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Strengthened Circle and Popov Criteria for the stability analysis of feedback systems with ReLU neural networks

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Abstract— **This paper considers the stability analysis of a Lurie system with a static repeated ReLU (rectified linear unit) nonlinearity. Properties of the ReLU function are leveraged to derive new tailored quadratic constraints (QCs) which are satisfied by the repeated ReLU. These QCs are used to strengthen the Circle and Popov Criteria for this specialised Lurie system. It is shown that the criteria can be cast as a set of linear matrix inequalities (LMIs) with less restrictive conditions on the matrix variables. Many systems involving a neural network (NN) with ReLU activations are important instances of this specialised Lurie system; for example, a continuous time recurrent neural network (RNN) or the interconnection of a linear system with a feedforward NN. Numerical examples show the strengthened criteria strike an appealing balance between reduced conservatism and complexity, compared to existing criteria.**

I. Introduction

Neural network (NN) based control has been growing in popularity due to recent successes in reinforcement learning (RL) and imitation learning (IL) [1]–[4]. To extend its range of application to safety critical systems, performance certificates such as closed loop stability must be verified.

Several articles have shown that different systems involving NNs can be modelled as a Lurie system, Fig. 1, where the nonlinearity, $\Phi(\cdot)$, is a vector of the NN activation functions. Examples are the interconnection of a linear time-invariant (LTI) system with a feedforward NN [5], [6] and a continuous time recurrent neural network (RNN) [7]. Stability analysis of a Lurie system lends itself to a range of criteria from absolute stability: the classical Circle and Povov Criteria [8]–[10], other Lyapunov-based criteria [11], [12] and Zames-Falb multipliers [13]–[16]. What separates the criteria is how they balance the trade-off between conservatism and computational complexity. The Circle and Popov Criteria have low complexity; in contrast, Park and Zames-Falb multipliers have higher complexity, but are generally less conservative.

The above absolute stability criteria can all be posed as semi-definite programming (SDP) problems involving linear matrix inequalities (LMIs) [17], which can be solved in polynomial time using efficient solvers e.g., MOSEK and the LMI toolbox [18], [19]. Therefore, recent work has used the absolute stability framework, and the associated SDP tools, for addressing a number of problems in NN analysis: estimation of the region of attraction [20], [21], synthesis of NN controllers [22], [23], and robustness analysis [24]–[26]. The main challenge in NN analysis is that the number of activation functions, m, is typically large, meaning that the resulting absolute stability problems suffer from greater computational complexity than traditional absolute stability problems where m is normally small. Some of the less conservative absolute stability tools, such as Zames-Falb multiplier analysis, scale poorly with m, and thus problems with computation arise. Conversely, some of the simpler tools, such as the Circle Criterion, become very conservative for large m, making them of limited use in NN analysis.

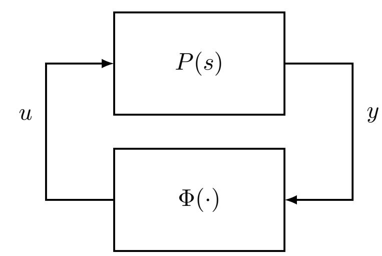

Fig. 1. Lurie system with static nonlinearity

Contribution: The main contribution of this paper is to address the trade-off between computational complexity and conservatism in absolute stability problems encountered in NN analysis. To do this, we focus on Lurie systems where the nonlinearity is of the ReLU type. ReLU is a popular choice of activation function in deep learning as it does not suffer from saturated outputs or increased computational inefficiency, unlike the Logistic sigmoid and Tanh functions [27]. Properties of the ReLU function are leveraged to derive tailored quadratic constraints (QCs) for the repeated ReLU. As these characterise the nonlinearity more accurately than sector/slope bounds, the stability analysis holds for few nonlinearities other than the repeated ReLU. Generality is therefore sacrificed to increase the freedom in the SDP optimisation and, as a consequence, reduce the conservatism of the stability analysis. Thus, our contribution is three-fold: tailored QCs are derived for the repeated ReLU; the low complexity Circle and Popov Criteria are strengthened for this specialised Lurie system; and, convex relaxations are proposed for converting the strengthened Popov Criterion into an SDP problem involving LMIs, which can be solved efficiently.

Paper structure: Section II presents the problem setup and highlights the properties of the ReLU function. Section III derives the tailored QCs for the repeated ReLU. Section IV presents the strengthened Circle and Popov Criteria for this specialised Lurie system, these are referred to as the Circle-

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like and Popov-like Criteria. Section V proposes convex relaxations for the Popov-like Criterion. Section VI presents some numerical examples.

II. Preliminaries

A. Notation

The sets of non-negative real numbers, m-dimensional vectors with non-negative elements, and square m-dimensional matrices with non-negative elements are, respectively, denoted by $\Re_{\geq 0}$, $\Re^m_{\geq 0}$, and $\Re^{m\times m}_{\geq 0}$. The set of square m-dimensional symmetric positive definite matrices is represented by $\mathcal{S}^m_+$, with the diagonal subset $\mathcal{D}^m_+ \subset \mathcal{S}^m_+$. The set of square m-dimensional matrices with non-positive off-diagonal elements is denoted by $\mathcal{Z}^m \subset \Re^{m\times m}$: these are known as *$Z$-matrices*. A matrix M of elements m_{ij} is sometimes expressed as $M = [m_{ij}]$, a negative definite matrix is denoted by $M \prec 0$ and $He(M) := M + M'$. The space of real rational transfer function matrices, analytic in the right-half complex plane, is represented by $\mathcal{RH}_\infty$.

