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Lyapunov-based avoidance controllers with stabilizing feedback

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***Abstract*— For control-affine nonlinear systems, we augment a predefined Lyapunov-based global stabilizer with a hybrid obstacle avoidance design preserving the Lyapunov decrease. While the method can be applied to the general class of control-affine systems, the size of the avoidance neighborhood is not a design parameter. Our design shows that a system can achieve global asymptotic stability with simultaneous unsafe set avoidance via hybrid feedback, which overcomes well-known issues of topological obstructions.**

***Index Terms*— Lyapunov methods, constrained control, stability of hybrid systems**

I. INTRODUCTION

In recent years, the classical Lyapunov-based stabilization problem for continuous-time dynamical systems has been endowed with additional constraints on forward invariance of certain safe (or viable) sets, by augmenting classical Lyapunov functions with barrier certificates (see, e.g., [1], [2], [12], [15], [16] and references therein). Barrier certificates, or barrier functions, ensure that certain unsafe or undesirable sets are never reached by the solutions, as long as the system is initialized outside those sets.

Combining (Lyapunov-based) stabilizing feedback and (barrier-based) avoidance laws ensuring safety is a challenging problem, as well clarified in [4], [5], where it is emphasized that discontinuous or hybrid feedback is generally required (see also [14, Fig. 7]). Classical and historical solutions, such as artificial potential fields [11], are limited to dynamics where setting the input to zero one can stop the state (the so-called driftless case). Obstacle avoidance with drift is more challenging. In fact, even for the linear case, global stabilizers with point avoidance are nontrivial and approaches necessarily depend on the position of the obstacle and on corresponding controllability/stabilizability assumptions (see the counterexamples in [7, Sec. 2]). Our recent works [6], [7], focusing on linear system dynamics, provide solutions when the obstacle to be avoided does not belong to the set of induced equilibria, whereas [3] provides a partial solution when the obstacle sits on an induced equilibrium. Nonlinear generalizations of these techniques do not provide general constructive techniques. For example, they are given as extensions of the driftless case developed in [17] with certain compatibility conditions, or they are just sketched in [8] without any stability proof.

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In this paper we exploit the "avoidance shell" construction proposed in [7] for the linear case and show that it can be exploited for stabilization with avoidance tasks also in the presence of nonlinear dynamics. The design that we propose is different from the one in [6], [7] as it exploits the gradient of the Lyapunov function to prevent any increase of the Lyapunov function during the avoidance maneuver. As a consequence, multiple obstacles, arbitrarily close to each other can be avoided, while preserving the stabilization task certified by a prescribed Lyapunov function. The downside of the proposed construction is that the size of the obstacles can not be specified as a parameter, making the proposed redesign in principle not suitable to address problems such as collision avoidance among robots. In light of these considerations, the main goal of this work is to propose sufficient conditions under which the general problem of simultaneous avoidance and stabilization can be solved with an appropriate construction of the avoidance controller. Section II defines the problem, Section III presents the novel avoidance mechanism, Section IV presents the hybrid automaton ensuring stabilization and avoidance, together with our main result, and a numerical example is discussed in Section V.

Notation: For $x, y \in \mathbb{R}^n$ we use the vector norm $|x| = \sqrt{\sum_{i=1}^n x_i^2}$. and $|x|_y = |x - y|$. For $\mathcal{A} \subset \mathbb{R}^n$ closed and $r > 0$ we define $\mathcal{B}_r(\mathcal{A}) = \{x \in \mathbb{R}^n | \min_{y \in \mathcal{A}} |x - y| \leq r\}$ and for $\mathcal{A} = \{0\}$ we use $\mathcal{B}_r = \mathcal{B}_r(0)$. The closure, the boundary and the interior of a set are denoted by $\overline{\mathcal{A}}$, $\partial\mathcal{A}$ and $\text{int}(\mathcal{A})$, respectively. For $1, \beta \in \mathbb{N}$, we use the notations $\mathbb{N}_\beta = \{1, \ldots, \beta\}$ and $\mathbb{Z}_\beta = \{-\beta, \ldots, 0, \ldots, \beta\}$. The class of positive definite functions is defined as $\mathcal{P} = \{\rho : \mathbb{R}^n \to \mathbb{R}_{\geq 0} | \ \rho \text{ continuous}, \ \rho(0) = 0, \ \rho(x) > 0 \ \forall \ x \neq 0\}$.

II. SETTING & PROBLEM FORMULATION

In this paper we consider control-affine dynamical systems

$$\dot{x} = f(x) + g(x)u, \qquad x_0 = x(0) \in \mathbb{R}^n \tag{1}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, non-trivial drift term $f : \mathbb{R}^n \to \mathbb{R}^n$, and $g : \mathbb{R}^n \to \mathbb{R}^{n \times m}$. We require that f and g be locally Lipschitz. Additionally, we use the notation $g_i : \mathbb{R}^n \to \mathbb{R}^m$, $i \in \mathbb{N}_m$, to refer to the individual columns of g, and write $g = [g_1(x) \cdots g_m(x)]$. Here, and throughout the paper, by a non-trivial drift term we mean that $f(x)$ cannot be cancelled via a feedback transformation $u = -\kappa(x) + v$ such that $f(x) - g(x)\kappa(x) = 0$ for all $x \in \mathbb{R}^n$. As stated in the introduction, the paper discusses a solution to the following problem, translating the results in [7] from the linear setting to the nonlinear setting in (1).

