lagrangian function for the optimization problem (4) as $\mathcal{L}_i(E_i^{\mathrm{d}}, E_i^{\mathrm{r}}, \mu_1, \mu_2, \nu_1, \nu_2) = -\mathcal{J}_{\sigma i}(E_i^{\mathrm{d}}, E_i^{\mathrm{r}}; z^{\mathrm{d}}, z^{\mathrm{r}}) + \mu_1(E_i^{\mathrm{d}} - E_i^{\max}) - \nu_1 E_i^{\mathrm{d}} + \mu_2(E_i^{\mathrm{r}} - E_i^{\mathrm{d}}) - \nu_2 E_i^{\mathrm{r}}$. Now, we apply the Karush–Kuhn–Tucker (KKT) conditions

$$\begin{aligned} \tfrac{\partial \mathcal{L}_i}{\partial E_i^{\mathrm{d}}} =& f^{\mathrm{d}}\big(\tfrac{1}{c^n}(z^{\mathrm{d}} + d^n)\big) - u_i + 2\sigma(E_i^{\mathrm{d}*} - z^{\mathrm{d}}) + \mu_1^* \\ &- \nu_1^* - \mu_2^* = 0 \end{aligned} \tag{7}$$

$$\begin{aligned} \tfrac{\partial \mathcal{L}_i}{\partial E_i^{\mathrm{r}}} =& u_i - f^{\mathrm{r}}\big(\tfrac{1}{c^n}(z^{\mathrm{d}} - z^{\mathrm{r}} + d^n)\big) + 2\sigma(E_i^{\mathrm{r}*} - z^{\mathrm{r}}) \\ &+ \mu_2^* - \nu_2^* = 0 \end{aligned} \tag{8}$$

$$0 \leq E_i^{\mathrm{d}*} \leq E_i^{\max}, \quad 0 \leq E_i^{\mathrm{r}*} \leq E_i^{\mathrm{d}*} \tag{9}$$

$$\mu_1^*, \ \mu_2^*, \ \nu_1^*, \ \nu_2^* \geq 0 \tag{10}$$

$$\begin{aligned} \mu_1^*(E_i^{\mathrm{d}*} - E_i^{\max}) = \mu_2^*(E_i^{\mathrm{r}*} - E_i^{\mathrm{d}*}) = \nu_1^* E_i^{\mathrm{d}*} \\ = \nu_2^* E_i^{\mathrm{r}*} = 0. \end{aligned} \tag{11}$$

Then, we investigate the following possible cases:

1. $E_i^{\mathrm{d}*} = 0$. By the complementary slackness condition (11) and dual feasibility condition (10), $\mu_1^* = 0$ and $\nu_1^* \geq 0$. By the primal feasibility condition (9), $E_i^{\mathrm{r}*} = 0$. Then, by the complementary slackness condition (11) and dual feasibility condition (10), $\mu_2^*, \nu_2^* \geq 0$. Then, by the stationary conditions (7), (8), we have $f^{\mathrm{d}}\big(\frac{1}{c^n}(z^{\mathrm{d}} + d^n)\big) \geq u_i + \big(f^{\mathrm{r}}\big(\frac{1}{c^n}(z^{\mathrm{d}} - z^{\mathrm{r}} + d^n)\big) - u_i\big) + 2\sigma(z^{\mathrm{d}} + z^{\mathrm{r}})$ and also $\frac{1}{2\sigma}\big(u_i - f^{\mathrm{d}}\big(\frac{1}{c^n}(z^{\mathrm{d}} + d^n)\big) + 2\sigma z^{\mathrm{d}} + \mu_2^*\big) = -\frac{1}{2\sigma}\nu_1^* \leq 0$, $\frac{1}{2\sigma}\big(f^{\mathrm{r}}\big(\frac{1}{c^n}(z^{\mathrm{d}} - z^{\mathrm{r}} + d^n)\big) - u_i + 2\sigma z^{\mathrm{r}} - \mu_2^*\big) = -\frac{1}{2\sigma}\nu_2^* \leq 0$.
2. $0 < E_i^{\mathrm{d}*} < E_i^{\max}$. By the complementary slackness condition (11), $\mu_1^* = \nu_1^* = 0$. Now, we investigate the following possible cases:
   a. $E_i^{\mathrm{r}*} = 0$. By the complementary slackness condition (11), $\mu_2^* = 0$ and $\nu_2^* \geq 0$. Then, by the stationary conditions (7), (8), we have $E_i^{\mathrm{d}*} = \frac{1}{2\sigma}\big(u_i - f^{\mathrm{d}}\big(\frac{1}{c^n}(z^{\mathrm{d}} + d^n)\big) + 2\sigma z^{\mathrm{d}}\big)$ and $\frac{1}{2\sigma}\big(f^{\mathrm{r}}\big(\frac{1}{c^n}(z^{\mathrm{d}} - z^{\mathrm{r}} + d^n)\big) - u_i + 2\sigma z^{\mathrm{r}}\big) = -\frac{1}{2\sigma}\nu_2^* \leq 0$.
   b. $0 < E_i^{\mathrm{r}*} < E_i^{\mathrm{d}*}$. By the complementary slackness condition (11), $\mu_2^* = \nu_2^* = 0$. Then, by the stationary conditions (7), (8), we have $E_i^{\mathrm{d}*} = \frac{1}{2\sigma}\big(u_i - f^{\mathrm{d}}\big(\frac{1}{c^n}(z^{\mathrm{d}} + d^n)\big) + 2\sigma z^{\mathrm{d}}\big)$ and $E_i^{\mathrm{r}*} = \frac{1}{2\sigma}\big(f^{\mathrm{r}}\big(\frac{1}{c^n}(z^{\mathrm{d}} - z^{\mathrm{r}} + d^n)\big) - u_i + 2\sigma z^{\mathrm{r}}\big)$.
   c. $E_i^{\mathrm{r}*} = E_i^{\mathrm{d}*}$. By the complementary slackness condition (11) and dual feasibility condition (10), $\mu_2^* \geq 0$ and $\nu_2^* = 0$. Then, by the stationary conditions (7), (8), we have $E_i^{\mathrm{d}*} = E_i^{\mathrm{r}*} = \frac{1}{2\sigma}\big(u_i - f^{\mathrm{d}}\big(\frac{1}{c^n}(z^{\mathrm{d}} + d^n)\big) + 2\sigma z^{\mathrm{d}} + \mu_2^*\big) = \frac{1}{2\sigma}\big(f^{\mathrm{r}}\big(\frac{1}{c^n}(z^{\mathrm{d}} - z^{\mathrm{r}} + d^n)\big) - u_i + 2\sigma z^{\mathrm{r}} - \mu_2^*\big)$ and