B. Problem setup

Consider the interconnection in Fig. 1, where $P(s) \in \mathcal{RH}_\infty$ is a finite dimensional LTI system, with state space realisation (A, B, C, D). The system is modelled by (1) with $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{m\times n}$, $D \in \Re^{m\times m}$ and $\Phi(\cdot) : \Re^m \to \Re^m$ being the repeated ReLU nonlinearity. Such an interconnection is a type of *Lurie system.* As the repeated ReLU satisfies $\Phi(0) = 0$, the origin is an equilibrium point of (1).

$$\begin{aligned} \dot{x} &= Ax + B\Phi(y) \\ y &= Cx + D\Phi(y) \end{aligned} \tag{1}$$

The following assumption is made throughout the paper.

Assumption 1 (Well-posedness): The Lurie system (1) with $\Phi(\cdot)$ being the repeated ReLU is well-posed.

Well-posedness is equivalent to the existence of a unique solution to the state space equations (1). Since $\Phi(\cdot)$ is globally Lipschitz (and differentiable almost everywhere), this is ensured if there exists a unique solution $y(\eta)$ to $F(y) := y - D\Phi(y) = \eta$. A sufficient condition for this is given by Lemma 1 below, adapted from [28, Section II].

Lemma 1: Assumption 1 holds if there exists $\mathbf{U} \in \mathcal{D}^m_+$ such that:

$$2\mathbf{U} - \mathbf{U}D - D'\mathbf{U} \succ 0 \tag{2}$$

In many absolute stability results (e.g., the standard Circle Criterion and Zames-Falb multipliers) the LMI (2) is an intrinsic part of the stability conditions and thus well-posedness is guaranteed. For the new results presented in this paper, this is not the case, so well-posedness needs to be verified by other means. Generally, this involves verifying (2) directly. Fortunately, this is straightforward for the case of many NN stability problems. For example, in an L-layer feedforward NN, the D-matrix in (1) takes the following form [5]:

$$D = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ W_1 & 0 & \dots & 0 & 0 \\ 0 & W_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & W_{L-1} & 0 \end{bmatrix} \in \Re^{m\times m} \tag{3}$$

where $W_i \in \Re^{m_i \times m_i}$ are weighting matrices. In this case, inequality (2) becomes:

$$\begin{bmatrix} 2\mathbf{U}_1 & -W_1'\mathbf{U}_2 & 0 & \dots & 0 & 0 \\ -\mathbf{U}_2 W_1 & 2\mathbf{U}_2 & -W_2'\mathbf{U}_3 & \dots & 0 & 0 \\ 0 & -\mathbf{U}_3 W_2 & 2\mathbf{U}_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2\mathbf{U}_{L-1} & -W_{L-1}'\mathbf{U}_L \\ 0 & 0 & 0 & \dots & -\mathbf{U}_L W_{L-1} & 2\mathbf{U}_L \end{bmatrix} \succ 0 \tag{4}$$

where $\mathbf{U}_i \in \mathcal{D}^{m_i}_+$ and $\sum_{i=1}^{L} m_i = m$. Hence, by a Schur complement argument (see e.g. [17]), for an arbitrary choice of $\mathbf{U}_L \in \mathcal{D}^{m_L}_+$, one can always choose $\mathbf{U}_{L-1} \in \mathcal{D}^{m_{L-1}}_+$ such that the lower-right block matrix is positive definite; the remaining $\mathbf{U}_i$ can then be chosen recursively *mutatis mutandis*. Thus, well-posedness for this class of Lurie systems is unconditionally guaranteed.

Similarly, for continuous time RNNs, the system (1) often has the form $A = -I$, $B = W$, $C = I$, $D = 0$ [7] and hence is, trivially, well-posed. A detailed discussion of well-posedness is beyond the scope of the paper; it suffices to say that many systems featuring NNs are naturally well-posed.

The following problem is addressed in the remainder of the manuscript.

Problem 1: Find convex Lyapunov-based conditions which ensure the origin of the Lurie system (1) is globally asymptotically stable when $\Phi(\cdot)$ is the repeated ReLU.

C. Properties of the ReLU function

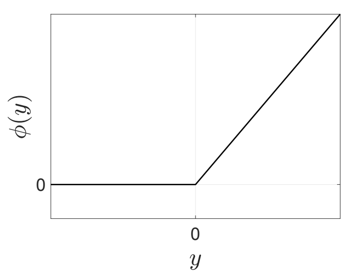

Fig. 2. ReLU function

The ReLU function is continuous over its domain. The repeated ReLU is a vectorised version of the ReLU function.

