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Parametrization of Linear Controllers for p-dominance

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Abstract— **Recently, the concept of p-dominance has been proposed as a unified framework to study rich behaviors of nonlinear systems. In this letter, we consider finding a set of linear dynamic output feedback controllers rendering the closed-loop systems p-dominant. We first derive an existence condition. Based on this condition, we then provide a parametrization of controllers. For Lure's systems, the proposed method can be applied only by solving a finite family of linear matrix inequalities, which is illustrated by achieving multi-stabilization and stabilization of a limit cycle.**

I. Introduction

Beyond mono-stability, various rich behaviors of nonlinear dynamical systems are interested in multiple disciplines such as bi-stability in bacteria [1] and stable limit cycles in circadian rhythms [2]. Typically, these problems have been studied independently, but there is a recent approach to develop a unified framework with the notion of p-dominance [3]. In this paper, our objective is to obtain a parametrization of linear dynamic output feedback controllers for p-dominance.

Literature Review: The concept of p-dominance is introduced via a differential Lyapunov matrix inequality for contraction (i.e., 0-dominance); see e.g. [4], [5] for contraction analysis. Relaxing the positive definiteness from its constant solution, it has been shown that if a solution has p negative eigenvalues, the system has the p-dimensional dominant behavior [3]. This unified framework for analysis and/or design of rich behaviors is applied to their robustness analysis [6] and model reduction for preserving them [7]. State feedback control design is studied in [3], [8], but a method for output feedback control is not well developed yet except for contraction [9], [10].

For stabilizing control, the Youla-Kučera parametrization [11] is a well known approach to parametrize output feedback controllers. Also, there is an alternative approach based on LMIs [12]. However, these approaches are not directly applicable to p-dominance. In particular, the latter relies on the positive definiteness of solutions to Lyapunov inequalities while the essence of p-dominance is to relax it.

Contribution: In this paper, we provide a parametrization of linear dynamic output feedback controllers rendering the closed-loop systems p-dominant by removing the requirement for solutions being positive definite from the aforementioned approach in [12]. First, we derive an existence condition for an output feedback controller as a natural extension of stabilizing control design for linear systems to the p-dominance of nonlinear systems. The proposed condition is based on a pair of control and filter Lyapunov type matrix inequalities. In general, the p-dominant behavior of the closed-loop system depends on this pair. However, under an additional condition, this is determined by the number of negative eigenvalues of the control inequality only.

Next, based on the derived existence condition, we present a parametrization of linear output feedback controllers for p-dominance. Utilizing the obtained parametrization, we further study stable control design as an extension of strong stabilization. Also, we study reduced-order control design. The proposed method consists of an infinite family of LMIs due to the state dependency, which can be relaxed into a finite set of LMIs by taking a similar approach as in [3], [8], [10]. Moreover, for Lure's systems, we can apply a different relaxation based on the sector condition, which is also demonstrated. That is, solving nonlinear partial differential equations/inequalities is not required for the proposed control design method.

Notation: The set of real numbers is denoted by $\mathbb{R}$. The set of $n \times n$ symmetric matrices is denoted by $\mathbb{S}_n$. The $n \times n$ identity matrix is denoted by I_n. For $P \in \mathbb{S}_n$, $P \succ 0$ ($P \succeq 0$) means that P is positive (semi) definite. For a matrix $B \in \mathbb{R}^{n \times m}$ with rank $B = r \leq m$, $B^\perp \in \mathbb{R}^{(n-r) \times n}$ denotes a matrix satisfying $B^\perp B = 0$ and $B^\perp (B^\perp)^\top \succ 0$. The vector 2-norm or the induced matrix 2-norm is denoted by $|\cdot|$.

II. Preliminaries

In this paper, our objective is to parametrize the set of linear controllers rendering the closed-loop systems p-dominant for nonlinear systems. First, we recall the notion of p-dominance and properties of p-dominant systems. Then, we state the considered problem.

A. p-dominance

For a closed nonlinear system $\dot{x} = f(x)$ with $f : \mathbb{R}^n \to \mathbb{R}^n$ of class C^1, the concept of *dominance* with rate $\lambda \geq 0$ is defined based on the following inequality with respect to $\varepsilon > 0$ and $P \in \mathbb{S}_n$:

$$\partial^\top f(x) P + P \partial f(x) \preceq -2\lambda P - \varepsilon I_n, \quad \forall x \in \mathbb{R}^n, \qquad (1)$$

where $\partial f(x) := \partial f(x)/\partial x$. Note that $P \in \mathbb{S}_n$ is not required to be positive definite. If it has p negative eigenvalues and $n - p$ positive eigenvalues, P is said to have the *inertia* p. Now, we are ready to mention the concept of p-dominance and properties of p-dominant systems.

Definition 2.1: [3, Definition 2] The closed nonlinear system $\dot{x} = f(x)$ is said to be *strictly p-dominant* with rate $\lambda \geq 0$ if there exist $\varepsilon > 0$ and $P \in \mathbb{S}_n$ with inertia p such that (1) holds. $\lhd$

Proposition 2.2: [3, Theorem 1] If a closed nonlinear system is strictly p-dominant with rate $\lambda \geq 0$, its flow on a compact limit set is topologically equivalent to a flow on a compact invariant set of a Lipshitz system in $\mathbb{R}^p$. $\lhd$

According to Proposition 2.2, realizing rich behavior can be formulated in terms of p-dominance. For instance, multi-stability and stable limit cycles are related to 1- and 2-dominance, respectively.