$\mu_2^* = \frac{1}{2}\big(f^{\mathrm{d}}\big(\frac{1}{c^n}(z^{\mathrm{d}} + d^n)\big) + f^{\mathrm{r}}\big(\frac{1}{c^n}(z^{\mathrm{d}} - z^{\mathrm{r}} + d^n)\big) - 2u_i + 2\sigma(z^{\mathrm{r}} - z^{\mathrm{d}})\big) \geq 0.$

3. $E_i^{\mathrm{d}*} = E_i^{\max}$. By the complementary slackness condition (11) and dual feasibility condition (10), $\mu_1^* \geq 0$ and $\nu_1^* = 0$. Now, we investigate the following possible cases:
   - *a.* $E_i^{\mathrm{r}*} = 0$. By the complementary slackness condition (11) and dual feasibility condition (10), $\mu_2^* = 0$ and $\nu_2^* \geq 0$. By the stationary conditions (7), (8), we have $\frac{1}{2\sigma}\big(u_i - f^{\mathrm{d}}\big(\frac{1}{c^n}(z^{\mathrm{d}} + d^n)\big) + 2\sigma z^{\mathrm{d}}\big) - E_i^{\max} = -\frac{1}{2\sigma}\mu_1^* \leq 0$ and $\frac{1}{2\sigma}\big(f^{\mathrm{r}}\big(\frac{1}{c^n}(z^{\mathrm{d}} - z^{\mathrm{r}} + d^n)\big) - u_i + 2\sigma z^{\mathrm{r}}\big) = -\frac{1}{2\sigma}\nu_2^* \leq 0$.
   - *b.* $0 < E_i^{\mathrm{r}*} < E_i^{\mathrm{d}*}$. By the complementary slackness condition (11), $\mu_2^* = \nu_2^* = 0$. By the stationary conditions (7), (8), we have $E_i^{\mathrm{r}*} = \frac{1}{2\sigma}\big(f^{\mathrm{r}}\big(\frac{1}{c^n}(z^{\mathrm{d}} - z^{\mathrm{r}} + d^n)\big) - u_i + 2\sigma z^{\mathrm{r}}\big)$ and $\frac{1}{2\sigma}\big(u_i - f^{\mathrm{d}}\big(\frac{1}{c^n}(z^{\mathrm{d}} + d^n)\big) + 2\sigma z^{\mathrm{d}}\big) - E_i^{\max} = \frac{1}{2\sigma}\mu_1^* \geq 0$.
   - *c.* $E_i^{\mathrm{r}*} = E_i^{\mathrm{d}*} = E_i^{\max}$. By the complementary slackness condition (11) and dual feasibility condition (10), $\mu_2^* \geq 0$ and $\nu_2^* = 0$. By the stationary conditions (7), (8), we have $\frac{1}{2\sigma}\big(u_i - f^{\mathrm{d}}\big(\frac{1}{c^n}(z^{\mathrm{d}} + d^n)\big) + 2\sigma z^{\mathrm{d}} + \mu_2^*\big) - E_i^{\max} = \frac{1}{2\sigma}\mu_1^* \geq 0$, $\frac{1}{2\sigma}\big(f^{\mathrm{r}}\big(\frac{1}{c^n}(z^{\mathrm{d}} - z^{\mathrm{r}} + d^n)\big) - u_i + 2\sigma z^{\mathrm{r}} - \mu_2^*\big) = E_i^{\max}$ and $\mu_2^* = f^{\mathrm{r}}\big(\frac{1}{c^n}(z^{\mathrm{d}} - z^{\mathrm{r}} + d^n)\big) - u_i + 2\sigma(z^{\mathrm{r}} - E_i^{\max})$.

Therefore, with $B = \mu_2^*$ we have $E_i^{\mathrm{d}*} = \mathrm{sat}_{[0,E_i^{\max}]}\big(\frac{1}{2\sigma}\big(u_i - f^{\mathrm{d}}\big(\frac{1}{c^n}(z^{\mathrm{d}} + d^n)\big) + 2\sigma z^{\mathrm{d}} + B\big)\big)$ and $E_i^{\mathrm{r}*} = \mathrm{sat}_{[0,E_i^{\max}]}\big(\frac{1}{2\sigma}\big(f^{\mathrm{r}}\big(\frac{1}{c^n}(z^{\mathrm{d}} - z^{\mathrm{r}} + d^n)\big) - u_i + 2\sigma z^{\mathrm{r}} - B\big)\big)$, where B must be chosen such that

$$g^{\mathrm{r}}(B, z) \leq g^{\mathrm{d}}(B, z), \tag{12}$$

with $g^{\mathrm{d}}(B, z)$ and $g^{\mathrm{r}}(B, z)$ given in (6). The mapping $g^{\mathrm{d}}(B, z)$ is continuous and non-decreasing and $g^{\mathrm{r}}(B, z)$ is continuous and non-increasing with B. Therefore, according to the possible cases 1−3, there exists B which satisfies (12). Moreover, in the possible cases 1−3, we have shown that $E_i^{\mathrm{d}*}$ and $E_i^{\mathrm{r}*}$ are unique such that $B^*(z)$ is the smallest non-negative B fulfilling (12). ■

Now, in the following lemma, we show that the optimal solution to the problem (4) is contractive. This result will be used later to show the existence and uniqueness of the Nash equilibrium associated with problem (5) for the infinite population case.