Problem 1: (Semiglobal $\hat{x}$-avoidance augmentation with GAS) Given an open set $\mathcal{U} \subset \mathbb{R}^n$, $0 \in \mathcal{U}$, a set of obstacle centroids $\{\hat{x}_1, \ldots, \hat{x}_\beta\} \in \mathcal{U} \setminus \{0\}$, $\beta \in \mathbb{N}$, that define spherical obstacles and a Lipschitz continuous stabilizing state feedback $u_{\rm s}(x) = \kappa_{\rm s}(x)$, for a sufficiently small $\delta > 0$, design a feedback selection of u that guarantees

(i) (Semiglobal preservation) the feedback $u(x)$ matches the original stabilizer $u(x) = \kappa_{\rm s}(x)$ for all $x \in \mathcal{U} \setminus \cup_{i=1}^{\beta} \mathcal{B}_\delta(\hat{x}_i)$;

(ii) (Semiglobal $\hat{x}$-avoidance) all solutions starting outside the balls $\cup_{i=1}^{\beta} \mathcal{B}_\delta(\hat{x}_i)$ never enter a suitable avoidance neighborhood $\mathcal{B}_{\chi_i}(\hat{x}_i)$, having measure greater than zero, around the centroids $\hat{x}_i$, $i \in \mathbb{N}_\beta$;

(iii) (GAS) uniform global asymptotic stability of the origin relative to $\mathcal{U}$, i.e., the origin is Lyapunov stable for the dynamics and solutions starting in $\mathcal{U}$ (including those starting at $\hat{x}_i$, $i \in \mathbb{N}_\beta$), converge uniformly to zero. ⌟

We point out that the results in this paper constitute theoretical results showing *global stabilizability with local avoidability properties* of general control affine systems where the avoidance neighborhood cannot be specified a priori. This is in contrast to papers whose controller designs are based on specific dynamics and specific obstacles and in contrast to papers that do not address deadlocks and accordingly can only guarantee local asymptotic stability or, at best, convergence from any arbitrary initial condition excluding a set of measure zero. Due to the nonlinear drift term, it is in general not possible to avoid obstacles of arbitrary size (see [7, Fig. 1]). For simplicity we focus on nominal system dynamics in (1) and Problem 1. Using similar techniques as in the linear setting in [7], the results can be extended to *robust stability* and *robust avoidance*.

To simplify the notation in the following, we use

$$f_{\rm s}(x) = f(x) + g(x)\kappa_{\rm s}(x) \tag{2}$$

to denote the right-hand side of the closed-loop system resulting from a prescribed stabilizing controller, whose existence is assumed in Problem 1. To solve Problem 1 we make the following standing assumption.

Assumption 1: Consider the dynamics (1) and obstacle centroids $\hat{x}_1, \ldots, \hat{x}_\beta \in \mathcal{U} \setminus \{0\}$, $\beta \in \mathbb{N}$, with $\mathcal{U}$ defined as in Problem 1.

(a) The function $V : \mathcal{D} \to \mathbb{R}_{\geq 0}$, $\mathcal{D} \subseteq \mathbb{R}^n$, is a continuously differentiable control Lyapunov function (radially unbounded for $\mathcal{D} = \mathbb{R}^n$), $\mathcal{U} \subset \mathcal{D}$ is a sub-level set of the Lyapunov function, and $\kappa_{\rm s} : \mathcal{U} \to \mathbb{R}^m$ is a corresponding stabilizing feedback law, i.e., there exists $\rho \in \mathcal{P}$ such that

$$\langle \nabla V(x), f(x) + g(x)\kappa_{\rm s}(x)\rangle \leq -\rho(x), \ \forall\, x \in \mathcal{U}. \tag{3}$$

(b) For every $i \in \mathbb{N}_\beta$, there exists $v \in \mathbb{R}^m$ such that

$$\nabla V(\hat{x}_i)^\top g(\hat{x}_i) v(\hat{x}_i) = 0 \text{ and } g(\hat{x}_i) v(\hat{x}_i) \neq 0 \tag{4}$$

are satisfied. ⌟

Assumption 1(a) simply states that the feedback law $u_{\rm s} = \kappa_{\rm s}(x)$ stabilizes the origin for system (1), without obstacles. Assumption 1(b) states that in a neighborhood around each $\hat{x}_i$, it is possible to select an input perpendicular to the gradient of the Lyapunov function. This degree of freedom will be used to design an avoidance controller. Assumption 1(b) holds if, for each $\hat{x}_i, i \in \mathbb{N}_\beta$, there exist $k, j \in \mathbb{N}_m$ such that $g_k(\hat{x}_i)$ and $g_j(\hat{x}_i)$ are linearly independent. This is for example satisfied if $m \geq 2$ and $g(\cdot)$ is constant.

III. Avoidance controller design

A. Lyapunov decrease property

Consider a Lyapunov function $V : \mathcal{D} \to \mathbb{R}_{\geq 0}$ and the stabilizing feedback law $\kappa_s : \mathcal{U} \to \mathbb{R}^m$ according to Assumption 1(a). To guarantee obstacle avoidance, we will locally modify the nominal control law $u(x) = \kappa_s(x)$ near each obstacle centroid $\hat{x}_i, i \in \mathbb{N}_\beta$ in the following way

$$u(x) = \kappa_s(x) + \alpha_p(x)\Lambda(\hat{x}_i), \quad p \in \{-1, 1\} \tag{5}$$

where $\Lambda(\hat{x}_i) \in \mathbb{R}^m$ defines an avoidance direction depending on the obstacle and $\alpha_1, \alpha_{-1} : \mathcal{B}_r(\hat{x}_i) \to \mathbb{R}$ are continuous scaling functions defined in a neighborhood of $\hat{x}_i$ having radius $r > 0$. The scaling can be positive as well as negative, which is determined by the parameter $p \in \{-1, 1\}$. The functions α_1, α_{-1}, as well as the direction $\Lambda(\hat{x}_i)$, are determined in the following sections to guarantee specific avoidance properties. In the following we use the notation

$$v_p^i(x) = \alpha_p(x)\Lambda(\hat{x}_i), \tag{6}$$

to capture the avoidance component of the feedback law (5), based on the centroid $i \in \mathbb{N}_\beta$ and the scaling $p \in \{-1, 1\}$.