B. Problem Formulation

Consider the following open nonlinear system:

$$\begin{cases} \dot{x} = f(x) + Bu \\ y = Cx \end{cases} \tag{2}$$

where f is of class C^1, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{q\times n}$.

Our objective is to design a controller making the closed-loop system p-dominant. Since linear controllers are easy-to-implement, investigating their limits and potentials are important. Thus, we focus on linear output feedback controllers:

$$\begin{bmatrix} u \\ \dot{x}_c \end{bmatrix} = \underbrace{\begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix}}_{G} \begin{bmatrix} y \\ x_c \end{bmatrix}, \tag{3}$$

where $A_c \in \mathbb{R}^{n_c\times n_c}$ $(0 \leq n_c \leq n)$, $B_c \in \mathbb{R}^{n_c\times q}$, $C_c \in \mathbb{R}^{m\times n_c}$, and $D_c \in \mathbb{R}^{m\times q}$. That is, we consider the following control design problem.

Problem 2.3: For a system (2), find a set of linear dynamic output feedback controllers (3) to render the closed-loop systems p-dominant with rate $\lambda \geq 0$. $\lhd$

We are also interested in the computational tractability of linear control design. For this reason, we consider constant B and C, which allows us to formulate design problems in terms of linear matrix inequalities (LMIs). Two different approaches are explained after Theorem 3.1 and in Section IV below, the latter of which focuses on Lure's systems.

Let us define

$$\hat{x} = \begin{bmatrix} x \\ x_c \end{bmatrix}, \hat{f}(\hat{x}) = \begin{bmatrix} f(x) \\ 0 \end{bmatrix}, \hat{B} = \begin{bmatrix} B & 0 \\ 0 & I_{n_c} \end{bmatrix}, \hat{C} = \begin{bmatrix} C & 0 \\ 0 & I_{n_c} \end{bmatrix}. \tag{4}$$

Then, the closed-loop system can be described as

$$\dot{\hat{x}} = \hat{f}(\hat{x}) + \hat{B}G\hat{C}\hat{x}.$$

Thus, for given $\lambda \geq 0$, Problem 2.3 reduces to finding $\varepsilon > 0$, $P \in \mathbb{S}_{n+n_c}$ with inertia p, and $G \in \mathbb{R}^{(m+n_c)\times(q+n_c)}$ such that

$$\begin{aligned} &(\partial \hat{f}(\hat{x}) + \hat{B}G\hat{C})^\top P + P(\partial \hat{f}(\hat{x}) + \hat{B}G\hat{C}) \\ &\preceq -2\lambda P - \varepsilon I_{n+n_c}, \quad \forall x \in \mathbb{R}^n \end{aligned} \tag{5}$$

holds. We design controllers based on (5).

A parametrization of stabilizing controllers is a well studied problem as represented by the Youla-Kučera parametrization, e.g., [11]. In contrast, literature on control design for p-dominance is still scarce. This paper is the first attempt to study dynamic output feedback control design and further their parametrization. As an advantages of parametrizing controllers, one can design controllers having additional performances, which is also investigated by finding stable controllers as a generalization of strong stabilization.

III. Main Results

In this section, we first investigate the existence of a linear dynamic output feedback controller achieving p-dominance with rate $\lambda \geq 0$ and then provide its parametrization. Also, we discuss controller reduction and stable control design based on the obtained parametrization.

A. Parametrization of Linear Controllers

We first derive a necessary condition for the existences of P and G satisfying (5). This also becomes a sufficient condition under a mild assumption, stated below.

Theorem 3.1: Given $\lambda \geq 0$, suppose that there exist $\varepsilon > 0$, non-singular $P \in \mathbb{S}_{n+n_c}$, and $G \in \mathbb{R}^{(m+n_c)\times(q\times n_c)}$ satisfying (5). Then, there exist $\varepsilon_X, \varepsilon_Y > 0$ and $X, Y \in \mathbb{S}_n$ such that

$$\begin{aligned} &B^\perp(\partial f(x)X + X\partial^\top f(x) \\ &\quad + 2\lambda X + \varepsilon_X I_n)(B^\perp)^\top \preceq 0, \quad \forall x \in \mathbb{R}^n \end{aligned} \tag{6a}$$

$$\begin{aligned} &(C^\top)^\perp(Y\partial f(x) + \partial^\top f(x)Y \\ &\quad + 2\lambda Y + \varepsilon_Y I_n)((C^\top)^\perp)^\top \preceq 0, \quad \forall x \in \mathbb{R}^n \end{aligned} \tag{6b}$$

hold. The converse is also true if X is non-singular.

Proof: The proof is in Appendix B. ∎

Corollary 3.2: Given $\lambda \geq 0$, suppose that (6) has solutions $\varepsilon_X, \varepsilon_Y > 0$ and $X, Y \in \mathbb{S}_n$ with non-singular X. Utilizing $P_{1,2} \in \mathbb{R}^{n\times n_c}$ and non-singular $P_{2,2} \in \mathbb{S}_{n_c}$ satisfying $Y - X^{-1} = P_{1,2}P_{2,2}^{-1}P_{1,2}^\top$, define

$$P := \begin{bmatrix} Y & P_{1,2} \\ P_{1,2}^\top & P_{2,2} \end{bmatrix}. \tag{7}$$

Then, P is non-singular for such arbitrary $P_{1,2}$ and $P_{2,2}$, and there exist $\varepsilon > 0$ and $G \in \mathbb{R}^{(m+n_c)\times(q\times n_c)}$ such that (5) holds. Moreover, if $Y - X^{-1} \succeq 0$, the inertias of P and X are the same.