Lemma 2: **(Contractiveness of** (6)**).** Let Assumptions 1 and 2 hold and $E_i^{\mathrm{d}*}(z)$ and $E_i^{\mathrm{r}*}(z)$ be the optimal solution to the problem (4) for each $i \in \mathcal{N}$. Then, the mappings $z \mapsto \bar{E}^{\mathrm{d},n*}(z) = \frac{1}{n}\sum_{i\in\mathcal{N}} E_i^{\mathrm{d}*}(z)$ and $z \mapsto \bar{E}^{\mathrm{r},n*}(z) = \frac{1}{n}\sum_{i\in\mathcal{N}} E_i^{\mathrm{r}*}(z)$ are continuous and contractive.

Proof: Following Lemma 1, the optimal solution to the problem (4) is given in (6). Now, we define the optimal solution with $\hat{z} = (\hat{z}^{\mathrm{d}}, \hat{z}^{\mathrm{r}})$ as $\hat{E}_i^{\mathrm{d}*} = \mathrm{sat}_{[0,E_i^{\max}]}\big(\frac{1}{2\sigma}\big(u_i - f^{\mathrm{d}}\big(\frac{1}{c^n}(\hat{z}^{\mathrm{d}} + d^n)\big) + 2\sigma\hat{z}^{\mathrm{d}} + B^*(\hat{z})\big)\big)$ and $\hat{E}_i^{\mathrm{r}*} = \mathrm{sat}_{[0,E_i^{\max}]}\big(\frac{1}{2\sigma}\big(f^{\mathrm{r}}\big(\frac{1}{c^n}(\hat{z}^{\mathrm{d}} - \hat{z}^{\mathrm{r}} + d^n)\big) - u_i + 2\sigma\hat{z}^{\mathrm{r}} - B^*(\hat{z})\big)\big)$ and the non-optimal solution with $\hat{z}$, but with $B^*(z)$ as $L_i^{\mathrm{d}} = \mathrm{sat}_{[0,E_i^{\max}]}\big(\frac{1}{2\sigma}\big(u_i - f^{\mathrm{d}}\big(\frac{1}{c^n}(\hat{z}^{\mathrm{d}} + d^n)\big) + 2\sigma\hat{z}^{\mathrm{d}} + B^*(z)\big)\big)$ and $L_i^{\mathrm{r}} = \mathrm{sat}_{[0,E_i^{\max}]}\big(\frac{1}{2\sigma}\big(f^{\mathrm{r}}\big(\frac{1}{c^n}(\hat{z}^{\mathrm{d}} - \hat{z}^{\mathrm{r}} + d^n)\big) - u_i + 2\sigma\hat{z}^{\mathrm{r}} - B^*(z)\big)\big)$. Now, consider the following lemmas.

Lemma 3: For all $i \in \mathcal{N}$, we have

$$|E_i^{\mathrm{d}*} - L_i^{\mathrm{d}}| \leq \Big|(z^{\mathrm{d}} - \hat{z}^{\mathrm{d}}) - \frac{1}{2\sigma}\Big(f^{\mathrm{d}}\big(\tfrac{1}{c^n}(z^{\mathrm{d}} + d^n)\big) - f^{\mathrm{d}}\big(\tfrac{1}{c^n}(\hat{z}^{\mathrm{d}} + d^n)\big)\Big)\Big| \tag{13}$$

$$|E_i^{\mathrm{r}*} - L_i^{\mathrm{r}}| \leq \Big|(z^{\mathrm{r}} - \hat{z}^{\mathrm{r}}) + \frac{1}{2\sigma}\Big(f^{\mathrm{r}}\big(\tfrac{1}{c^n}(z^{\mathrm{d}} - z^{\mathrm{r}} + d^n)\big) - f^{\mathrm{r}}\big(\tfrac{1}{c^n}(\hat{z}^{\mathrm{d}} - \hat{z}^{\mathrm{r}} + d^n)\big)\Big)\Big|. \tag{14}$$

Proof: Let $W_{E_i^{\mathrm{d}}} = \frac{1}{2\sigma}\big(u_i - f^{\mathrm{d}}\big(\frac{1}{c^n}(z^{\mathrm{d}} + d^n)\big) + 2\sigma z^{\mathrm{d}} + B^*(z)\big)$, $W_{L_i^{\mathrm{d}}} = \frac{1}{2\sigma}\big(u_i - f^{\mathrm{d}}\big(\frac{1}{c^n}(\hat{z}^{\mathrm{d}} + d^n)\big) + 2\sigma\hat{z}^{\mathrm{d}} + B^*(z)\big)$, $W_{E_i^{\mathrm{r}}} = \frac{1}{2\sigma}\big(f^{\mathrm{r}}\big(\frac{1}{c^n}(z^{\mathrm{d}} - z^{\mathrm{r}} + d^n)\big) - u_i + 2\sigma z^{\mathrm{r}} - B^*(z)\big)$ and $W_{L_i^{\mathrm{r}}} = \frac{1}{2\sigma}\big(f^{\mathrm{r}}\big(\frac{1}{c^n}(\hat{z}^{\mathrm{d}} - \hat{z}^{\mathrm{r}} + d^n)\big) - u_i + 2\sigma\hat{z}^{\mathrm{r}} - B^*(z)\big)$. Therefore, we can rewrite (13), (14) as $|E_i^{\mathrm{d}*} - L_i^{\mathrm{d}}| \leq |W_{E_i^{\mathrm{d}}} - W_{L_i^{\mathrm{d}}}|$ and $|E_i^{\mathrm{r}*} - L_i^{\mathrm{r}}| \leq |W_{E_i^{\mathrm{r}}} - W_{L_i^{\mathrm{r}}}|$, respectively. Then, we have the following possible cases for $E_i^{\mathrm{d}*}$:

1. $E_i^{\mathrm{d}*} = 0 \Rightarrow W_{E_i^{\mathrm{d}}} \leq 0 = E_i^{\mathrm{d}*}$;
2. $0 < E_i^{\mathrm{d}*} < E_i^{\max} \Rightarrow W_{E_i^{\mathrm{d}}} = E_i^{\mathrm{d}*}$;
3. $E_i^{\mathrm{d}*} = E_i^{\max} \Rightarrow W_{E_i^{\mathrm{d}}} \geq E_i^{\max} = E_i^{\mathrm{d}*}$.

Also, we have the following possible cases for L_i^{d}:

a. $L_i^{\mathrm{d}} = 0 \Rightarrow W_{L_i^{\mathrm{d}}} \leq 0 = L_i^{\mathrm{d}}$;
b. $0 < L_i^{\mathrm{d}} < E_i^{\max} \Rightarrow W_{L_i^{\mathrm{d}}} = L_i^{\mathrm{d}}$;
c. $L_i^{\mathrm{d}} = E_i^{\max} \Rightarrow W_{L_i^{\mathrm{d}}} \geq E_i^{\max} = L_i^{\mathrm{d}}$.

Now, we investigate the case that $E_i^{\mathrm{d}*} = 0$ and the other combinations of the above cases can be proved similarly:

a. $L_i^{\mathrm{d}} = 0 \Rightarrow |E_i^{\mathrm{d}*} - L_i^{\mathrm{d}}| = 0 \leq |W_{E_i^{\mathrm{d}}} - W_{L_i^{\mathrm{d}}}|$;
b. $0 < L_i^{\mathrm{d}} < E_i^{\max} \Rightarrow |E_i^{\mathrm{d}*} - L_i^{\mathrm{d}}| = L_i^{\mathrm{d}} - E_i^{\mathrm{d}*} = W_{L_i^{\mathrm{d}}} - E_i^{\mathrm{d}*} \leq W_{L_i^{\mathrm{d}}} - W_{E_i^{\mathrm{d}}} \leq |W_{E_i^{\mathrm{d}}} - W_{L_i^{\mathrm{d}}}|$;
c. $L_i^{\mathrm{d}} = E_i^{\max} \Rightarrow |E_i^{\mathrm{d}*} - L_i^{\mathrm{d}}| = L_i^{\mathrm{d}} - E_i^{\mathrm{d}*} \leq W_{L_i^{\mathrm{d}}} - E_i^{\mathrm{d}*} \leq W_{L_i^{\mathrm{d}}} - W_{E_i^{\mathrm{d}}} \leq |W_{E_i^{\mathrm{d}}} - W_{L_i^{\mathrm{d}}}|$.

The analogous analysis can be used for the different cases of $E_i^{\mathrm{r}*}$ and L_i^{r}, then (14) can be proved similarly. ■

Lemma 4: For all $i \in \mathcal{N}$, we have

$$|E_i^{\mathrm{d}*} - \hat{E}_i^{\mathrm{d}*}| + |E_i^{\mathrm{r}*} - \hat{E}_i^{\mathrm{r}*}| \leq 2|E_i^{\mathrm{d}*} - L_i^{\mathrm{d}}| + 2|E_i^{\mathrm{r}*} - L_i^{\mathrm{r}}|. \tag{15}$$

Proof: Using the definition of $B^*(z)$ and $B^*(\hat{z})$, we have for all $i \in \mathcal{N}$, $V(B^*(z)) := E_i^{\mathrm{d}*} - E_i^{\mathrm{r}*} \geq 0$, $\hat{V}(B^*(\hat{z})) := \hat{E}_i^{\mathrm{d}*} - \hat{E}_i^{\mathrm{r}*} \geq 0$; however, the sign of $\hat{V}(B^*(z)) := L_i^{\mathrm{d}} - L_i^{\mathrm{r}}$ is not known. Now, we investigate the following possible cases:

1. $\hat{V}(B^*(z)) = L_i^{\mathrm{d}} - L_i^{\mathrm{r}} \geq 0$. L_i^{d} and L_i^{r} have the form of the optimal solution with $\hat{z}$ and $\hat{V}(B^*(z)) \geq 0$, then it is the unique optimal solution with $\hat{z}$. Therefore, $L_i^{\mathrm{d}} = \hat{E}_i^{\mathrm{d}*}$ and $L_i^{\mathrm{r}} = \hat{E}_i^{\mathrm{r}*}$ and $|E_i^{\mathrm{d}*} - \hat{E}_i^{\mathrm{d}*}| + |E_i^{\mathrm{r}*} - \hat{E}_i^{\mathrm{r}*}| = |E_i^{\mathrm{d}*} - L_i^{\mathrm{d}}| + |E_i^{\mathrm{r}*} - L_i^{\mathrm{r}}| \leq 2|E_i^{\mathrm{d}*} - L_i^{\mathrm{d}}| + 2|E_i^{\mathrm{r}*} - L_i^{\mathrm{r}}|$.
2. $\hat{V}(B^*(z)) = L_i^{\mathrm{d}} - L_i^{\mathrm{r}} < 0 \leq \hat{V}(B^*(\hat{z}))$. $\hat{V}$ is monotonically non-decreasing, then we have $B^*(\hat{z}) > B^*(z)$, which leads to $L_i^{\mathrm{d}} \leq \hat{E}_i^{\mathrm{d}*}$ and $L_i^{\mathrm{r}} \geq \hat{E}_i^{\mathrm{r}*}$. According to the possible cases 1−3 in the proof of Lemma 1, when $B^*(z) = 0$ ($B^*(\hat{z}) = 0$) we have $E_i^{\mathrm{d}*} \geq E_i^{\mathrm{r}*}$ ($\hat{E}_i^{\mathrm{d}*} \geq \hat{E}_i^{\mathrm{r}*}$) and when $B^*(z) \geq 0$ ($B^*(\hat{z}) \geq 0$) we have $E_i^{\mathrm{d}*} = E_i^{\mathrm{r}*}$ ($\hat{E}_i^{\mathrm{d}*} = \hat{E}_i^{\mathrm{r}*}$). Since $B^*(\hat{z}) > B^*(z)$, we have the following possible cases:
   - *a.* $B^*(\hat{z}) > B^*(z) > 0$. Then, $E_i^{\mathrm{d}*} = E_i^{\mathrm{r}*}$ and $\hat{E}_i^{\mathrm{d}*} = \hat{E}_i^{\mathrm{r}*}$. Therefore, we have $|E_i^{\mathrm{d}*} - \hat{E}_i^{\mathrm{d}*}| + |E_i^{\mathrm{r}*} - \hat{E}_i^{\mathrm{r}*}| \leq |E_i^{\mathrm{d}*} - L_i^{\mathrm{d}}| + |\hat{E}_i^{\mathrm{d}*} - L_i^{\mathrm{d}}| + |E_i^{\mathrm{r}*} - L_i^{\mathrm{r}}| + |\hat{E}_i^{\mathrm{r}*} - L_i^{\mathrm{r}}| =$

$|E_i^{\mathrm{d}*}-L_i^{\mathrm{d}}|+|E_i^{\mathrm{r}*}-L_i^{\mathrm{r}}|+\hat{E}_i^{\mathrm{d}*}-L_i^{\mathrm{d}}+L_i^{\mathrm{r}}-\hat{E}_i^{\mathrm{r}*}=|E_i^{\mathrm{d}*}-L_i^{\mathrm{d}}|+|E_i^{\mathrm{r}*}-L_i^{\mathrm{r}}|+|L_i^{\mathrm{r}}-L_i^{\mathrm{d}}+E_i^{\mathrm{r}*}-E_i^{\mathrm{r}*}|\leq|E_i^{\mathrm{d}*}-L_i^{\mathrm{d}}|+2|E_i^{\mathrm{r}*}-L_i^{\mathrm{r}}|+|E_i^{\mathrm{r}*}-L_i^{\mathrm{d}}|=2|E_i^{\mathrm{d}*}-L_i^{\mathrm{d}}|+2|E_i^{\mathrm{r}*}-L_i^{\mathrm{r}}|$.

b. $B^*(\hat{z})>B^*(z)=0$. Similar to the possible case *a.*, this case can be proved analogously.