The directions $\Lambda(\hat{x}_i)$ in (6) will be defined in Section III-B so that the conditions (4) hold for all $i \in \mathbb{N}_\beta$. Then, along solutions of (1), (5), for all $i \in \mathbb{N}_\beta$ one has

$$\begin{aligned}&\langle \nabla V(\hat{x}_i), f(\hat{x}_i) + g(\hat{x}_i)(\kappa_s(\hat{x}_i) + v_p^i(\hat{x}_i))\rangle \\ &\quad = \langle \nabla V(\hat{x}_i), f(\hat{x}_i) + g(\hat{x}_i)\kappa_s(\hat{x}_i)\rangle \leq -\rho(\hat{x}_i),\end{aligned} \tag{7}$$

with ρ defined in Assumption 1(a). Since, ∇V, f, g and κ_s are continuous by assumption, there exist $\eta_i \in \mathbb{R}_{>0}$ such that

$$\begin{aligned}&\langle \nabla V(x), f(x) + g(x)(\kappa_s(x) + v_p^i(\hat{x}_i))\rangle \\ &= \langle \nabla V(x), f(x) + g(x)\kappa_s(x)\rangle + \langle \nabla V(x), g(x) v_p^i(x)\rangle \\ &= -\rho(x) + \alpha_p(x)\langle \nabla V(x), g(x)\Lambda(\hat{x}_i)\rangle < 0,\end{aligned} \tag{8}$$

is satisfied for all $x \in \mathcal{B}_{\eta_i}(\hat{x}_i)$. While the size of the η_i-neighborhood depends on the selection of the function α_p, for every continuous function α_p, a corresponding $\eta_i > 0$ with the decrease property in (8) exists.

For this reason, we choose to carry out the design of the avoidance control input $v_p^i(x)$ without accounting for the size of the avoidance neighborhood. Once $\alpha_p(x)$ and $\Lambda(\hat{x}_i)$ have been determined, the largest admissible value of η_i can be determined, for example, as the solution of the optimization problem

$$\begin{aligned}\eta_i = \min_{x \in \mathbb{R}^n}\ & |x - \hat{x}_i| - \varepsilon, \\ \text{s.t. }\ & -\rho(x) + \alpha_p(x)\nabla V(x)^\top g(x)\Lambda(\hat{x}_i) \geq 0\end{aligned} \tag{9}$$

for each $\hat{x}_i, i \in \mathbb{N}_\beta$ and for an arbitrarily small $\varepsilon > 0$.

B. Avoidance shell and avoidance direction

The avoidance controller (6) is activated in a shell-shaped neighborhood around the obstacle. While the shell-shaped set follows the same definition as in the linear setting in [6, Sec. IV], the definition of the avoidance controller deviates from the linear setting. To effectively present the avoidance controller (6), we briefly recall the definitions of the shell-shaped avoidance neighborhood introduced in [6], to make the paper self-contained. Recall that an avoidance neighborhood with a non-smooth boundary is necessary to ensure that, despite the smoothness of f, g and u, for each point on the boundary of the avoidance neighborhood, there exists an input $u \in \mathbb{R}^m$ such that $\dot{x} = f(x) + g(x)u$ does not point inside the avoidance neighborhood.

Definition 1 (Avoidance shell, [6, Sec. IV]): Consider an obstacle centroid $\hat{x} \in \{\hat{x}_1, \ldots, \hat{x}_\beta\}$ together with parameters:

1) $\delta \in \mathbb{R}_{>0}$ (size of the shell);
2) $\mu \in (0, 2)$ (aspect ratio of the shell);
3) $b \in \mathbb{R}^n$ (orientation of the shell).

Moreover, let

$$\delta_\mu := \delta \left(\tfrac{1}{\mu} - \tfrac{\mu}{4} \right), \tag{10}$$

$$\mathcal{O}_p := \mathcal{B}_{\left(\frac{\mu\delta}{2} + \delta_\mu\right)}(\hat{x} - p\delta_\mu b), \quad p \in \{+1, -1\}, \tag{11}$$

$$\mathcal{S}(\delta) := \mathcal{O}_{+1} \bigcap \mathcal{O}_{-1}. \tag{12}$$

Then, (12) defines an eye-shaped set centered at $\hat{x}$, called *avoidance shell* in the following. ♠

Note that $\mu \in (0, 2)$ fixes the aspect ratio of the shell, whose height corresponds to $\mu\delta$. The set $\mathcal{S}(\delta)$ is visualized in Figure 1, which also shows the role of the shell parameters.

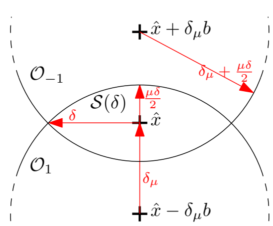

Fig. 1. Visualization of the avoidance shell defined in Definition 1.

Before we proceed with the construction of the avoidance controller we recall the following result, proven in [7].

Lemma 1: [Avoidance shell set-inclusion; [7, Lemma 1]] Given an aspect ratio $\mu \in (0, 2)$ and an orientation $b \in \mathbb{R}^n$, $|b| = 1$, for each $\delta > 0$, the following inclusions hold for the shell $\mathcal{S}(\delta)$ defined in (12):

$$\mathcal{B}_{\frac{\mu\delta}{2}}(\hat{x}) \subset \mathcal{S}(\delta) \subset \mathcal{B}_\delta(\hat{x}). \tag{13}$$

By selecting $\delta = \eta_i$ (where η_i is defined in (9)) in $\mathcal{S}(\delta) \subset \mathcal{B}_\delta(x)$ allows us to focus on $\mathcal{S}(\delta) \subset \mathcal{B}_{\eta_i}(\hat{x})$ in the avoidance controller design.