Proof: The proof is in Appendix C. ∎

For fixed $\lambda \geq 0$, (6) is an infinite family of LMIs with respect to $\varepsilon_X, \varepsilon_Y > 0$ and $X, Y \in \mathbb{S}_n$. To make it feasible, similar convex relaxations as in [3], [8], [10] can be applied. Let $\hat{A}_i \in \mathbb{R}^{(n+n_c)\times(n+n_c)}$, $i = 1, \dots, r$ be such that for each $\hat{x} \in \mathbb{R}^{n+n_c}$, there exist $\theta_i(\hat{x})$ satisfying $\partial \hat{f}(\hat{x}) = \sum_{i=1}^r \theta_i(\hat{x})\hat{A}_i$ and $\sum_{i=1}^r \theta_i(\hat{x}) = 1$. Then, (6) holds if

$$\begin{aligned} B^\perp(A_iX + XA_i^\top + 2\lambda X + \varepsilon_X I_n)(B^\perp)^\top \preceq 0 \\ (C^\top)^\perp(YA_i + A_i^\top Y + 2\lambda Y + \varepsilon_Y I_n)((C^\top)^\perp)^\top \preceq 0 \\ \forall i = 1, \dots, r \end{aligned}$$

are satisfied. This is a finite family of LMIs with respect to $\varepsilon_X, \varepsilon_Y > 0$ and $X, Y \in \mathbb{S}_n$.

According to Corollary 3.2, achieving p-dominance reduces to finding a suitable pair of X and Y (also ε_X and ε_Y) such that P in (7) has inertia p, which can require iteratively solving (6) for different rates $\lambda \geq 0$. This process can be simplified to finding X with inertia p by imposing $Y - X^{-1} \succeq 0$ when solving (6b). In general, increasing $\lambda \geq 0$ corresponds to making p larger.

In this letter, we are interested in not only finding a stabilizing controller but also parametrizing a set of controllers achieving p-dominance with rate $\lambda \geq 0$. Such a parametrization is presented as follows.

Theorem 3.3: Given $\lambda \geq 0$, suppose that all conditions in Corollary 3.2 hold, and consider P in (7). Then, for this P, all G parametrized below satisfy (5) for some $\varepsilon > 0$:

$$\begin{aligned} G = &-R^{-1}\hat{B}^\top P\hat{Q}^{-1}\hat{C}^\top(\hat{C}\hat{Q}^{-1}\hat{C}^\top)^{-1} \\ &+ S^{1/2}L(\hat{C}\hat{Q}^{-1}\hat{C}^\top)^{-1/2} \\ S := &\, R^{-1} - R^{-1}\hat{B}^\top P\hat{Q}^{-1} \\ &(\hat{Q} - \hat{C}^\top(\hat{C}\hat{Q}^{-1}\hat{C}^\top)^{-1}\hat{C})\hat{Q}^{-1}P\hat{B}R^{-1}, \end{aligned} \tag{8}$$

where parameters $L \in \mathbb{R}^{(n_c+m)\times(q+n_c)}$, $0 \prec R \in \mathbb{S}_{n_c+m}$, and $0 \prec \hat{Q} \in \mathbb{S}_n$ are arbitrary as long as $|L| < 1$ and

$$\begin{aligned} &-\partial^\top \hat{f}(\hat{x})P - P\partial\hat{f}(\hat{x}) - 2\lambda P \\ &\quad - \hat{\varepsilon}I_{n+n_c} + P\hat{B}R^{-1}\hat{B}^\top P \succ \hat{Q}, \quad \forall x \in \mathbb{R}^n \end{aligned} \tag{9a}$$

$$(\hat{C}^\top)^\perp(\hat{Q} - P\hat{B}R^{-1}\hat{B}^\top P)((\hat{C}^\top)^\perp)^\top \succ 0 \tag{9b}$$

hold for some $\hat{\varepsilon} > 0$.

Proof: The proof is in Appendix D. ■

Note that (9) is linear with respect to $\hat{\varepsilon} > 0$, $R \succ 0$, and $\hat{Q} \succ 0$. Thus, a similar convex relaxation as for (6) can be applied to (9) for deriving a finite family of LMIs.

If (6) and (9) hold for all $x \in \mathbb{R}^n$, we obtain a parametrization of controllers rendering global p-dominance. If one is interested in achieving local p-dominance on a convex $D \subset \mathbb{R}^n$, one only has to consider these conditions on D.

B. Controller Reduction

From Corollary 3.2, the dimension n_c of a designed controller is equivalent to the rank of $Y - X^{-1}$. Thus, given $n_c < n$, a reduced-order control design can be formulated as finding $X, Y \in \mathbb{S}_n$ such that (6) and $\mathrm{rank}\,(XY - I_n) = \mathrm{rank}\,(Y - X^{-1}) = n_c$ hold. To make this problem computationally tractable, we relax reducing $\mathrm{rank}\,(XY - I_n)$ into making $\mathrm{trace}(XY - I_n)$ smaller. To this end, we alternatively update X and Y.