■

Now, according to Lemmas 3 and 4 and Assumptions 1 and 2, we continue the proof of Lemma 2. Let $x^{\mathrm{d}}=\frac{1}{c^n}(z^{\mathrm{d}}+d^n)$, $\hat{x}^{\mathrm{d}}=\frac{1}{c^n}(\hat{z}^{\mathrm{d}}+d^n)$, $x^{\mathrm{r}}=\frac{1}{c^n}(z^{\mathrm{d}}-z^{\mathrm{r}}+d^n)$ and $\hat{x}^{\mathrm{r}}=\frac{1}{c^n}(\hat{z}^{\mathrm{d}}-\hat{z}^{\mathrm{r}}+d^n)$, then $x^{\mathrm{d}}-\hat{x}^{\mathrm{d}}=\frac{1}{c^n}(z^{\mathrm{d}}-\hat{z}^{\mathrm{d}})$ and $x^{\mathrm{r}}-\hat{x}^{\mathrm{r}}=\frac{1}{c^n}(z^{\mathrm{d}}-\hat{z}^{\mathrm{d}}-(z^{\mathrm{r}}-\hat{z}^{\mathrm{r}}))$. Thus, we have $|f^{\mathrm{d}}(x^{\mathrm{d}})-f^{\mathrm{d}}(\hat{x}^{\mathrm{d}})|+|f^{\mathrm{r}}(x^{\mathrm{r}})-f^{\mathrm{r}}(\hat{x}^{\mathrm{r}})|\leq\max_{x^{\mathrm{d}}\in[\underline{x}^{\mathrm{d}},\bar{x}^{\mathrm{d}}]}\frac{\partial f^{\mathrm{d}}(x^{\mathrm{d}})}{\partial x^{\mathrm{d}}}\times|x^{\mathrm{d}}-\hat{x}^{\mathrm{d}}|+\max_{x^{\mathrm{r}}\in[\underline{x}^{\mathrm{r}},\bar{x}^{\mathrm{r}}]}\frac{\partial f^{\mathrm{r}}(x^{\mathrm{r}})}{\partial x^{\mathrm{r}}}\times|x^{\mathrm{r}}-\hat{x}^{\mathrm{r}}|=\max_{x^{\mathrm{d}}\in[\underline{x}^{\mathrm{d}},\bar{x}^{\mathrm{d}}]}\frac{\partial f^{\mathrm{d}}(x^{\mathrm{d}})}{\partial x^{\mathrm{d}}}\times\frac{1}{c^n}|z^{\mathrm{d}}-\hat{z}^{\mathrm{d}}|+\max_{x^{\mathrm{r}}\in[\underline{x}^{\mathrm{r}},\bar{x}^{\mathrm{r}}]}\frac{\partial f^{\mathrm{r}}(x^{\mathrm{r}})}{\partial x^{\mathrm{r}}}\times\frac{1}{c^n}|z^{\mathrm{d}}-\hat{z}^{\mathrm{d}}-(z^{\mathrm{r}}-\hat{z}^{\mathrm{r}})|\leq 4\sigma|z^{\mathrm{d}}-\hat{z}^{\mathrm{d}}|+2\sigma|z^{\mathrm{r}}-\hat{z}^{\mathrm{r}}|$, where the last inequality is based on Assumption 2. Analogously, we can obtain similar inequality in terms of $\min_{x^{\mathrm{d}}\in[\underline{x}^{\mathrm{d}},\bar{x}^{\mathrm{d}}]}\frac{\partial f^{\mathrm{d}}(x^{\mathrm{d}})}{\partial x^{\mathrm{d}}}$ and $\min_{x^{\mathrm{r}}\in[\underline{x}^{\mathrm{r}},\bar{x}^{\mathrm{r}}]}\frac{\partial f^{\mathrm{r}}(x^{\mathrm{r}})}{\partial x^{\mathrm{r}}}$. Hence, both together result in $\frac{1}{2\bar{\gamma}}(|z^{\mathrm{d}}-\hat{z}^{\mathrm{d}}|+|z^{\mathrm{r}}-\hat{z}^{\mathrm{r}}|)\leq\frac{1}{2\sigma}\big(|f^{\mathrm{d}}(x^{\mathrm{d}})-f^{\mathrm{d}}(\hat{x}^{\mathrm{d}})|+|f^{\mathrm{r}}(x^{\mathrm{r}})-f^{\mathrm{r}}(\hat{x}^{\mathrm{r}})|\big)\leq 2|z^{\mathrm{d}}-\hat{z}^{\mathrm{d}}|+|z^{\mathrm{r}}-\hat{z}^{\mathrm{r}}|$, where $\bar{\gamma}=\max(\gamma_2,\frac{\gamma_1\gamma_2}{\gamma_2-\gamma_1})$. Then, we can obtain $(1-\frac{1}{2\bar{\gamma}})(|z^{\mathrm{d}}-\hat{z}^{\mathrm{d}}|+|z^{\mathrm{r}}-\hat{z}^{\mathrm{r}}|)\geq|z^{\mathrm{d}}-\hat{z}^{\mathrm{d}}|-\frac{1}{2\sigma}|f^{\mathrm{d}}(x^{\mathrm{d}})-f^{\mathrm{d}}(\hat{x}^{\mathrm{d}})|+|z^{\mathrm{r}}-\hat{z}^{\mathrm{r}}|-\frac{1}{2\sigma}|f^{\mathrm{r}}(x^{\mathrm{r}})-f^{\mathrm{r}}(\hat{x}^{\mathrm{r}})|\geq 0$. Since $f^{\mathrm{d}}(x^{\mathrm{d}})$ and $f^{\mathrm{r}}(x^{\mathrm{r}})$ are strictly increasing functions, we have $(1-\frac{1}{2\bar{\gamma}})(|z^{\mathrm{d}}-\hat{z}^{\mathrm{d}}|+|z^{\mathrm{r}}-\hat{z}^{\mathrm{r}}|)\geq|(z^{\mathrm{d}}-\hat{z}^{\mathrm{d}})-\frac{1}{2\sigma}(f^{\mathrm{d}}(x^{\mathrm{d}})-f^{\mathrm{d}}(\hat{x}^{\mathrm{d}}))|+|(z^{\mathrm{r}}-\hat{z}^{\mathrm{r}})+\frac{1}{2\sigma}(f^{\mathrm{r}}(x^{\mathrm{r}})-f^{\mathrm{r}}(\hat{x}^{\mathrm{r}}))|$. Then, according to Lemmas 3 and 4, we obtain

$$|E_i^{\mathrm{d}*}(z)-E_i^{\mathrm{d}*}(\hat{z})|+|E_i^{\mathrm{r}*}(z)-E_i^{\mathrm{r}*}(\hat{z})| \leq(2-\tfrac{1}{\bar{\gamma}})(|z^{\mathrm{d}}-\hat{z}^{\mathrm{d}}|+|z^{\mathrm{r}}-\hat{z}^{\mathrm{r}}|). \quad (16)$$

Thus, since $\frac{1}{2}<\bar{\gamma}<1$, the mappings $z\mapsto\bar{E}^{\mathrm{d},n*}(z)=\frac{1}{n}\sum_{i\in\mathcal{N}}E_i^{\mathrm{d}*}(z)$ and $z\mapsto\bar{E}^{\mathrm{r},n*}(z)=\frac{1}{n}\sum_{i\in\mathcal{N}}E_i^{\mathrm{r}*}(z)$ are continuous and contractive. ■

The following theorem represents the main result of the paper, where the existence and uniqueness of the Nash equilibrium associated with problem (5) when the population size tends to infinity, is proved.

Theorem 2: **(Existence and uniqueness of the Nash equilibrium).** Let Assumption 1 hold. Then, a set of strategies $\{E_i^{\mathrm{d}*},E_i^{\mathrm{r}*}\}_{i\in\mathcal{N}}$ is a Nash equilibrium associated with (5) for the infinite population case if $E_i^{\mathrm{d}*}(z)$, $E_i^{\mathrm{r}*}(z)$ is the optimal solution to (4) for all $i\in\mathcal{N}$ with $z=(z^{\mathrm{d}},z^{\mathrm{r}})=(\bar{E}^{\mathrm{d}*},\bar{E}^{\mathrm{r}*})$. If Assumption 2 holds, then the Nash equilibrium is unique.