In the avoidance controller design, we first choose the orientation $b_i = b(\hat{x}_i)$ of the avoidance shell based on the obstacle centroid of interest. First, we define the directions

$$D(\hat{x}_i) := \tfrac{\nabla V(\hat{x}_i)}{|\nabla V(\hat{x}_i)|}, \; i \in \mathbb{N}_\beta. \tag{14}$$

Then, for each obstacle centroid $\hat{x}_i, i \in \mathbb{N}_\beta$, it is desirable to select $b(\hat{x}_i)$ as the direction perpendicular to $D(\hat{x}_i)$ in (14) that maximizes the inner product $b(\hat{x}_i)^\top g(\hat{x}_i)\Lambda(\hat{x}_i)$, i.e., $b(\hat{x}_i)$ is the vector orthogonal to $D(\hat{x}_i)$ maximally aligned with the subspace spanned by the columns of $g(\hat{x}_i)$. To guarantee that the avoidance control input defined in (6) satisfies (4) for each $\hat{x}_i$, $i \in \mathbb{N}_\beta$, we restrict the choice of $\Lambda(\hat{x}_i)$ to non-zero vectors satisfying $D(\hat{x}_i)^\top g(\hat{x}_i)\Lambda(\hat{x}_i) = 0$. For each $\hat{x}_i, i \in \mathbb{N}_\beta$, we define the kernel of $D(\hat{x}_i)^\top g(\hat{x}_i)$, i.e., we define

$$V_\perp(\hat{x}_i) = [v_1 \; \cdots \; v_p] \in \mathbb{R}^{m \times p}, \tag{15}$$

such that $D(\hat{x}_i)^\top g(\hat{x}_i)V_\perp(\hat{x}_i) = 0$ and where $v_1, .., v_p$, $p \in \mathbb{N}$, denote orthonormal basis vectors of the kernel. Assumption 1(b) guarantees that $p \geq 1$. Then we select

$$\Lambda(\hat{x}_i) = V_\perp(\hat{x}_i)\bar{\Lambda}(\hat{x}_i), \tag{16}$$

where the vector $\bar{\Lambda}(\hat{x}_i) \in \mathbb{R}^p$ defines a linear combination of the columns of $V_\perp(\hat{x}_i)$. Based on these definitions, we are interested in solving the following optimization problem

$$\begin{aligned} (b(\hat{x}_i), \bar{\Lambda}(\hat{x}_i)) \in \underset{\mathfrak{b} \in \mathbb{R}^n, \Lambda \in \mathbb{R}^p}{\operatorname{argmax}} \; & \mathfrak{b}^\top g(\hat{x}_i)V_\perp(\hat{x}_i)\Lambda, \\ \text{s.t. } & \mathfrak{b}^\top D(\hat{x}_i) = 0, |\mathfrak{b}| = 1, \quad |\Lambda| = 1, \end{aligned} \tag{17}$$

for each $\hat{x}_i, i \in \mathbb{N}_\beta$. It follows from the definition of the optimization problem that $(b(\hat{x}_i), \bar{\Lambda}(\hat{x}_i))$ is optimal if and only if $(-b(\hat{x}_i), -\bar{\Lambda}(\hat{x}_i))$ is optimal. Even though the optimization problem (17) is non-convex and solutions are not unique, it can be solved explicitly.

Lemma 2: Consider the dynamics (1), let Assumption 1 be satisfied and $D(\hat{x}_i)$, $V_\perp(\hat{x}_i)$ be defined as in (14), (15) respectively. Let $\lambda_M(\hat{x}_i)$ be a normalized eigenvector of the symmetric matrix $(g(\hat{x}_i)V_\perp(\hat{x}_i))^\top g(\hat{x}_i)V_\perp(\hat{x}_i)$ corresponding to the largest eigenvalue, denoted by $\sigma_M(\hat{x}_i)$. Then it holds that $\sigma_M(\hat{x}_i) > 0$, a solution to (17) is given by

$$(b(\hat{x}_i), \bar{\Lambda}(\hat{x}_i)) = \left(\tfrac{g(\hat{x}_i)V_\perp(\hat{x}_i)\lambda_M(\hat{x}_i)}{\sqrt{\sigma_M(\hat{x}_i)}}, \lambda_M(\hat{x}_i) \right) \tag{18}$$

and $D(\hat{x}_i)^\top g(\hat{x}_i)V_\perp(\hat{x}_i)\bar{\Lambda}(\hat{x}_i) = 0$ for each $\hat{x}_i, i \in \mathbb{N}_\beta$. ⌟

Proof: For ease of notation, we drop the dependence on $\hat{x}_i$ in the proof. Since in (17), b and $gV_\perp\bar{\Lambda}$ belong to the hyperplane orthogonal to D (due to the definition of $V_\perp$ in (15)), the maximum of the inner product $b^\top gV_\perp\bar{\Lambda}$ is attained when b is aligned with $gV_\perp\bar{\Lambda}$, and thus $b = \frac{gV_\perp\bar{\Lambda}}{|gV_\perp\bar{\Lambda}|}$. Substituting this selection of b into the objective function of (17) gives

$$\tfrac{(gV_\perp\bar{\Lambda})^\top gV_\perp\bar{\Lambda}}{|gV_\perp\bar{\Lambda}|} = |gV_\perp\bar{\Lambda}|. \tag{19}$$

Since $\bar{\Lambda}$, with $|\bar{\Lambda}| = 1$, maximizes (19) if and only if it maximizes $|gV_\perp\bar{\Lambda}|^2$, we now focus on the maximization of the function

$$|gV_\perp\bar{\Lambda}|^2 = \bar{\Lambda}^\top V_\perp^\top g^\top gV_\perp\bar{\Lambda}. \tag{20}$$

Let us consider the matrix $M := V_\perp^\top g^\top gV_\perp$. Since M is a symmetric positive semi-definite matrix, substituting the definition of M into (20) one has the following upper bound $|gV_\perp\bar{\Lambda}|^2 = \bar{\Lambda}^\top M\bar{\Lambda} \leq \sigma_M|\bar{\Lambda}|^2 = \sigma_M$ for all $\bar{\Lambda} \in \mathbb{R}^m$, where $|\bar{\Lambda}| = 1$ and σ_M denotes the largest eigenvalue of M, i.e., $M\lambda_M = V_\perp^\top g^\top gV_\perp\lambda_M = \sigma_M\lambda_M$. Due to Assumption 1(b), $gV_\perp \neq 0$ and thus $\mathrm{rank}(gV_\perp) > 0$. Using the properties of the $\mathrm{rank}(\cdot)$ operator, one has $\mathrm{rank}(M) = \mathrm{rank}(V_\perp^\top g^\top gV_\perp) = \mathrm{rank}((gV_\perp)^\top gV_\perp) = \mathrm{rank}(gV_\perp)$, which implies that $\mathrm{rank}(M) = \mathrm{rank}(gV_\perp) > 0$ for all $\hat{x}_i, i \in \mathbb{N}_\beta$. Then, $\sigma_M > 0$ and substituting $\bar{\Lambda} = \lambda_M$ into (20) gives $|gV_\perp\lambda_M|^2 = \lambda_M^\top V_\perp^\top g^\top gV_\perp\lambda_M = \sigma_M$, implying that the maximum of (19), which coincides with the maximum of (17), is $|gV_\perp\lambda_M| = \sqrt{\sigma_M}$. Vector b can therefore be written as $b = \frac{gV_\perp\lambda_M}{|gV_\perp\lambda_M|} = \frac{gV_\perp\lambda_M}{\sqrt{\sigma_M}}$ (giving (18)) and $D^\top gV_\perp\bar{\Lambda} = 0$ follows from the definition of $V_\perp$ in (15). ■