For the initial solution $X_k, Y_k \in \mathbb{S}_n$ with $k = 0$, suppose that $\mathrm{rank}\,(X_kY_k - I_n) > n_c$. Then, we first update X by solving the following optimization problem:

$$\begin{aligned} &\min_{\substack{\varepsilon_k \geq 0,\, \varepsilon_{X_{k+1}} > 0 \\ X_{k+1} \in \mathbb{S}_n}} \varepsilon_k \\ \text{s.t.} \quad &-\varepsilon_k \leq \mathrm{trace}(X_{k+1}Y_k - I_n) \leq \varepsilon_k \\ &\text{(6a) holds for } X = X_{k+1} \text{ and } \varepsilon_X = \varepsilon_{X_{k+1}} \\ &X_{k+1} \text{ has inertia } p. \end{aligned} \tag{10}$$

The last constrain is not convex. However, it is expected that this holds if the update $X_{k+1} - X_k$ is small, e.g., $-cI_n \preceq X_{k+1} - X_k \preceq cI_n$ with sufficiently small $c > 0$. Next, we update Y based on the following optimization problem:

$$\begin{aligned} &\min_{\substack{\varepsilon_k \geq 0,\, \varepsilon_{Y_{k+1}} > 0 \\ Y_{k+1} \in \mathbb{S}_n}} \varepsilon_k \\ \text{s.t.} \quad &-\varepsilon_k \leq \mathrm{trace}(X_{k+1}Y_{k+1} - I_n) \leq \varepsilon_k \\ &\text{(6b) holds for } Y = Y_{k+1} \text{ and } \varepsilon_Y = \varepsilon_{Y_{k+1}} \\ &Y_{k+1} - X_{k+1}^{-1} \succeq 0. \end{aligned} \tag{11}$$

The last constraint is to make the inertias of X and P are the same; recall Corollary 3.2.

We repeat the updates of X_k and Y_k until $\mathrm{rank}\,(X_kY_k - I_n) \leq n_c$. The optimization problems (10) and (11) are always feasible because they have trivial solutions $\varepsilon_{X_{k+1}} = \varepsilon_{X_k}$, $\varepsilon_{Y_{k+1}} = \varepsilon_{Y_k}$, $X_{k+1} = X_k$, and $Y_{k+1} = Y_k$. However, when $X_{k+1} - X_k$ and $Y_{k+1} - Y_k$ are too marginal, we may need to terminate the algorithm (fail).

C. Stable Control Design

In Theorem 3.3, we have obtained a parametrization (8) of linear controllers G rendering the closed-loop systems p-dominant, where G is linear with respect to L. We derive an LMI condition for L such that a controller becomes stable.

Proposition 3.4: Given $\lambda \geq 0$, suppose that all conditions in Corollary 3.2 hold. Consider P in (7) and the corresponding parametrization G in (8) for fixed $\hat{\varepsilon} > 0$, $R \succ 0$, and $\hat{Q} \succ 0$ satisfying (9). Define $G_{2,2} := \begin{bmatrix} 0 & I_{n_c} \end{bmatrix} G \begin{bmatrix} 0 & I_{n_c} \end{bmatrix}^\top$. If $P_{2,2} \succ 0$, and the following set of LMIs:

$$G_{2,2}^\top P_{2,2} + P_{2,2}G_{2,2} \prec 0 \tag{12a}$$

$$\begin{bmatrix} I_{n_c+m} & L \\ L^\top & I_{p+n_c} \end{bmatrix} \succ 0 \tag{12b}$$

has a solution $L \in \mathbb{R}^{(n_c+m)\times(q+n_c)}$, then A_c is Hurwitz while P and G satisfy (5) for some $\varepsilon > 0$.

Proof: From (3), $G_{2,2} = A_c$. Thus, (12a) with $P_{2,2} \succ 0$ implies that A_c is Hurwitz. From Theorem 3.3, if L satisfies $|L| < 1$, i.e., (12b), P and G satisfy (5) for some $\varepsilon > 0$. ■

From the construction (7) of P, $Y - X^{-1} \succeq 0$ implies $P_{2,2} \succ 0$. Noting this, we summarize an algorithm for reduced-order stable linear control design achieving p-dominance with rate $\lambda \geq 0$ in Algorithm 1 below. The constructed G in line 23 gives a parametrization of (non-necessarily stable) linear controllers rendering the closed-loop systems p-dominant.

IV. For Lure's Systems

To design a controller based on the proposed approach, we need to solve an infinite family of LMIs (6), which can be relaxed into a finite one as mentioned above. In this section, focusing on Lure's systems, we consider another relaxation. In particular, we derive a sufficient LMI condition for (6) and provide a parametrization of controllers.

Lure's system is a system described by

$$\begin{cases} \dot{x} = Ax + Bu + B_zg(z) \\ y = Cx \\ z = C_zx, \end{cases} \tag{13}$$

where $A \in \mathbb{R}^{n\times n}$, $B_z \in \mathbb{R}^{n\times n_2}$, $C_z \in \mathbb{R}^{n_1\times n}$, and $g : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ is of class C^1 such that for some $\gamma > 0$,

$$|\partial g(z)| \leq \gamma, \quad \forall z \in \mathbb{R}^{n_1} \tag{14}$$

holds. If there are parameter uncertainties, these can be covered by selecting γ larger.