Proof: In analogy with Lemmas 3 and 4, we have $|E_i^{\mathrm{d}*}(z)-E_i^{\mathrm{d}*}(\hat{z})|+|E_i^{\mathrm{r}*}(z)-E_i^{\mathrm{r}*}(\hat{z})|\leq 2|E_i^{\mathrm{d}*}(z)-L_i^{\mathrm{d}}(z,\hat{z})|+2|E_i^{\mathrm{r}*}(z)-L_i^{\mathrm{r}}(z,\hat{z})|\leq 2|(z^{\mathrm{d}}-\hat{z}^{\mathrm{d}})-\frac{1}{2\sigma}(f^{\mathrm{d}}(\frac{1}{c^n}(z^{\mathrm{d}}+d^n))-f^{\mathrm{d}}(\frac{1}{c^n}(\hat{z}^{\mathrm{d}}+d^n)))|+2|(z^{\mathrm{r}}-\hat{z}^{\mathrm{r}})+\frac{1}{2\sigma}(f^{\mathrm{r}}(\frac{1}{c^n}(z^{\mathrm{d}}-z^{\mathrm{r}}+d^n))-f^{\mathrm{r}}(\frac{1}{c^n}(\hat{z}^{\mathrm{d}}-\hat{z}^{\mathrm{r}}+d^n)))|$. Therefore, $E_i^{\mathrm{d}*}(z)$ and $E_i^{\mathrm{r}*}(z)$ are continuous in z if $f^{\mathrm{d}}(x^{\mathrm{d}})$ and $f^{\mathrm{r}}(x^{\mathrm{r}})$ are continuous in x^{d} and x^{r}, respectively. Consequently, $\bar{E}^{\mathrm{d}*}(z)$ and $\bar{E}^{\mathrm{r}*}(z)$, which are the average of continuous functions, are also continuous in z. Now, we define the convex compact set $\chi:=\big\{(v^{\mathrm{d}},v^{\mathrm{r}})|\ 0\leq v^{\mathrm{d}}\leq\max_{i\in\mathcal{N}}\{E_i^{\max}\},0\leq v^{\mathrm{r}}\leq\max_{i\in\mathcal{N}}\{E_i^{\max}\}\big\}$. Thus, the constraint set of the problem (4) is a subset of χ and by extension $(\bar{E}^{\mathrm{d}*}(z),\bar{E}^{\mathrm{r}*}(z))\in\chi$. Then, for any $z\in\chi$, we have $(\bar{E}^{\mathrm{d}*}(z),\bar{E}^{\mathrm{r}*}(z))\in\chi$, so $(\bar{E}^{\mathrm{d}*}(\cdot),\bar{E}^{\mathrm{r}*}(\cdot))$ maps a convex compact set to itself. Hence, according to the Brouwer fixed point theorem [10], a fixed point $z\in\chi$ exists such that $(\bar{E}^{\mathrm{d}*}(z),\bar{E}^{\mathrm{r}*}(z))=z$. Following Theorem 1, $\big\{E_i^{\mathrm{d}*}(z),E_i^{\mathrm{r}*}(z)\big\}_{i\in\mathcal{N}}$ is the set of optimal strategies, then the fixed point $(\bar{E}^{\mathrm{d}*}(z),\bar{E}^{\mathrm{r}*}(z))=z$ is a Nash equilibrium.

For the second part of the theorem (uniqueness), note that the applicability of Lemma 1 can be expanded to the infinite population case. Furthermore, in analogy with Lemma 2, the mapping $z\mapsto(\bar{E}^{\mathrm{d}*}(z),\bar{E}^{\mathrm{r}*}(z))$ is continuous and contractive. Hence, following the contraction mapping theorem [11, Theorem 1.2.2], the mapping $z\mapsto(\bar{E}^{\mathrm{d}*}(z),\bar{E}^{\mathrm{r}*}(z))$ has a unique fixed point that is the Nash equilibrium with respect to (5) when the population size tends to infinity. ■

Algorithm 1: Picard-Banach iteration

Select $\varepsilon>0$ and $z^{(1)}$; set $\epsilon>\varepsilon$ and $l=1$;
while $\epsilon>\varepsilon$ **do**
  $z^{(l+1)}:=\big(\bar{E}^{\mathrm{d},n*}(z^{(l)}),\bar{E}^{\mathrm{r},n*}(z^{(l)})\big)$;
  $\epsilon:=\|z^{(l+1)}-z^{(l)}\|_1$; $l:=l+1$;
end

Following Theorem 2, when the population size tends to infinity, the Nash equilibrium associated with (5) can be obtained from (4) exploiting the fixed point of the mapping $(\bar{E}^{\mathrm{d}*},\bar{E}^{\mathrm{r}*})$ as reference z. If Assumption 2 holds, this mapping is continuous and contractive (the proof is the same as the proof of Lemma 2). Folllowing [7]–[9], we consider Picard-Banach iteration, Algorithm 1, for computing the unique Nash equilibrium with respect to (5). Then, in analogy with [12, Theorem 2.1], we show in the following proposition the convergence of Algorithm 1.

Proposition 1: **(Convergence of Algorithm 1).** Let Assumptions 1 and 2 hold. Then, the convergence of Algorithm 1 to the unique fixed point of the mapping $(\bar{E}^{\mathrm{d},n*},\bar{E}^{\mathrm{r},n*})$ is guaranteed for any initial condition $z^{(1)}$. For the infinite population case, this point is the unique Nash equilibrium associated with (5).