C. Avoidance controller

For each obstacle centroid $\hat{x}_i, i \in \mathbb{N}_\beta$, Lemma 2 provides vectors $b(\hat{x}_i)$ and $\bar{\Lambda}(\hat{x}_i)$ defining the orientation of the avoidance shell in (12) and the direction of the avoidance controller in (6), (16), respectively. With $\bar{\Lambda}(\hat{x}_i)$ defined as in Lemma 2, $v_p^i(\hat{x}_i)$ defined in (6), (16) satisfies (4).

In this section we focus on the derivation of $\alpha_p(x)$ for $p \in \{-1, 1\}$, which is the missing component in the definition of $v_p^i(x)$. To this end, let

$$c_p := \hat{x} - p\delta_\mu b, \qquad p \in \{-1, +1\} \tag{21}$$

denote the centers of the balls $\mathcal{O}_p$, $p \in \{-1, +1\}$ in (11) for a generic obstacle centroid $\hat{x} \in \{\hat{x}_1, \ldots, \hat{x}_\beta\}$ and corresponding $b(\hat{x})$ defined in (18).

The avoidance controller is defined so that

$$\tfrac{d}{dt}|x(t) - c_p|^2 = 0, \qquad p \in \{-1, 1\}, \tag{22}$$

holds for any $x \in \mathcal{S}(\delta)$, ensuring that the distance between the state $x(t)$ and the point c_p does not decrease when performing an avoidance maneuver. Imposing (22) will give us the definition of $\alpha_p(\hat{x}_i)$ in (6). We now show that by selecting $\bar{\Lambda}(\hat{x}_i)$ and $b(\hat{x}_i)$ according to Lemma 2 there always exists a sufficiently small neighborhood for each obstacle centroid inside of which condition (22) holds.

Lemma 3: Consider the dynamics (1), (5). Let Assumption 1 be satisfied, $\hat{x}_i, i \in \mathbb{N}_\beta$ be an obstacle centroid, $b(\hat{x}_i), \sigma_M(\hat{x}_i)$, $\bar{\Lambda}(\hat{x}_i)$ be defined as in Lemma 2, $V_\perp(\hat{x}_i)$ and $\Lambda(\hat{x}_i)$ be defined as in (15) and (16). Then, for all $\bar{\mu}_i \in (0, 2)$ there exists $\bar{\delta}_i \in \mathbb{R}_{>0}$ defining the shell $\mathcal{S}(\bar{\delta}_i)$ such that the avoidance control input (6) with

$$\alpha_p(x) = \frac{-f_s(x)^\top(x - \hat{x}_i + p\delta_\mu b(\hat{x}_i))}{(g(x)\Lambda(\hat{x}_i))^\top(x - \hat{x}_i + p\delta_\mu b(\hat{x}_i))}, \tag{23}$$

is well defined and satisfies (22) for all $x \in \mathcal{S}(\bar{\delta}_i)$. ⌟

Proof: We start by rewriting (22), substituting (1), the closed-loop dynamics (2), and the control input selection $u_a(x) = \kappa_s(x) + \alpha_p(x)\Lambda(\hat{x}_i)$, with $\Lambda(\hat{x}_i)$ defined as in (16),

$$\begin{aligned}&\tfrac{d}{dt}|x(t) - c_p|^2 = \dot{x}^\top(x - c_p) = (f(x) + g(x)u_a)^\top(x - c_p)\\ &= f_s(x)^\top(x - \hat{x}_i + p\delta_\mu b(\hat{x}_i))\\ &\quad + \alpha_p(x)(g(x)\Lambda(\hat{x}_i))^\top(x - \hat{x}_i + p\delta_\mu b(\hat{x}_i)).\end{aligned} \tag{24}$$

Imposing (24) to be zero results in the condition on α_p (23). We now show that for each $\bar{\mu}_i \in (0, 2)$ there exists $\bar{\delta}_i > 0$ such that the denominator of (23) is non-zero for all $x \in \mathcal{S}(\bar{\delta}_i)$. Observe that for $x = \hat{x}_i$ the denominator of (23) simplifies to $(g(\hat{x}_i)\Lambda(\hat{x}_i))^\top p\delta_\mu b(\hat{x}_i) = p\delta_\mu\sqrt{\sigma_M(\hat{x}_i)}$, which is unequal to zero for all $\delta > 0, \mu \in (0, 2)$. Moreover, for each $\delta > 0$, Lipschitz continuity of g implies the existence of $L_g > 0$ such that $|g(x)\Lambda(\hat{x}_i) - g(\hat{x}_i)\Lambda(\hat{x}_i)| \leq L_g|x - \hat{x}_i|$ for all $x \in \mathcal{B}_\delta(\hat{x}_i)$, which in turn implies $g(x)\Lambda(\hat{x}_i) \in g(\hat{x}_i)\Lambda(\hat{x}_i) + \mathcal{B}_{L_g\delta}$ for all $x \in \mathcal{B}_\delta(\hat{x}_i)$. Let us now prove that there always exist $\hat{\delta}, \hat{\mu} > 0$ sufficiently small such that the denominator of (23) is non-zero for all $x \in \mathcal{S}(\hat{\delta})$, where $\mathcal{S}(\hat{\delta})$ is the avoidance shell defined by $\hat{\delta}, \hat{\mu}$ according to (12). Using the inclusion derived above for $g(x)\Lambda(\hat{x}_i)$ and the fact that $|b(\hat{x}_i)| = 1$ and $|g(\hat{x}_i)\Lambda(\hat{x}_i)| = \sqrt{\sigma_M(\hat{x}_i)}$, one obtains that the lower bound on the denominator of (23)