Algorithm 1 Control design for p-dominance with rate $\lambda \geq 0$

Require: System (2), p, $n_c \leq n$
Ensure: $\hat{n}_c$-dimensional ($\hat{n}_c \leq n_c$) stable linear controller (3) achieving p-dominance with rate $\lambda \geq 0$ or **Fail**

$\lambda = 0$
while Inertia of X **is equivalent to** p **do**
  Solve (6a) with respect to $\varepsilon_X > 0$ and $X \in \mathbb{S}_n$
  if Inertia of X is less than p **then**
    Increase $\lambda \geq 0$
  end if
  if Inertia of X is greater than p **then**
    Return Fail
  end if
end while
Solve (6b) and $Y - X^{-1} \succeq 0$ with respect to $\varepsilon_Y > 0$ and $Y \in \mathbb{S}_n$
$k \leftarrow 0$, $X_0 \leftarrow X$, $Y_0 \leftarrow Y$
while rank $(X_k Y_k - I_n) \leq n_c$ **do**
  Solve (10) with respect to $\varepsilon_{X_{k+1}} > 0$ and $X_{k+1} \in \mathbb{S}_n$
  Solve (11) with respect to $\varepsilon_{Y_{k+1}} > 0$ and $Y_{k+1} \in \mathbb{S}_n$
  if $X_{k+1} - X_k$ and $Y_{k+1} - Y_k$ are marginal **then**
    Return Fail
  end if
  $k \leftarrow k + 1$
end while
$\hat{n}_c \leftarrow$ rank $(X_k Y_k - I_n)$, $X \leftarrow X_k$, $Y \leftarrow Y_k$
Compute P in (7), and solve (9) with respect to $\hat{\varepsilon} > 0$, $0 \prec R \in \mathbb{S}_{\hat{n}_c+m}$, and $0 \prec \hat{Q} \in \mathbb{S}_n$
Construct G in (8) with a tuning parameter $|L| < 1$
if (12) does not have a solution L **then**
  Return Fail
end if
Return G **with a solution** L **to** (12)

Utilizing the structure of Lure's systems, we have the following sufficient condition for (6).

Proposition 4.1: For Lure's system (13), if given $\lambda \geq 0$, there exist $\varepsilon_X, \varepsilon_Y > 0$ and $X, Y \in \mathbb{S}_n$ such that

$$\begin{aligned} &B^{\perp}(AX + XA^{\top} + XC_z^{\top}C_zX \\ &\qquad + \gamma^2 B_z B_z^{\top} + 2\lambda X + \varepsilon_X I_n)(B^{\perp})^{\top} \preceq 0 \quad (15a) \\ &(C^{\top})^{\perp}(YA + A^{\top}Y + C_z^{\top}C_z + \gamma^2 Y B_z B_z^{\top} Y \\ &\qquad + 2\lambda Y + \varepsilon_Y I_n)((C^{\top})^{\perp})^{\top} \preceq 0 \quad (15b) \end{aligned}$$

hold, then they satisfy (6).

Proof: For Lure's system, (6a) becomes

$$\begin{aligned} &B^{\perp}(AX + XA^{\top} + B_z \partial g(z) C_z X \\ &\qquad + XC_z^{\top}\partial^{\top} g(z) B_z^{\top} + 2\lambda X + \varepsilon_X I_n)(B^{\perp})^{\top} \preceq 0, \end{aligned}$$

which can be rearranged as

$$B^{\perp}\left(AX + XA^{\top} + 2\lambda X + \varepsilon_X I_n + \begin{bmatrix} B_z & XC_z^{\top} \end{bmatrix} \begin{bmatrix} 0 & \partial g(z) \\ \partial^{\top} g(z) & 0 \end{bmatrix} \begin{bmatrix} B_z^{\top} \\ C_z X \end{bmatrix}\right)(B^{\perp})^{\top} \preceq 0.$$

By the Schur complement with (14), this holds if (15a) holds. The proof for Y is similar. ■

Note that (15) is equivalent to the following set of LMIs with respect to $X, Y \in \mathbb{S}_n$:

$$\begin{bmatrix} \bar{X}_{1,1} & B^{\perp}XC_z^{\top} \\ C_z X (B^{\perp})^{\top} & -I_{n_1} \end{bmatrix} \preceq 0 \quad (16a)$$

$$\begin{bmatrix} \bar{Y}_{1,1} & \gamma (C^{\top})^{\perp} Y B_z \\ \gamma B_z^{\top} Y ((C^{\top})^{\perp})^{\top} & -I_{n_2} \end{bmatrix} \preceq 0 \quad (16b)$$

$$\begin{aligned} \bar{X}_{1,1} &:= B^{\perp}(AX + XA^{\top} + \gamma^2 B_z B_z^{\top} + 2\lambda X + \varepsilon_X I_n)(B^{\perp})^{\top} \\ \bar{Y}_{1,1} &:= (C^{\top})^{\perp}(YA + A^{\top}Y + C_z^{\top}C_z \\ &\qquad + 2\lambda Y + \varepsilon_Y I_n)((C^{\top})^{\perp})^{\top}. \end{aligned}$$

Thus, for Lure's systems, the construction of P in (7) can be relaxed into solving a finite set of LMIs.

Also for a parametrization of linear controllers, we have the following LMI condition.

