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Exponential Stability of Continuous-Time Piecewise Affine Systems

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***Abstract*— This work addresses the problem of global exponential stability analysis of the origin of continuous-time Continuous Piecewise Affine (CPWA) systems. The stability analysis in this paper considers Piecewise Quadratic (PWQ) Lyapunov Functions (LF) and a ramp-based implicit representation of PWA systems. Sufficient convex stability conditions are obtained in the form of a Semidefinite Programming (SDP) problem. Two major benefits arise from the proposed results: i) the need for equality constraints to ensure the continuity of the LF across the boundaries of the sets of the partition is withdrawn; ii) there is no need to consider separate SDP conditions for each set of the partition, which simplifies the application of the conditions. The effectiveness of the proposed method is illustrated in numerical examples.**

I. INTRODUCTION

Piecewise Affine (PWA) systems are obtained by partitioning the state space and associating each set of the partition with an affine dynamic equation. They are useful, for instance, in modeling nonlinear circuits [14], mechanical systems [21], and systems subject to input saturation [6]. Moreover, they are equivalent to some other classes of hybrid systems [7] and can also be used to approximate some nonlinear dynamics [5]. For continuous-time systems, one possible model with polyhedral partition is the *explicit representation* [15]

$$\dot{x} = F_i x + f_i \qquad \forall x \in \Gamma_i \subseteq \mathbb{R}^n \tag{1a}$$

$$\Gamma_i = \{x \in \mathbb{R}^n \mid \Theta_i x \succeq \theta_i\}, \tag{1b}$$

for $i \in \{1, \dots, N_\Gamma\}$, where N_Γ is the total number of sets of the state space partition. In (1), for each set Γ_i, $F_i \in \mathbb{R}^{n\times n}$ and $f_i \in \mathbb{R}^n$ define an affine dynamics while $\Theta_i \in \mathbb{R}^{\ell_i \times n}$ and $\theta_i \in \mathbb{R}^{\ell_i}$ define the sets of the polyhedral *partition*, that is, $\cup_i \Gamma_i = \mathbb{R}^n$ and $\mathrm{int}(\Gamma_i) \cap \mathrm{int}(\Gamma_j) = \emptyset \ \forall i, j \in \{1, \dots, N_\Gamma\}$, $i \neq j$.

When the explicit representation (1) is used to analyze the Global Exponential Stability (GES) of the origin of PWA systems, Piecewise Quadratic (PWQ) Lyapunov Function (LF) candidates are often considered, as in [8], [9], and [11]. Specifically, [9] introduced Linear Matrix Inequality (LMI) conditions to assess the GES of the origin of PWA systems. Local results were later proposed by [11]. Moreover, such conditions can be relaxed using cone-copositivity tests proposed in [8].

However, using the explicit representation (1) has two drawbacks. When parameterizing PWQ LF candidates, additional equality constraints are required to ensure continuity at the boundaries of the partition sets [8], [9], [11]. Moreover, the stability conditions are written for each set and must distinguish whether or not the set contains the origin, resulting in convoluted numerical implementation of these conditions [9], [11]. To avoid these drawbacks, we consider the following *ramp-based implicit representation*, proposed in [6]

$$\dot{x} = Ax + B\phi(y(x)) \tag{2a}$$

$$y(x) = Cx + D\phi(y(x)) + e, \tag{2b}$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{m\times m}$, and $e \in \mathbb{R}^m$, where $\phi : \mathbb{R}^m \to \mathbb{R}^m$ is the vector-valued ramp function, defined elementwise in terms of the scalar ramp function $r : \mathbb{R} \to \mathbb{R}$ for $j \in \{1, \dots, m\}$ as

$$\phi_{(j)}(y(x)) = r(y_{(j)}(x)) = \begin{cases} 0 & \text{if } \ y_{(j)}(x) \leq 0 \\ y_{(j)}(x) & \text{if } \ y_{(j)}(x) > 0. \end{cases} \tag{3}$$

The relation between the representations (1) and (2) can be found in [2], where the stability of uncertain continuous-time PWA systems was analyzed with quadratic LFs. The stability analysis of discrete-time PWA systems was reported in [6], where the proposed methods outperformed methods based on the representation (1) in numerical examples.

In this paper, we use the ramp-based implicit representation (2) to formulate LMI conditions for the GES of the origin of continuous-time Continuous PWA (CPWA) systems considering PWQ LF candidates. The use of the implicit representation yields directly the continuity of the LFs, which follows from the continuity of the ramps and the well-posedness of an algebraic loop similar to (2b). Moreover, it also allows the formulation of the stability conditions as a reduced set of LMIs. Differently from [6], we consider here the more general case where the partition of the PWQ LF can differ from the partition of the system dynamics, and we also introduce copositive matrices in the stability conditions, instead of elementwise nonnegative ones.

The paper is structured as follows. Section II states the problem under consideration. In Section III, the properties of the ramp function are recalled and an SDP test to verify the positive semidefiniteness of PWQ functions is provided.