$$\begin{aligned}&|(g(x)\Lambda(\hat{x}_i))^\top(x - \hat{x}_i + pb(\hat{x}_i)\delta_\mu)|\\ &\geq -|(g(\hat{x}_i)\Lambda(\hat{x}_i))^\top(x - \hat{x}_i)| - L_g\delta^2\\ &\quad + |(g(\hat{x}_i)\Lambda(\hat{x}_i))^\top p\delta_\mu b(\hat{x}_i)| - L_g\delta\delta_\mu\\ &\geq \sqrt{\sigma_M(x)}(\delta_\mu - \delta) - L_g\delta^2 - L_g\delta\delta_\mu\end{aligned} \tag{25}$$

holds for all $x \in \mathcal{B}_\delta(\hat{x}_i)$. One can verify that, for every $\mu \in (0, 2(\sqrt{2}-1))$, $\delta_\mu - \delta$ is positive for every $\delta > 0$. Then, given $\hat{\mu} \in (0, 2(\sqrt{2} - 1))$, it is possible to show that there always exists a sufficiently small $\hat{\delta} > 0$ such that (25) is positive. Define $\bar{\delta}_i := \frac{\hat{\mu}\hat{\delta}}{2}$. From (13) one has $\mathcal{B}_{\bar{\delta}_i}(\hat{x}_i) \subset \mathcal{S}(\hat{\delta})$ and $\mathcal{S}(\bar{\delta}_i) \subset \mathcal{B}_{\bar{\delta}_i}(\hat{x}_i)$, which show that for every $\bar{\mu}_i \in (0, 2)$ there exists a sufficiently small $\bar{\delta}_i > 0$ such that the denominator of (23) is non-zero for all $x \in \mathcal{S}(\bar{\delta}_i)$. ■

Lemma 3 guarantees the existence of $\bar{\delta}_i > 0$ such that $\alpha_p(x)$ is well-defined for all $x \in \mathcal{B}_{\bar{\delta}_i}(\hat{x}_i)$. Unfortunately, to be able to use $\alpha_p(x)$ in the avoidance controller design, η_i defined through (9) needs to satisfy $\eta_i \geq \bar{\delta}_i$, where (9) depends on the selection of $\bar{\delta}_i$ and $\bar{\mu}_i$. To avoid the need to compute η_i, we consider the optimization problem

$$\bar{\delta}_{i,p} = \min_{x \in \mathbb{R}^n \, \delta \geq 0} \delta, \tag{26}$$

$$\text{s.t.} \ -\rho(x) + \alpha_p(x)\nabla V(x)^\top g(x)\Lambda(\hat{x}_i) \geq 0 \tag{27}$$

$$|x - c_p| \leq \delta_\mu + \tfrac{\mu\delta}{2}, \ p(x - \hat{x}_i)^\top b_i(\hat{x}_i) \geq 0 \tag{28}$$

for $p \in \{-1, 1\}$ and a fixed aspect ratio $\mu \in (0, 2)$. Here, $\bar{\delta}_{i,p}$ defines the largest δ such that the decrease condition (8) is not satisfied for all x in the half shell $x \in \mathcal{S}(\delta) \cap \{x \in \mathbb{R}^n : pb_i(\hat{x})^\top(x - \hat{x}_i) \geq 0\}$. Accordingly, the decrease condition (8) and the properties of Lemma 3 are satisfied for

$$\bar{\delta}_i \in (0, \min\{\bar{\delta}_{i,1}, \bar{\delta}_{i,-1}\}). \tag{29}$$

Remark 1: Since the half shell in (28) is bounded, an approximate solution of the optimization problem (26) can be computed numerically, by discretizing the half shell for different values of δ and by evaluating (27) over the set of discretized states.

Remark 2: As in [7] condition (22) can be changed to $\frac{d}{dt}|x(t) - c_p|^2 > 0$, $p \in \{-1, 1\}$, to obtain robust avoidance properties. Here, we restrict our attention to the nominal setting covered through Lemma 3.

IV. Hybrid controller design

In this section we follow the construction presented in [7, Sec. IV], based on the formalism of hybrid dynamical systems, to orchestrate the switching between the nominal control law and the modified control law (5). To keep the paper self-contained, we briefly recall the definitions given in [7, Sec. IV] of the inner avoidance shell and upper/lower parts of the avoidance shell.

Definition 2 ([6, Sec. IV]): Consider an obstacle centroid $\hat{x} \in \{\hat{x}_1, \dots, \hat{x}_\beta\}$ together with the parameter $h \in (0,1)$, called hysteresis factor, and let

$$\mathcal{O}_{h,p} := \mathcal{B}_{h\frac{\mu\delta}{2}+\delta_\mu}(c_p), \quad p \in \{-1,1\}, \tag{30}$$

$$\mathcal{S}_h(\delta) := \mathcal{O}_{h,+1} \cap \mathcal{O}_{h,-1}. \tag{31}$$

Then, (31) defines a smaller eye-shaped set centered around $\hat{x}_i$ called *inner avoidance shell* in the following, and

$$\mathcal{S}_p := \mathcal{S}(\delta) \cap \{x \in \mathbb{R}^n : pb^\top (x - \hat{x}) \geq 0\}, \; p \in \{-1,1\} \tag{32}$$

defines the lower and upper halves of the avoidance shell with respect to the orientation b. ♠

To account for multiple obstacles, we substitute $p \in \{-1,1\}$ used in the previous section with the logic variable $q \in \mathbb{Z}_\beta$ and write

$$c_q := \hat{x}_{|q|} - \tfrac{q}{|q|}\delta_{\mu_{|q|}} b_{|q|}, \quad q \in \mathbb{Z}_\beta \backslash \{0\}. \tag{33}$$