Definition 1 (ReLU function): $\phi(\cdot) : \Re \to \Re_{\geq 0}$

$$\phi(y_i) = \begin{cases} y_i & y_i \geq 0 \\ 0 & y_i < 0 \end{cases} \tag{5}$$

Definition 2 (Repeated ReLU): If $\phi(\cdot) : \Re \to \Re_{\geq 0}$ is the ReLU function, the repeated ReLU is $\Phi(\cdot) : \Re^m \to \Re^m_{\geq 0}$

$$\Phi(\cdot) = \begin{bmatrix} \phi(\cdot) \\ \vdots \\ \phi(\cdot) \end{bmatrix} \tag{6}$$

It is well known that the ReLU function satisfies a number of properties [26], [29]; some of these are summarised in Table I. Although the final two properties in Table I hold for many common static nonlinearities, the first four are less typical. In fact, the complementarity condition holds for few activation functions other than ReLU.

TABLE I
PROPERTIES OF THE RELU FUNCTION

$\phi(y_i) \geq 0$	$\forall y_i \in \Re$	positivity
$\phi(\beta y_i) = \beta\phi(y_i)$	$\forall y_i \in \Re, \beta \in \Re_{\geq 0}$	positive homogeneity
$\phi(y_i) - y_i \geq 0$	$\forall y_i \in \Re$	positive complement
$\phi(y_i)(y_i - \phi(y_i)) = 0$	$\forall y_i \in \Re$	complementarity
$0 \leq \frac{\phi(y_i)}{y_i} \leq 1$	$\forall y_i \in \Re$	sector-boundedness
$0 \leq \frac{\phi(y_i)-\phi(\tilde{y}_i)}{y_i - \tilde{y}_i} \leq 1$	$\forall y_i, \tilde{y}_i \neq y_i \in \Re$	slope-restriction

III. QUADRATIC CONSTRAINTS

This section derives two tailored QCs for the repeated ReLU, using properties from Table I. In the next section, the presented QCs are leveraged to prove the strengthened criteria.

Fact 1 (Sector-like QC): Let $\Phi(\cdot) : \Re^m \to \Re^m_{\geq 0}$ be the repeated ReLU. If $\mathbf{V} \in \mathcal{Z}^m$ then the following QC holds:

$$\Phi(y)'\mathbf{V}[y - \Phi(y)] \geq 0 \quad \forall y \in \Re^m \tag{7}$$

Proof: By scaling the product of the positivity and positive complement properties, inequality (8) follows. When $i = j$ the less restrictive equation (9) is implied.

$$\phi(y_i)v_{ij}(y_j - \phi(y_j)) \geq 0 \quad \forall y_i, y_j \in \Re \quad \text{and} \quad v_{ij} \leq 0 \tag{8}$$

$$\phi(y_i)v_{ii}(y_i - \phi(y_i)) = 0 \quad \forall y_i \in \Re \quad \text{and} \quad v_{ii} \in \Re \tag{9}$$

Summing these inequalities yields:

$$\sum_{i=1}^{m}\sum_{\substack{j=1\\ j\neq i}}^{m} \phi(y_i)v_{ij}(y_j - \phi(y_j)) + \sum_{i=1}^{m} \phi(y_i)v_{ii}(y_i - \phi(y_i)) \geq 0 \tag{10}$$

Majorizing this expression leads to (7) with $\mathbf{V} = [v_{ij}]$. □

Remark 1: The sector-like QC (7) takes the same form as the QC associated with the Sector$[0, I]$, which holds for more general static nonlinearities, when $\mathbf{V} \in \mathcal{D}^m_+$. It should be observed that the sector-like QC introduces considerably more freedom in the choice of $\mathbf{V}$. □□

Fact 2 (Slope restricted QC): Let $\Phi(\cdot) : \Re^m \to \Re^m_{\geq 0}$ be the repeated ReLU and $\Psi(y, \tilde{y}) := \Phi(y) - \Phi(\tilde{y})$. If $\mathbf{W} \in \mathcal{D}^m_+$ then the following QC is satisfied:

$$\Psi(y, \tilde{y})'\mathbf{W}[y - \tilde{y} - \Psi(y, \tilde{y})] \geq 0 \quad \forall y, \tilde{y} \in \Re^m \tag{11}$$

Proof: The slope-restricted QC is well known (e.g. [11], [26]) but can be derived for the ReLU function by simply noting that $\text{sign}\{y_i - \tilde{y}_i\} = \text{sign}\{\phi(y_i) - \phi(\tilde{y}_i)\}$ and $\text{sign}\{y_i - \tilde{y}_i\} = \text{sign}\{y_i - \tilde{y}_i - (\phi(y_i) - \phi(\tilde{y}_i))\}$. □

Fact 3 (Positivity QC): Let $\Phi(\cdot) : \Re^m \to \Re^m_{\geq 0}$ be the repeated ReLU. If $\mathbf{Q}_{11}, \mathbf{Q}_{12}, \mathbf{Q}_{21}, \mathbf{Q}_{22} \in \Re^{m\times m}_{\geq 0}$ then the following QC holds:

$$\begin{bmatrix} \Phi(y) \\ \Phi(y) - y \end{bmatrix}' \begin{bmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} \end{bmatrix} \begin{bmatrix} \Phi(\tilde{y}) \\ \Phi(\tilde{y}) - \tilde{y} \end{bmatrix} \geq 0 \quad \forall y, \tilde{y} \in \Re^m \tag{12}$$