Proposition 4.2: Given $\lambda \geq 0$, suppose that all the conditions in Proposition 4.1 hold, and consider P in (7). Then, for this P, all G parametrized by (8) satisfy (5), where $L \in \mathbb{R}^{(n_c+m)\times(q+n_c)}$ and $0 \prec R \in \mathbb{S}_{n_c+m}$ are arbitrary as long as $|L| < 1$ and

$$\begin{aligned} \hat{Q} := &-\hat{A}P - P\hat{A} - \hat{C}_z^{\top}\hat{C}_z - \gamma^2 P \hat{B}_z \hat{B}_z^{\top} P \\ &- 2\lambda P - \hat{\varepsilon} I_{n+n_c} + P\hat{B}R^{-1}\hat{B}^{\top}P \succ 0 \end{aligned} \quad (17)$$

for some $\hat{\varepsilon} > 0$, where

$$\hat{A} := \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \ \hat{B}_z := \begin{bmatrix} B_z \\ 0 \end{bmatrix}, \ \hat{C}_z := \begin{bmatrix} C_z & 0 \end{bmatrix}.$$

Proof: The proof is in Appendix E. ■

V. Examples

Consider the following Lure's system:

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ -3 & -5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \sin(z) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x \\ z = \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{cases} \quad (18)$$

For this system, $\gamma > 0$ in (14) is 1. This system is 0-dominant when $u = 0$.

We design two 1-dimensional controllers: one is for 1-dominance, and the other is for 2-dominance. First, we consider 1-dominance. Applying a similar algorithm to Algorithm 1 for the Lure's system, a set of solutions to (16) is obtained by $\lambda = 1.1$, $\varepsilon_X = 0.01$, $\varepsilon_Y = 0.01$, and

$$X = \begin{bmatrix} -0.600 & 0 \\ 0 & 7.00 \end{bmatrix}, \quad Y = \begin{bmatrix} -0.667 & 0 \\ 0 & 0.143 \end{bmatrix}.$$

Note that the inertia of X is 1, and $Y - X^{-1} \succeq 0$. From Corollary 3.2, P in (7) has inertia 1 also. Since $1/7 \approx 0.143$, the rank of $Y - X^{-1}$ is viewed as 1, and P with inertial 1 is constructed as follows:

$$P = \begin{bmatrix} -0.667 & 0 & 1 \\ 0 & 0.143 & 0 \\ 1 & 0 & 0.1000 \end{bmatrix}.$$

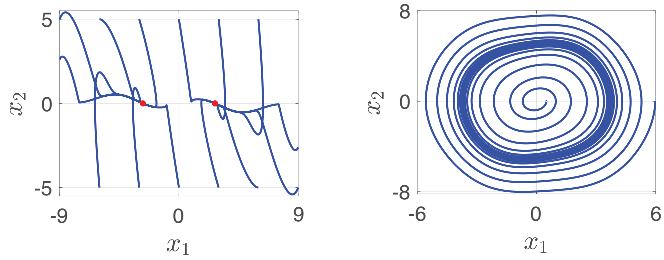

Fig. 1. Closed-loop trajectories (left) 1-dominance (right) 2-dominance with rates $\lambda = 1.1$

To obtain a parametrization of controllers, we compute $0 \prec R \in \mathbb{S}_2$ satisfying (17), which is

$$R = \begin{bmatrix} 0.0620 & -0.190 \\ -0.190 & 0.962 \end{bmatrix}.$$

Then, a parametrization of controllers achieving 1-dominance with rate $\lambda = 1.1$ is given by G in (8) with a tuning parameter $|L| < 1$. From the obtained parametrization, we select a stable controller. In this case, (12) has a trivial solution $L = 0$, and the corresponding G in (8) is

$$G = \begin{bmatrix} -0.997 & -7.58 \\ -1.24 & -2.54 \end{bmatrix}.$$

From (3), $A_c = -2.54$ is Hurwitz. Figure 1 (left) shows the closed-loop trajectories in the x-plane. As expected from 1-dominance with rate $\lambda = 1.1$, the closed-loop system is multi-stable.

Next, we design a controller achieving 2-dominance. To increase p, we select larger $\varepsilon_X = 2.5$ than the one for $p = 1$, where $\lambda = 1.1$ is the same as $p = 1$. Then, a solution $X \in \mathbb{S}_2$ to the LMI (16a) is obtained by

$$X = \begin{bmatrix} -0.600 & 0 \\ 0 & -0.500 \end{bmatrix}.$$

The inertia of X is 2. Also, the set solutions to (16b) and $Y - X^{-1} \succeq 0$ is obtained by $\varepsilon_Y = 0.01$ and

$$Y = \begin{bmatrix} -1.62 & -0.310 \\ -0.310 & -0.110 \end{bmatrix}.$$

Based on X and Y, we construct P in (7) as

$$P = \begin{bmatrix} -1.62 & -0.310 & 0.162 \\ -0.310 & -0.110 & -0.987 \\ 0.162 & -0.987 & 0.515 \end{bmatrix}$$

whose inertia is 2. For this P, a matrix $0 \prec R \in \mathbb{S}_2$ satisfying (17) and a controller G with $L = 0$ are obtained by

$$R = \begin{bmatrix} 0.145 & 0.0772 \\ 0.0772 & 0.993 \end{bmatrix}, \quad G = \begin{bmatrix} 2.06 & 7.57 \\ -1.38 & -0.250 \end{bmatrix}.$$

Note that $A_c = -0.250$ is Hurwitz. Figure 1 (right) shows the closed-loop trajectories in the x-plane. As expected from 2-dominance with rate $\lambda = 1.1$, the closed-loop system has a stable Limit cycle.