Moreover, when referring to an obstacle centroid $\hat{x}_i, i \in \mathbb{N}_\beta$, we will use the notation $\cdot^i, i \in \mathbb{N}_\beta$, to refer to quantities corresponding to the i-th obstacle centroid $\hat{x}_i, i \in \mathbb{N}_\beta$. Using q we define the overall hybrid state as $\xi := [x^\top \; q]^\top \in \Xi := \mathbb{R}^n \times \mathbb{Z}_\beta$, and write the new feedback law as

$$u(\xi) := \begin{cases} \kappa_s(x), & \text{if } q = 0, \\ \kappa_s(x) + v_a^{|q|}(x), & \text{if } q \in \mathbb{Z}_\beta \backslash \{0\}. \end{cases} \tag{34}$$

The jump dynamics, modeling the switching between nominal and modified control law, can be written as

$$\begin{aligned} \mathcal{D}_{+i} &:= \left(\mathcal{S}^i_{h_i}(\delta_i) \cap \mathcal{S}^i_{+1}\right) \times \{0\}, \quad i \in \mathbb{N}_\beta \\ \mathcal{D}_{-i} &:= \left(\mathcal{S}^i_{h_i}(\delta_i) \cap \mathcal{S}^i_{-1}\right) \times \{0\}, \quad i \in \mathbb{N}_\beta \\ \mathcal{D}_0 &:= \overline{\mathbb{R}^n \setminus \cup_{i \in \mathbb{N}_\beta} \mathcal{S}^i(\delta_i)} \times (\mathbb{Z}_\beta \setminus \{0\}) \\ q^+ \in G_q(x,q) &:= \begin{cases} i, & \text{if } (x,q) \in \mathcal{D}_{+i} \setminus \mathcal{D}_{-i}, \; i \in \mathbb{N}_\beta \\ -i, & \text{if } (x,q) \in \mathcal{D}_{-i} \setminus \mathcal{D}_{+i}, \; i \in \mathbb{N}_\beta \\ \{i,-i\}, & \text{if } (x,q) \in \mathcal{D}_{+i} \cap \mathcal{D}_{-i}, \; i \in \mathbb{N}_\beta \\ 0, & \text{if } (x,q) \in \mathcal{D}_0, \end{cases} \end{aligned} \tag{35}$$

where the flow and jump sets are compactly written as

$$\mathcal{C} := \overline{\Xi \setminus (\cup_{q \in \mathbb{Z}_\beta} \mathcal{D}_q)}, \quad \mathcal{D} := \cup_{q \in \mathbb{Z}_\beta} \mathcal{D}_q, \tag{36}$$

and the overall hybrid dynamical system can be written as

$$\dot{\xi} = \begin{bmatrix} \dot{x} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} f(x)+g(x)u(\xi) \\ 0 \end{bmatrix}, \qquad \xi \in \mathcal{C} \tag{37}$$

$$\xi^+ = \begin{bmatrix} x^+ \\ q^+ \end{bmatrix} \in \begin{bmatrix} x \\ G_q(x,q) \end{bmatrix}, \qquad \xi \in \mathcal{D}. \tag{38}$$

When the state $\xi = [x^\top \; q]^\top$ enters the jump set $\mathcal{D}$, the jump map (35) updates q while keeping x constant. The variable q is responsible for the selection of the nominal or modified control law, as it can be seen from (34). When the x component of the state ξ enters the inner avoidance shell $\mathcal{S}^i_h(\delta)$, q is set to either i or $-i$. When x leaves the avoidance shell $\mathcal{S}^i(\delta)$, q is reset to 0.

We are now ready to state the main result of this paper.

Theorem 1: Let Assumption 1 be satisfied and $h \in (0,1)$, $\mu_i \in (0,2)$, $i \in \mathbb{N}_\beta$, be arbitrary. Then, there exist $\delta_i > 0$, $i \in \mathbb{N}_\beta$, such that the hybrid system (36), (37), (38), solves Problem 1 for $\chi_i = \frac{h\mu_i\delta_i}{2}$. ⌟

Proof: For each obstacle centroid $\hat{x}_i, i \in \mathbb{N}_\beta$, consider δ_i defined from (26)-(29). From the inclusion (13) we have $\mathcal{S}^i(\delta_i) \subset \mathcal{B}_{\delta_i}(\hat{x}_i)$, and since the modified control law is only used inside the avoidance shell $\mathcal{S}^i(\delta_i)$ of an obstacle, this proves that the hybrid system satisfies item (i) of Problem 1 by taking $\delta = \max_{i \in \mathbb{N}_\beta} \delta_i$.

Let us now focus on item (ii) of Problem 1. Consider an arbitrary obstacle centroid $\hat{x}_i, i \in \mathbb{N}_\beta$, and let $(t,j) \mapsto \xi(t,j)$ be a solution of (36)-(38) such that $\xi(0,0) \notin \mathcal{D}$, which implies $x(0,0) \notin \cup_{i=1}^\beta \mathcal{B}_\delta(\hat{x}_i)$, with δ defined as above. Assume that there exists a time $(t_0, j_0) \in \text{dom } \xi$ at which the solution reaches the inner avoidance shell of the obstacle $\hat{x}_i$, $\xi(t_0,j_0) \in \mathcal{D}_i \cup \mathcal{D}_{-i}$. Then, the solution will jump according to the jump map (35), with q^+ set to either $-i$ or i, and, according to (34), the control law $u(\xi) = \kappa_s(x) + v_a^i(x)$ will be selected to perform the avoidance maneuver. Lemma 3 ensures that the x component of the state ξ never enters the interior of the avoidance shell $\mathcal{S}^i_{h_i}(\delta_i)$ during the avoidance maneuver, thus implying, using the inclusion (13), that x never enters the set $\mathcal{B}_{\frac{h\mu_i\delta_i}{2}}(\hat{x}_i)$, proving (ii) of Problem 1.