Proof: The positivity QC is derived by scaling the product of the positivity and positive complement properties with different arguments. Thus, for any $q_{11,ij}, q_{12,ij}, q_{21,ij}, q_{22,ij} \in \Re_{\geq 0}$ it follows that:

$$\phi(y_i)q_{11,ij}\phi(\tilde{y}_j) \geq 0 \quad \forall y_i, \tilde{y}_j \in \Re \tag{13}$$

$$\phi(y_i)q_{12,ij}(\phi(\tilde{y}_j) - \tilde{y}_j) \geq 0 \quad \forall y_i, \tilde{y}_j \in \Re \tag{14}$$

$$(\phi(y_i) - y_i)q_{21,ij}\phi(\tilde{y}_j) \geq 0 \quad \forall y_i, \tilde{y}_j \in \Re \tag{15}$$

$$(\phi(y_i) - y_i)q_{22,ij}(\phi(\tilde{y}_j) - \tilde{y}_j) \geq 0 \quad \forall y_i, \tilde{y}_j \in \Re \tag{16}$$

Summing over these inequalities and majorizing the resulting expressions leads to (12) with $\mathbf{Q}_k = [q_{k,ij}]$ for $k = \{11, 12, 21, 22\}$. □

Remark 2: It is clear that the positivity QC (12) contains four individual QCs. In particular, (17) is the majorized expression of (13) included as the top left element of (12).

$$\Phi(y)'\mathbf{Q}_{11}\Phi(\tilde{y}) \geq 0 \quad \forall y, \tilde{y} \in \Re^m \tag{17}$$

Also, note that the positivity QC (12) holds for $\tilde{y} = y$. Two extra QCs (18), (19) can be extracted from this case; the other two possible QCs are redundant since they would take the same form as the sector-like QC, but with matrices from a more constrained set.

$$\Phi(y)'\mathbf{Q}_{11}\Phi(y) \geq 0 \quad \forall y \in \Re^m \tag{18}$$

$$[\Phi(y) - y]'\mathbf{Q}_{22}[\Phi(y) - y] \geq 0 \quad \forall y \in \Re^m \tag{19}$$

Later in the paper, it will become clear that inequality (12) in its entirety is rather unwieldy and the special cases (17), (18) are easier to apply. □□

IV. MAIN RESULTS

This section derives the strengthened stability criteria. Two Lyapunov candidates are proposed and the tailored QCs are used to derive matrix inequalities which, by the Barbashin-Krasovskii Theorem [8], are sufficient to verify the origin of the specialised Lurie system (1) is globally asymptotically stable (GAS). The first Lyapunov candidate has a quadratic form, as in the Circle Criterion, and the second is of the Lurie-type, as in the Popov Criterion. Due to the integral term in the Lurie-type Lyapunov candidate, the strengthened Popov Criterion may only be applied to systems with $D = 0$, as is the case with the standard Popov Criterion.

Theorem 1 (Circle-like Criterion): Consider the Lurie system (1) with $\Phi(\cdot)$ the repeated ReLU. Let Assumption 1 be satisfied. If there exists $\mathbf{P} \in \mathcal{S}^n_+$, $\mathbf{V} \in \mathcal{Z}^m$, $\mathbf{Q}_{11} \in \Re^{m\times m}_{\geq 0}$ such that:

$$\begin{bmatrix} He(A'\mathbf{P}) & \mathbf{P}B + C'\mathbf{V}' \\ \star & He(\mathbf{Q}_{11} - \mathbf{V}(I - D)) \end{bmatrix} \prec 0 \tag{20}$$

then the origin of (1) is GAS.

Proof: The quadratic Lyapunov candidate (21) is radially unbounded and satisfies $V_c(x) > 0 \;\; \forall x \neq 0$ if $\mathbf{P} \in \mathcal{S}^n_+$.

$$V_c(x) = x'\mathbf{P}x \tag{21}$$

Since $\Phi(\cdot)$ is the repeated ReLU, QCs (7) and (18) are satisfied. Appending these to the time derivative of $V_c(\cdot)$ leads to (22).

$$\dot{V}_c(x) \leq \dot{x}'\mathbf{P}x + x'\mathbf{P}\dot{x} + 2\Phi(y)'\mathbf{V}[y - \Phi(y)] + 2\Phi(y)'\mathbf{Q}_{11}\Phi(y) \tag{22}$$

Subbing in (1) and putting into quadratic form results in (23). Therefore, $\dot{V}_c(x) < 0 \;\; \forall x \neq 0$ if (20) holds.

$$\dot{V}_c(x) \leq \begin{bmatrix} x \\ \Phi \end{bmatrix}' \begin{bmatrix} He(A'\mathbf{P}) & \mathbf{P}B + C'\mathbf{V}' \\ \star & He(\mathbf{Q}_{11} - \mathbf{V}(I - D)) \end{bmatrix} \begin{bmatrix} x \\ \Phi \end{bmatrix} \tag{23}$$