VI. Conclusion

In this letter, we have proposed a parametrization of linear dynamic output feedback controllers rendering the closed-loop systems p-dominant. Utilizing the proposed parametrization, we have further discussed how to impose the stability for controller dynamics. Also, we have mentioned reduced-order control design. Based on the proposed method, we have designed reduced-order stable controllers for Lure's system, which achieve 1- and 2-dominance. Future work includes applying the proposed method to real-life systems by addressing problems caused in practical environment.

Appendix

A. Lemmas for Matrices

Before proving the theorems, we list existing results.

Lemma 1.1: [13, A.5.5, C.4.1] Consider a symmetric block matrix P in (7). If $P_{2,2}$ is non-singular, it can be decomposed into

$$P = \begin{bmatrix} I & P_{1,2}P_{2,2}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} Y - P_{1,2}P_{2,2}^{-1}P_{1,2}^\top & 0 \\ 0 & P_{2,2} \end{bmatrix} \begin{bmatrix} I & P_{1,2}P_{2,2}^{-1} \\ 0 & I \end{bmatrix}^\top. \quad (19)$$

Moreover, if P is non-singular, the first $n \times n$ block element of P^{-1} is $X := (Y - P_{1,2}P_{2,2}^{-1}P_{1,2}^\top)^{-1}$. $\triangleleft$

Lemma 1.2: [12, Lemma 2.1] Let matrices B and $H = H^\top$ with compatible dimensions be given. Then, the following two statements are equivalent:

(i) there exists $R \succ 0$ satisfying $H + BRB^\top \succ 0$;
(ii) $B^\perp H (B^\perp)^\top \succ 0$ or $BB^\top \succ 0$. $\triangleleft$

Lemma 1.3: [12, Lemma 2.2] Let matrices K, C, $Q = Q^\top$, and $R \succ 0$ with compatible dimensions be given. If $CC^\top \succ 0$, the following two statements are equivalent:

(i) there exists G satisfying $(K + GC)^\top R(K + GC) \prec Q$;
(ii) a) $Q \succ 0$ and b) $(C^\top)^\perp (Q - K^\top RK)((C^\top)^\perp)^\top \succ 0$ or $C^\top C \succ 0$.

If (ii) holds, all G satisfying (i) are given by

$$G = -KQ^{-1}C^\top(CQ^{-1}C^\top)^{-1} + S^{1/2}L(CQ^{-1}C^\top)^{-1/2}$$
$$S := R^{-1} - K(Q^{-1} - Q^{-1}C^\top(CQ^{-1}C^\top)^{-1}CQ^{-1})K^\top$$

where $|L| < 1$. $\triangleleft$

B. Proof of Theorem 3.1

As a preliminary step, we rearrange (5). Since (5) contains $\varepsilon > 0$, (5) holds if and only if there exist $\hat{\varepsilon} > 0$ and $0 \prec R \in \mathbb{S}_{n_c+m}$ such that

$$(\partial \hat{f}(\hat{x}) + \hat{B}G\hat{C})^\top P + P(\partial \hat{f}(\hat{x}) + \hat{B}G\hat{C}) + (G\hat{C})^\top RG\hat{C} \prec -2\lambda P - \hat{\varepsilon} I_{n+n_c},$$

holds, where note that the inequality is strict. Completing the square with respect to $G\hat{C}$ yields

$$(R^{-1}\hat{B}^\top P + G\hat{C})^\top R(R^{-1}\hat{B}^\top P + G\hat{C}) \prec Q(\hat{x}), \quad (20)$$

where

$$Q(\hat{x}) := -\partial^\top \hat{f}(\hat{x})P - P\partial \hat{f}(\hat{x})$$

$$- 2\lambda P - \hat{\varepsilon} I_{n+n_c} + P\hat{B}R^{-1}\hat{B}^\top P. \tag{21}$$

Note that $Q(\hat{x})$ depends on x only because $\hat{f}(\hat{x})$ does so. According to Lemma 1.3, (20) is equivalent to

$$Q(\hat{x}) \succ 0 \tag{22a}$$
$$(\hat{C}^\top)^\perp (Q(\hat{x}) - P\hat{B}R^{-1}\hat{B}^\top P)((\hat{C}^\top)^\perp)^\top \succ 0. \tag{22b}$$

Note that every transformation is equivalent until here. Namely, there exist P and G satisfying (5) if and only if there exist P and $R \succ 0$ such that (22) holds for $Q(\hat{x})$ in (21). Thus, we show the statements based on (22).