We focus on item (iii) of Problem 1. Consider the Lyapunov function defined in Assumption 1(a). Observe that since $\hat{x}_i \neq 0$ and $\xi \in \mathcal{U}$ for all $i \in \mathbb{N}_\beta$, it is always possible to scale the parameters δ_i to ensure that $0 \notin \mathcal{D}$ and $\cup_{i=1}^\beta \mathcal{S}^i(\delta_i) \subset \mathcal{U}$. Then, the set $\cup_{i=1}^\beta \mathcal{S}^i(\delta_i)$ is bounded away from the origin, and there exists a neighborhood of the origin inside of which the solutions evolve continuously according to (2). Then, Lyapunov stability of the origin for (2) implies Lyapunov stability of the origin for (36)–(38). We proceed by proving global attractivity of the origin relative to $\mathcal{U}$ for (36)–(38). Since V does not depend on q, we write $V(\xi) = V(x)$. When the solution jumps, one has $V(\xi^+) - V(\xi) = 0$ for all $\xi \in \mathcal{D}$. Moreover, due to the definition of the jump sets (36), after a jump the solutions must evolve continuously for some non-zero time. By choosing δ_i small enough for all $i \in \mathbb{N}_\beta$ so that $\delta_i \leq \bar{\delta}_i$, with $\bar{\delta}_i$ defined as in (29), and $\cup_{i=1}^\beta \mathcal{S}^i(\delta_i) \subset \mathcal{U}$, we have that the decrease property in (8) holds for all $x \in \mathcal{S}^i(\delta_i)$, and, together with the decrease (3) guaranteed by the nominal control law $u(\xi) = \kappa_s(\xi)$, this shows that $\dot{V} < 0$ for every $\xi \in \mathcal{C} \cap (\mathcal{U} \times \mathbb{Z}_\beta)$ with $x \neq 0$. Then, we have that the Lyapunov function V does not increase when the solutions of (36)–(38) jump or flow for all $\xi \in \mathcal{U} \times \mathbb{Z}_q$. Recalling that $\mathcal{U}$ is defined in Assumption 1 as a sub-level set of V, we can conclude that all solutions of (36)-(38) starting from $\mathcal{U} \times \mathbb{Z}_\beta$ remain inside it. Moreover, since the hybrid system (36)–(38) satisfies the hybrid basic conditions [10, As. 6.5], we can apply the invariance principle for hybrid systems [9, Thm. S13] to prove that all solutions converge to the largest set in which V is constant. Since V always decreases when solutions flow, and solutions must flow for some nonzero time between two consecutive jumps, we have that all solutions converge to the origin, proving global attractivity of the origin relative to $\mathcal{U}$. From Lyapunov stability and global attractivity of the origin, relative to $\mathcal{U}$, we can conclude GAS of the origin relative to $\mathcal{U}$. Finally, since the hybrid system $(36)-(38)$ satisfies the hybrid basic conditions [10, As. 6.5], from [10, Thm. 7.21] one has that GAS implies uniform GAS, thus proving item (iii) of Problem 1. ■

V. Numerical Simulations

To illustrate the effectiveness of the proposed avoidance architecture, we consider the input-affine nonlinear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = f(x) + g(x)u = \begin{bmatrix} x_2 + x_1^2 \\ x_1^2 + x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

which is a modified version of the nonlinear system presented in [13, Sec. 3.3]. We assume that the stabilization task has to be performed by u_2, while u_1 is an additional input that can be used during avoidance. Through backstepping, the stabilizing controller $u_s = [0\ u_n]^\top$, $u_n = -3x_1 - 5x_2 - 3x_3 - 8x_1x_2 - 2x_1x_3 - 8x_1^2x_2 - 8x_1^2 - 10x_1^3 - 2x_2^2 - 6x_1^4$, and the Lyapunov function $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 + x_1 + x_1^2)^2 + \frac{1}{2}(2x_1 + 2x_2 + x_3 + 2x_1(x_1^2 + x_2) + 3x_1^2)^2$ are obtained. To illustrate the properties of our hybrid redesign, in Fig. 2 we show numerical simulations obtained by considering three obstacle centroids, $\hat{x}_1 = [0.54, -0.42, -0.6]^\top$, $\hat{x}_2 = [0.5, 0.3, -0.8]^\top$, $\hat{x}_3 = [0.54, -0.22, -1.2]^\top$. The parameters used for the construction of the shells are $h_i = 0.75$, $\mu_i = 1$, for $i \in \mathbb{N}_3$, and $\delta_1 = 0.3$, $\delta_2 = 0.5$, $\delta_3 = 0.15$. In Fig. 2, left, one can see three trajectories, two encountering more than one obstacle along their path. The dashed line is used to indicate when the avoidance controller is active. In Fig. 2, right, one can see the time evolution of V, confirming that stability of the origin is preserved, and $u = [u_1\ u_2]^\top$ along one of the trajectories shown in the right plot (comparable plots can be obtained for the other two trajectories).

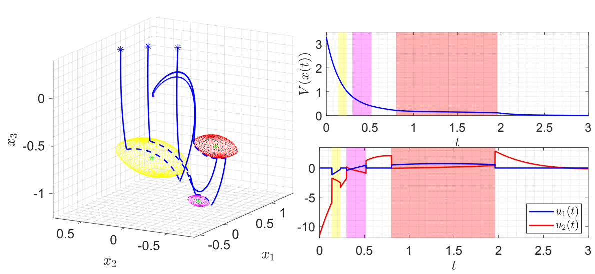

Fig. 2. (Left) State trajectories (blue) avoiding the interior of avoidance shells $\mathcal{S}_h(\delta_i)$. (Right) Time evolution of the input u and Lyapunov function V along one trajectory. The colored patches indicate when the avoidance controller is active.

VI. Conclusions

This paper proposes a controller design with local avoidance and global asymptotic stability properties. Under mild assumptions on the control matrix $g(\cdot)$, an avoidance controller is defined to preserve the decrease property of a Lyapunov function, corresponding to a predefined stabilizing controller without avoidance properties. The augmented avoidance controller is defined based on the solution of a non-convex optimization problem, for which an explicit optimal solution is derived in the paper. Future work will focus on investigating the maximum size of the obstacles for which avoidance and asymptotic stability of the closed loop system can be guaranteed.

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