□

Remark 3: The Circle-like Criterion is a specialisation of the Circle Criterion when $\Phi(\cdot)$ is the repeated ReLU. Since the solution space of the Circle Criterion is a subset of (20)

$$F(\cdot)=\begin{bmatrix} He(A'\mathbf{P}) & \mathbf{P}B+C'\mathbf{V}'+A'C'\mathbf{H}'\mathbf{\Lambda} & A'C'\mathbf{H}'\mathbf{\Lambda}+C'(\mathbf{H}'-I)\mathbf{W} \\ \star & He(\tilde{\mathbf{Q}}_{11}+\mathbf{Q}_{11}-\mathbf{V}+\mathbf{\Lambda}\mathbf{H}CB) & B'C'\mathbf{H}'\mathbf{\Lambda}+\tilde{\mathbf{Q}}_{11} \\ \star & \star & -2\mathbf{W} \end{bmatrix} \tag{25}$$

when $\mathbf{V}\in\mathcal{Z}^m$, $\mathbf{Q}_{11}\in\Re^{m\times m}_{\geq 0}$ are reduced to $\mathbf{V}\in\mathcal{D}^m_+$, $\mathbf{Q}_{11}=0$, one expects Theorem 1 to be less conservative than the Circle Criterion. □□

Theorem 2 (Popov-like Criterion): Consider the Lurie system (1) with $\Phi(\cdot)$ the repeated ReLU and let $D=0$. If there exists $\mathbf{P}\in\mathcal{S}^n_+$; $\mathbf{H}\in\Re^{m\times m}$; $\mathbf{\Lambda},\mathbf{W}\in\mathcal{D}^m_+$; $\mathbf{V}\in\mathcal{Z}^m$; $\mathbf{Q}_{11},\tilde{\mathbf{Q}}_{11}\in\Re^{m\times m}_{\geq 0}$ such that:

$$F(\mathbf{P},\mathbf{H},\mathbf{\Lambda},\mathbf{W},\mathbf{V},\mathbf{Q}_{11},\tilde{\mathbf{Q}}_{11})\prec 0 \tag{24}$$

where $F(\cdot)$ is defined in (25), then the origin of (1) is GAS.

Proof: The following Lyapunov candidate is considered:

$$V_p(x)=x'\mathbf{P}x+2\int_0^{\mathbf{H}y}\mathbf{\Lambda}\Phi(\sigma)\cdot d\sigma \tag{26}$$

$$=x'\mathbf{P}x+\Phi(\mathbf{H}y)'\mathbf{\Lambda}\Phi(\mathbf{H}y) \tag{27}$$

This is a more general version of the standard Lurie-type Lyapunov candidate since the line integral features an additional, *unstructured* matrix $\mathbf{H}\in\Re^{m\times m}$, to be determined.

A necessary and sufficient condition for the line integral to be path independent is the existence of a scalar function of which the integrand is the gradient [30]. Since $\Phi(\cdot)$ is the repeated ReLU, the scalar function $\theta(\cdot)$ can be chosen as:

$$\theta(\sigma)=\Phi(\sigma)'\mathbf{\Lambda}\Phi(\sigma) \tag{28}$$

The gradient of θ can be calculated as:

$$\nabla_\sigma\big(\theta(\sigma)\big)=2\frac{\partial\Phi}{\partial\sigma}\mathbf{\Lambda}\Phi(\sigma) \tag{29}$$

where $$\frac{\partial\Phi}{\partial\sigma}=\mathrm{diag}\Big(\frac{\partial\phi(\sigma_1)}{\partial\sigma_1},\dots,\frac{\partial\phi(\sigma_m)}{\partial\sigma_m}\Big)$$

Since $\frac{\partial\Phi}{\partial\sigma}$ and $\mathbf{\Lambda}$ are both diagonal it follows that:

$$\nabla_\sigma\big(\theta(\sigma)\big)=2\mathbf{\Lambda}\frac{\partial\Phi}{\partial\sigma}\Phi(\sigma)=2\mathbf{\Lambda}\Phi(\sigma)\quad\text{a.e.} \tag{30}$$

The final equality holds because $\frac{\partial\phi_i(\sigma_i)}{\partial\sigma_i}\phi_i(\sigma_i)=\phi_i(\sigma_i)$ almost everywhere (to see this set $\sigma_i<0$ and then $\sigma_i\geq 0$). Therefore, by the Gradient Theorem, the Lyapunov candidate (26) can equally be expressed as (27). This is clearly radially unbounded and satisfies $V_p(x)>0\ \ \forall x\neq 0$, if $\mathbf{P}\in\mathcal{S}^n_+$ and $\mathbf{\Lambda}\in\mathcal{D}^m_+$. Furthermore, it is *independent of the choice of* $\mathbf{H}$.