Proof: If Assumption 2 holds, the mapping $(\bar{E}^{\mathrm{d},n*},\bar{E}^{\mathrm{r},n*})$ (as well as $(\bar{E}^{\mathrm{d}*},\bar{E}^{\mathrm{r}*})$) is continuous and contractive. Thus, following [12, Theorem 2.1], Picard-Banach iteration (Algorithm 1) can compute the unique fixed point of such mapping. In analogy with Theorem 2, when $n\to\infty$, the unique fixed point of this mapping is the unique Nash equilibrium associated with (5). ■

We note that Assumption 2 is a sufficient condition and for sufficiently small σ, algorithm 1 still converges in simulation. Even under Assumption 2, σ is often small in practice.

IV. Simulation Results

In this section, the performance of the proposed method is verified by simulation in Python 3.9.13. We use the total inflexible load profile given in [8, Figure 1]. The time horizon covers the 12-hour period and we use the supply price functions $f^{\mathrm{d}}(x)=0.15x^{1.5}$ \$/kWh and

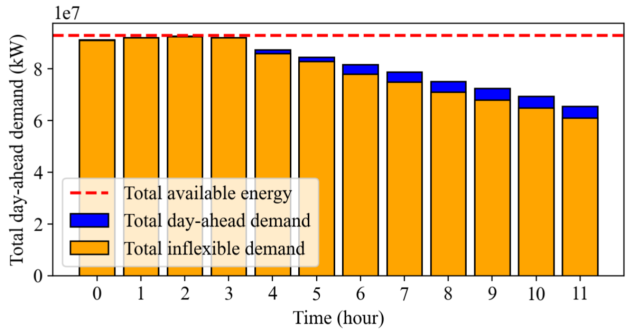

Fig. 1. The consumption schedules of energy consumers when they cannot anticipate the redispatch market.

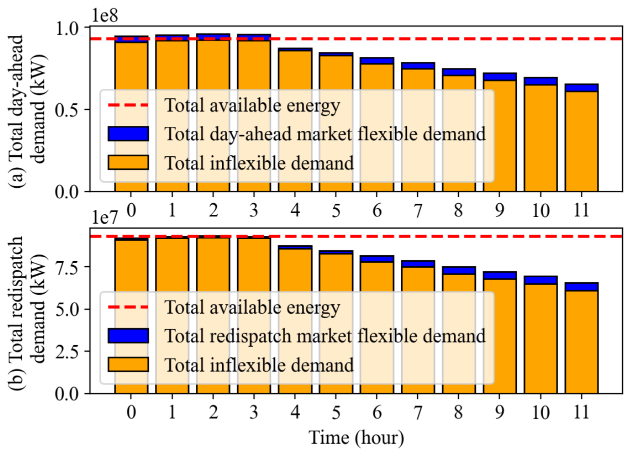

Fig. 2. The consumption schedules of energy consumers when they can anticipate the redispatch market: (a) total day-ahead market demand and (b) total demand after redispatch market.

$f^{\mathrm{r}}(x) = 0.18x^{1.5}$ \$/kWh from [7]–[9]. Also, we select $\sigma = 10^{-5}$ \$/(kWh)^2 and the utility and the maximum consumption of each agent are chosen randomly from the uniform distributions on the intervals $[0.04, 0.15]$ \$/kWh and $[3 \times 10^4, 10^5]$ kWh, respectively. The other parameters are identical to those used in [7]–[9].

Now, we consider Algorithm 1 for this simulation setup. Fig. 1 shows that the congestion does not occur when the consumers cannot anticipate the redispatch market. However, Fig. 2(a) demonstrates that the congestion occurs at times $t = 0, 1, 2, 3$ when the consumers can anticipate the redispatch market. Indeed, the consumers bid high consumption schedules on the day-ahead market at the times when they anticipate the congestion may occur to make profits on the redispatch market. We can observe from Fig. 2(b) that DSO asks the consumers to reduce their consumptions. By comparing Fig. 2 with Fig. 1, we can see that the congestion problem is aggravated when the consumers can anticipate the redispatch market. Moreover, we can notice from Fig. 3 that consumers make more profits from the DSO when they can anticipate the redispatch market in comparison with the case that they cannot. Thus, the increase-decrease game aggravates the congestion and allows the consumers to

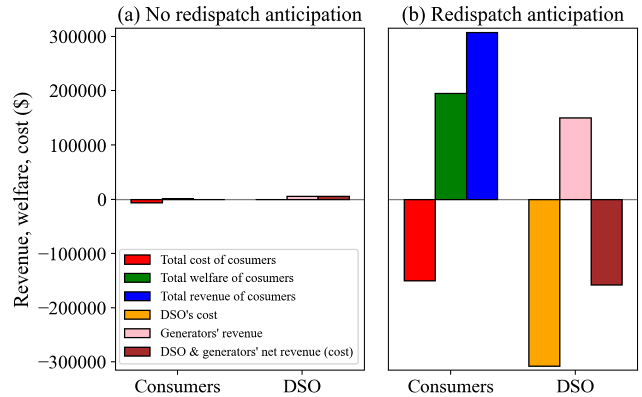

Fig. 3. The revenue, welfare and cost at time $t = 1$: (a) energy consumers cannot anticipate the redispatch market and (b) energy consumers can anticipate the redispatch market.

extract windfall profits from the DSO.

V. Conclusions and Future Work

In this paper, we have modeled the increase-decrease game for large populations of energy consumers using deterministic mean field game. We have shown that a Nash equilibrium exists for this game and used an algorithm that converges to this Nash equilibrium for the infinite population case. Future works include modeling and analyzing the increase-decrease game using stochastic mean field game.

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