First, we show that (22) implies (6). From (21), pre- and post-multiplying P^{-1} to (22a) lead to

$$\begin{aligned} &- P^{-1}\partial^\top \hat{f}(\hat{x}) - \partial \hat{f}(\hat{x}) P^{-1} \\ &\quad - 2\lambda P^{-1} - \hat{\varepsilon} P^{-2} + \hat{B}R^{-1}\hat{B}^\top \succ 0. \end{aligned} \tag{23}$$

Since P is non-singular and symmetric, we have $P^{-2} \succ 0$. Thus, (23) holds if and only if for some $\varepsilon_X > 0$,

$$\begin{aligned} &- P^{-1}\partial^\top \hat{f}(\hat{x}) - \partial \hat{f}(\hat{x}) P^{-1} \\ &\quad - 2\lambda P^{-1} - \varepsilon_X I_{n+n_c} + \hat{B}R^{-1}\hat{B}^\top \succ 0. \end{aligned}$$

According to Lemma 1.2, this is equivalent to

$$\begin{aligned} &\hat{B}^\perp(-P^{-1}\partial^\top \hat{f}(\hat{x}) - \partial \hat{f}(\hat{x}) P^{-1} \\ &\quad - 2\lambda P^{-1} - \varepsilon_X I_{n+n_c})(\hat{B}^\perp)^\top \succ 0. \end{aligned} \tag{24}$$

The definition of $\hat{B}$ in (4) implies $\hat{B}^\perp = \begin{bmatrix} B^\perp & 0 \end{bmatrix}$. Also from the definition of $\hat{f}(\hat{x})$ in (4), (24) holds if and only if

$$B^\perp(-X\partial^\top f(x) - \partial f(x) X - 2\lambda X - \varepsilon_X I_n)(B^\perp)^\top \succ 0,$$

holds, where X denotes the first $n \times n$ block diagonal element of P^{-1}. Since $B^\perp(B^\perp)^\top \succ 0$ and $\varepsilon_X > 0$, this is equivalent to the non-strict inequality (6a).

Similarly, the definition of $\hat{C}$ in (4) implies $(\hat{C}^\top)^\perp = \begin{bmatrix} (C^\top)^\perp & 0 \end{bmatrix}$. Then, from (21), (22b) is equivalent to

$$\begin{aligned} &(C^\top)^\perp(-\partial^\top f(x) Y - Y\partial f(x) \\ &\quad - 2\lambda Y - \hat{\varepsilon} I_n)((C^\top)^\perp)^\top \succ 0, \end{aligned}$$

where $Y = P_{1,1}$. Again, this is equivalent to the non-strict inequality (6b).

Next, we prove the converse under the non-singularity of X. Define $n_c = \mathrm{rank}\,(Y - X^{-1})$. Then, there exist $P_{1,2} \in \mathbb{R}^{n \times n_c}$ and non-singular $P_{2,2} \in \mathbb{S}_{n_c}$ such that $Y - X^{-1} = P_{1,2}P_{2,2}^{-1}P_{1,2}^\top$ holds. Using these matrices, we construct P as in (7). Since $P_{2,2}$ is non-singular, P can be decomposed as in (19), where $P_{1,1} - P_{1,2}P_{2,2}^{-1}P_{1,2}^\top = X^{-1}$. The non-singularity of X and $P_{2,2}$ imply that of P; when $n_c = 0$, $P = Y = X^{-1}$ is non-singular.

From Lemma 1.1, the first $n \times n$ block element of P^{-1} is X. Therefore, if X satisfies (6a), P^{-1} does (24). Namely, there exists $R \succ 0$ satisfying (23), i.e., (22a). Similarly, if Y satisfies (6b), P does (22b). In summary, for the constructed P, there exists $R \succ 0$ satisfying (22). ■

C. Proof of Corollary 3.2

The first statement follows from the proof of Theorem 3.1 above. We show the second statement. From the constructions of $P_{1,2}$ and $P_{2,2}$, $Y - X^{-1} \succeq 0$ implies $P_{2,2} \succ 0$. Furthermore, from the decomposition (19) of P, if $P_{2,2} \succ 0$ then the inertia of P is equivalent to that of $P_{1,1} - P_{1,2}P_{2,2}^{-1}P_{1,2}^\top = X^{-1}$. Note that the inertias of X^{-1} and X are the same. ■

D. Proof of Theorem 3.3

From Lemma 1.3 with $\hat{Q} \succ 0$ and (9b), all G in (8) satisfy

$$(R^{-1}\hat{B}^\top P + G\hat{C})^\top R(R^{-1}\hat{B}^\top P + G\hat{C}) \prec \hat{Q}. \tag{25}$$

From (9a) and (21), we have $\hat{Q} \preceq Q(x)$ and thus (20). From the proof of Theorem 3.1, all G satisfy (5). ■

E. Proof of Proposition 4.2

As in the proof of Theorem 3.1, it is possible to show that the constructed P satisfies $\hat{Q} \succ 0$ and

$$(\hat{C}^\top)^\perp(\hat{Q} - P\hat{B}R^{-1}\hat{B}^\top P)((\hat{C}^\top)^\perp)^\top \succ 0.$$

for some $R \succ 0$, $\lambda \geq 0$, and $\varepsilon > 0$, where $\hat{Q}$ is defined in (17). Also, similarly to the proof of Theorem 3.3, it is possible to show that all G in (8) with $\hat{Q}$ in (17) satisfy (25). Again from the proof of Theorem 3.1, (25) implies

$$\begin{aligned} &(\hat{A} + \hat{B}G\hat{C})^\top P - P(\hat{A} + \hat{B}G\hat{C}) + \hat{C}_z^\top \hat{C}_z + \gamma^2 P\hat{B}_z\hat{B}_z^\top P \\ &+ 2\lambda P + \hat{\varepsilon} I_{n+n_c} \preceq 0 \end{aligned}$$

As in the proof of Proposition 4.1, one can show that this implies (5) for the Lure's system (13). ■

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