The time derivative of $V_p(\cdot)$ is given by (31)-(33) where, for convenience, $\tilde{y}:=\mathbf{H}y$.

$$\dot{V}_p(x)=\dot{x}'\mathbf{P}x+x'\mathbf{P}\dot{x}+\nabla_{\tilde{y}}\Big(\theta(\tilde{y})\Big)\cdot\frac{\partial\tilde{y}}{\partial x}\dot{x} \tag{31}$$

$$=\dot{x}'\mathbf{P}x+x'\mathbf{P}\dot{x}+(2\mathbf{\Lambda}\Phi(\tilde{y}))'\mathbf{H}C\dot{x} \tag{32}$$

$$=2\Big(x'\mathbf{P}+\Phi(\tilde{y})'\mathbf{\Lambda}\mathbf{H}C\Big)(Ax+B\Phi(y)) \tag{33}$$

The final equality features the repeated ReLU with two different arguments. However, $\Phi(\tilde{y})$ can be expressed as:

$$\Phi(\tilde{y})=\underbrace{\Phi(\tilde{y})-\Phi(y)}_{=:\Psi(y)}+\Phi(y) \tag{34}$$

Both the sector-like QC (7) and the slope-restricted QC (11) can be appended to equation (33) to give:

$$\dot{V}_p(x)\leq 2\Big(x'\mathbf{P}+\Phi(\tilde{y})'\mathbf{\Lambda}\mathbf{H}C\Big)(Ax+B\Phi(y))$$
$$+2\Phi(y)'\mathbf{V}[y-\Phi(y)]+2\Psi(y)'\mathbf{W}[\tilde{y}-y-\Psi(y)] \tag{35}$$

Using (34) and $\tilde{\mathbf{Q}}_{11}\in\Re^{m\times m}_{\geq 0}$ the positivity QC (17) can be re-written as:

$$\Phi(y)'\tilde{\mathbf{Q}}_{11}\Big(\Psi(y)+\Phi(y)\Big)\geq 0\quad\forall y,\tilde{y}\in\Re^m \tag{36}$$

Appending this and QC (18) to (35) gives:

$$\dot{V}_p(x)\leq 2\Big(x'\mathbf{P}+\Phi(\tilde{y})'\mathbf{\Lambda}\mathbf{H}C\Big)(Ax+B\Phi(y))$$
$$+2\Phi(y)'\mathbf{V}[y-\Phi(y)]+2\Psi(y)'\mathbf{W}[\tilde{y}-y-\Psi(y)]$$
$$+2\Phi(y)'\tilde{\mathbf{Q}}_{11}[\Psi(y)+\Phi(y)]+2\Phi(y)'\mathbf{Q}_{11}\Phi(y) \tag{37}$$

This can be majorized to get:

$$\dot{V}_p(x)\leq\begin{bmatrix}x\\\Phi\\\Psi\end{bmatrix}'F(\cdot)\begin{bmatrix}x\\\Phi\\\Psi\end{bmatrix} \tag{38}$$

where $F(\cdot)$ is given by (25). □

Remark 4: The Popov-like Criterion is a specialisation of the Popov Criterion when $\Phi(\cdot)$ is the repeated ReLU. Since the solution space of the Popov Criterion is a subset of (24) when $\mathbf{V}\in\mathcal{Z}^m$, $\mathbf{H}\in\Re^{m\times m}$, $\mathbf{W}\in\mathcal{D}^m_+$, $\mathbf{Q}_{11},\tilde{\mathbf{Q}}_{11}\in\Re^{m\times m}_{\geq 0}$ are reduced to $\mathbf{V}\in\mathcal{D}^m_+$, $\mathbf{H}=I$, $\mathbf{W}=\mathbf{Q}_{11}=\tilde{\mathbf{Q}}_{11}=0$, one expects Theorem 2 to be less conservative than the standard Popov Criterion. □□

Remark 5: Although the Popov-like Criterion is less conservative than the Popov Criterion, the arising matrix inequality (24) is *not linear* in the matrix variables. □□

Remark 6: To further tailor the stability analysis to the repeated ReLU, the positivity QC (12) could have replaced the special case positivity QC (17) in (37). However, this would have introduced an additional nonlinear term to (25) and increased the complexity of the matrix inequality. □□

Remark 7: The Circle-like Criterion can be recovered from the Popov-like Criterion. As the solution space of (20) is a subset of (24) when $\mathbf{H}\in\Re^{m\times m}$, $\mathbf{\Lambda}\in\mathcal{D}^m_+$, $\tilde{\mathbf{Q}}_{11}\in\Re^{m\times m}_{\geq 0}$ are reduced to $\mathbf{H}=I$, $\mathbf{\Lambda}=\tilde{\mathbf{Q}}_{11}=0$, one expects Theorem 2 to be less conservative than Theorem 1. The trade-off for this reduced conservatism is increased complexity. □□

V. Convex relaxations

Inequality (24) is a bilinear matrix inequality (BMI) which is difficult to convexify. Consequently, two convex relaxations are described below, which enable the matrix inequality to be expressed as an LMI. The approaches suggested below make specific choices for certain variables; however, less conservative relaxations may exist.

A. Specific choice of $\mathbf{\Lambda}$ and $\mathbf{W}$

Since $\mathbf{\Lambda}$ always appears in a product with $\mathbf{H}$, the choice $\mathbf{\Lambda}=I$ may be made without loss of generality. However, some conservatism is introduced since $\mathbf{H}$ appears in a product with $\mathbf{W}$ in the upper right element, without $\mathbf{\Lambda}$. As the only other appearance of $\mathbf{W}$ is in the lower right element, the choice $\mathbf{W} = \eta I$ is made, with the choice of η guided by the Schur complement conditions needed to satisfy (24) (see e.g. [17]).

Corollary 1 (Relaxed Popov-like Criterion 1): Consider the Lurie system (1) with $\Phi(\cdot)$ being the repeated ReLU and $D = 0$. If there exists $\mathbf{P} \in \mathcal{S}_+^n$; $\mathbf{H} \in \Re^{m\times m}$; $\mathbf{V} \in \mathcal{Z}^m$; $\mathbf{Q}_{11}, \tilde{\mathbf{Q}}_{11} \in \Re_{\geq 0}^{m\times m}$ such that:

$$F(\mathbf{P}, \mathbf{H}, I, \eta I, \mathbf{V}, \mathbf{Q}_{11}, \tilde{\mathbf{Q}}_{11}) \prec 0 \tag{39}$$

where $F(\cdot)$ is defined in (25), then the origin of (1) is GAS.

The advantage of this corollary, over the one presented below, is that it makes use of $\mathbf{H}$ being a *full matrix*. The disadvantage is that restrictive choices for $\mathbf{\Lambda}$ and $\mathbf{W}$ have been made which reduce the solution space.

B. Specific choice of H

If the restriction $\mathbf{H} \in \mathcal{D}_+^m$ is made, the product of $\mathbf{H}$ and $\mathbf{\Lambda}$ will always be a member of $\mathcal{D}_+^m$. Therefore, the product of $\mathbf{H}$ and $\mathbf{\Lambda}$ may equivalently be represented by the matrix $\mathbf{\Lambda} \in \mathcal{D}_+^m$; $\mathbf{H} = I$ may be chosen without loss of generality.

Corollary 2 (Relaxed Popov-like Criterion 2): Consider the Lurie system (1) with $\Phi(\cdot)$ being the repeated ReLU and $D = 0$. If there exists $\mathbf{P} \in \mathcal{S}_+^n$, $\mathbf{\Lambda} \in \mathcal{D}_+^m$, $\mathbf{V} \in \mathcal{Z}^m$, $\mathbf{Q}_{11} \in \Re_{\geq 0}^{m\times m}$ such that:

$$\begin{bmatrix} He(A'\mathbf{P}) & \mathbf{P}B + C'\mathbf{V}' + A'C'\mathbf{\Lambda} \\ \star & He(\mathbf{Q}_{11} + \mathbf{\Lambda}CB - \mathbf{V}) \end{bmatrix} \prec 0 \tag{40}$$

then the origin of (1) is GAS.

Proof: By setting $\mathbf{H} = I$ the matrix $F(\cdot)$ in (25) collapses to a 2 by 2 block matrix. This is a result of $\Phi(\tilde{y}) = \Phi(y) \Rightarrow \Psi(y) = 0$, which reduces equation (37) to:

$$\begin{aligned} \dot{V}_p(x) &\leq 2\Big(x'\mathbf{P} + \Phi(y)'\mathbf{\Lambda}C\Big)(Ax + B\Phi(y)) \\ &+ 2\Phi(y)'\mathbf{V}[y - \Phi(y)] + 2\Phi(y)'\Big(\tilde{\mathbf{Q}}_{11} + \mathbf{Q}_{11}\Big)\Phi(y) \end{aligned} \tag{41}$$

Since $\tilde{\mathbf{Q}}_{11} + \mathbf{Q}_{11} \in \Re_{\geq 0}^{m\times m}$, $\tilde{\mathbf{Q}}_{11}$ may be set as $\tilde{\mathbf{Q}}_{11} = 0$ without loss of generality. Putting (41) into quadratic form shows (38) has been relaxed to:

$$\dot{V}_p(x) \leq \begin{bmatrix} x \\ \Phi \end{bmatrix}' \begin{bmatrix} He(A'\mathbf{P}) & \mathbf{P}B + C'\mathbf{V}' + A'C'\mathbf{\Lambda} \\ \star & He(\mathbf{Q}_{11} + \mathbf{\Lambda}CB - \mathbf{V}) \end{bmatrix} \begin{bmatrix} x \\ \Phi \end{bmatrix} \tag{42}$$

The advantage of this corollary, over the one above, is that it has lower complexity and does not require any parameters to be chosen. Clearly, the disadvantage is that the flexibility introduced by $\mathbf{H} \in \Re^{m\times m}$ has been completely removed.

VI. Numerical Examples

The maximum series gain (also known as the maximum sector/slope size) was used to compare the conservatism of the criteria in this paper against criteria with low (Circle and Popov [8]), medium (Park [11]) and high (Zames-Falb [31]) complexity. The Projective method [35] is implemented in Matlab to pose such stability criteria as SDP problems involving LMIs and to solve them efficiently. As the number of floating point operations per iteration is proportional to N^3, the total number of decision variables, N, was used to compare the complexity of the criteria.

A. Experimental setup

The setup involved the replacement $\Phi(y) \to \alpha\Phi(y)$ in (1) where $\alpha \in \Re_{\geq 0}$. It is clear from (1) that this is equivalent to replacing $B \to \alpha B$ and $D \to \alpha D$ in the LMIs of each criterion. The maximum series gain is the largest α for which each criterion can certify the origin of (1) is GAS.

Table II gives a list of the examples and values (n, m) where $x \in \Re^n$ and $\Phi(\cdot) : \Re^m \to \Re^m$. Each example is a different state space model (A, B, C, D) from the literature, where $D = 0$ was used to allow comparison with the Popov and Popov-like Criteria. The examples used can be found by referring to the references or looking at the related code[1].

For each example, Corollary 1 was applied with $\eta = 10,000$. The diagonal matrices of the Zames-Falb method varied between examples and were often set to the identity due to lack of guidance in selection. Naturally, different choices of η and the Zames-Falb parameters may lead to different results. Furthermore, to reduce computation time of the Zames-Falb method, the maximum α from the Circle Criterion was used as the initial lower bound of the α search.

B. Discussion

Table II presents the maximum series gain and the number of decision variables for each criterion. The following observations were noted:

- Theorem 1 and Corollary 2 were of similar complexity, slightly higher than Popov. Corollary 1 had higher complexity than Corollary 2 and Park. Zames-Falb had notably higher complexity than Corollary 1, excluding Examples 3 and 8.
- Example 8: a positive system for which the multivariable Aizerman Conjecture holds [32]. In this case, all methods achieved the linear upper bound on α.
- Corollary 2 is of equal or less conservatism than all existing criteria in 7 out of 8 examples. Corollary 2 strikes a very appealing balance of reduced conservatism and complexity compared to existing criteria. Corollary 1 is less conservative than Corollary 2 in Examples 1 and 3; however, the balance of reduced conservatism and complexity is less appealing.
- Theorem 1 is of equal or less conservatism than all existing criteria in 75% of examples, although still inferior to Corollary 2. Theorem 1 also strikes an appealing balance between reduced conservatism and complexity. An advantage over Corollary 2 is that it can be applied when $D \neq 0$.
- Corollary 1 is more conservative than Popov in Examples 2 and 5. This highlights the effect of the convex relaxation to Theorem 2. This effect is not observed for Corollary 2.

A limitation of this setup is that $m \leq 5$. In absolute stability analysis, this is regarded as a high dimensional nonlinearity, but not in the NN literature. Nevertheless, since Theorem 1 and Corollary 2 are less conservative than all existing criteria in most examples, this gives one confidence that the strengthened criteria should perform well when used in cases where m may be much larger. Furthermore, both criteria scale much better than the Park or Zames-Falb approaches, which are the "next best" criteria.

[1] https://github.com/CR-Richardson/Max-Series-Gain

TABLE II

COMPARISON OF THE MAXIMUM SERIES GAIN AND THE NUMBER OF DECISION VARIABLES FOR VARIOUS CRITERIA

				Maximum series gain (left) and number of decision variables (right)													
Ex	n	m	Source	Circle [8]		Theorem 1		Popov [8]		Corollary 1		Corollary 2		Park [11]		Zames-Falb [31]	
1	9	3	[11] Ex. 3	20.8770	48	39.5246	63	434.3034	51	52430.1224	81	9990.0000	66	448.7543	87	435.8305	252
2	3	3	[32] Ex. 3	89.9025	9	89.9025	24	89.9025	12	89.8981	42	89.9106	27	89.9025	30	89.9025	45
3	3	4	[33] Ex. 4.9	0.5236	10	0.6818	38	0.5236	14	0.6819	70	0.6818	42	0.5236	40	0.5526	48
4	8	4	[34] Ex. 22	0.0010	40	0.0012	68	0.0010	44	0.0012	100	0.0015	72	0.0010	90	0.0010	208
5	6	4	[34] Ex. 17	0.0813	25	0.0814	53	0.0824	29	0.0814	85	0.0830	57	0.0845	67	0.0845	129
6	6	4	[34] Ex. 19	0.1946	25	0.3901	53	0.1947	29	0.4158	85	0.5048	57	0.2266	67	0.2785	129
7	8	4	[34] Ex. 23	0.0966	40	0.1232	68	0.0968	44	0.1232	100	0.1462	72	0.1035	90	0.1040	208
8	5	5	[32] Ex. 2	2.0221	20	2.0221	65	2.0221	25	2.0221	115	2.0221	70	2.0221	70	2.0221	100

VII. CONCLUSION

This paper has proposed strengthened Circle and Popov Criteria for the analysis of Lurie systems with repeated ReLU nonlinearities. The criteria are built upon new, tailored quadratic constraints which have been derived for the ReLU function. The new criteria have, potentially, much lower levels of conservatism than the standard Circle and Popov Criteria whilst retaining much of their computational appeal. This appeal has been demonstrated with numerical examples, where Corollary 2 in particular, provides low levels of conservatism without the high computational overhead of approaches such as Zames-Falb analysis. The main deficiency of the new results is their limitation to the ReLU nonlinearity. Despite this, it is hoped that these results may help bring NN based control into the domain of safety critical systems.

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