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Furthermore, the concept of almost everywhere differentiability presented in [3], [4], [10], [13], [19] is extended to PWA systems. In Section IV, the main result of this work is presented, with examples of its application given in Section V. Finally, Section VI provides final remarks.

Notation: For a matrix M, we denote $\mathbb{S}^n = \{M \in \mathbb{R}^{n\times n} \mid M = M^\top\}$, $\mathbb{D}^n = \{M \in \mathbb{S}^n \mid M_{(i,j)} = 0\ \forall i \neq j\}$, $\mathbb{P}^n$ is the set of copositive matrices, *i.e.*, $\mathbb{P}^n = \{M \in \mathbb{S}^n \mid v^\top M v \geq 0\ \forall v \succeq 0\}$, $\mathrm{He}\{M\} := M + M^\top$ and $\mathrm{diag}(M_1, M_2)$ is the block diagonal matrix composed by M_1 and M_2. $1_m \in \mathbb{R}^m$ represents a vector with each element equal to 1.

II. Problem Statement

Consider a continuous-time system as in (2). Let the system trajectories be denoted as $x(t, x_0)$, for $t \geq 0$, where $x(0, x_0) = x_0$. The following assumption is made.

Assumption 1. *The origin is an equilibrium, which imposes* $B\phi(y(0)) = 0$.

Note that, regarding the explicit representation (1), Assumption 1 implies that $f_i = 0\ \forall i$ such that $0 \in \Gamma_i$ (either in its interior or on its boundary).

The completeness of the trajectories does not follow directly from the representation (2) since (2b) defines an algebraic loop which, in some cases, may not be well-posed. This could render the vector field not defined for some sets of the state space, thus preventing the solution interval from being $t \in \mathbb{R}_{\geq 0}$ for some initial condition $x_0 \in \mathbb{R}^n$. To ensure the completeness of the trajectories, we consider the following assumption.

Assumption 2 (Global well-posedness of the algebraic loop)**.** *The algebraic loop* (2b) *is well-posed, that is, for any* $x \in \mathbb{R}^n$ *the solution* $y(x)$ *of* (2b) *exists and is unique.*

From the well-posedness of (2b) we have the continuity of y on x, which is a PWA function, therefore globally Lipschitz, thus guaranteeing the completeness and continuity of solutions. Moreover, as a consequence of Assumption 2, the dynamics of the PWA system (2) are continuous across the boundaries of the partition, that is, (2) is a Continuous PWA (CPWA) system. Therefore, no sliding modes can occur [18], and the solutions of (2) are in the Carathéodory sense. To test whether Assumption 2 holds, a sufficient condition is given in the following result from [6], [20].

Proposition 1 (Test for the well-posedness of (2b))**.** *If there exists a matrix* $W \in \mathbb{D}^m$ *such that*

$$W > 0, \quad -2W + He\{WD\} < 0,$$

then (2b) *is well-posed.*

The stability analysis problem considered in this work is stated as follows:

Problem 1 (Stability Analysis)**.** *Given a CPWA system* (2)*, determine whether the origin is GES, that is, whether there exist strictly positive real numbers* κ *and* γ *such that*

$$\|x(t, x_0)\|_2 \leq \kappa \|x_0\|_2 e^{-\gamma t} \quad \forall t > 0,\ \forall x_0 \in \mathbb{R}^n.$$

Observe that the GES of the origin does not require that the matrix A in (2) to be Hurwitz. To assess the stability of the origin of system (2) we consider PWQ LF candidates defined as

$$V(x) = \begin{bmatrix} x \\ \phi(\eta(x)) \end{bmatrix}^\top \begin{bmatrix} P_1 & P_2 \\ P_2^\top & P_3 \end{bmatrix} \begin{bmatrix} x \\ \phi(\eta(x)) \end{bmatrix}, \tag{4a}$$

$$\eta(x) = C_\eta x + D_\eta \phi(\eta(x)) + e_\eta, \tag{4b}$$

with $P_1 \in \mathbb{S}^n$, $P_2 \in \mathbb{R}^{n\times q}$, $P_3 \in \mathbb{S}^q$, $C_\eta \in \mathbb{R}^{q\times n}$, $D_\eta \in \mathbb{R}^{q\times q}$, and $e_\eta \in \mathbb{R}^q$. Note that the partitions of the state space induced by (2b) and (4b) may differ. When imposing $C_\eta = C$, $D_\eta = D$, and $e_\eta = e$, the partition of the LF candidate and the system dynamics become the same, as considered in [6]. Moreover, we assume that the algebraic loop (4b) is well-posed, which can be verified using Proposition 1. From the well-posedness of (4b) and the continuity of the ramp function (3), we have that V in (4) is a continuous function.

III. Preliminaries

In this section, we recall the properties of the vector-valued ramp function, which leads to conditions for the positive semidefiniteness of PWQ functions. To address the fact that the ramp function is not differentiable, we use the results from [10], [13] showing how to obtain an expression of the derivative of the ramp function that holds almost everywhere. The results presented in this section will be instrumental in obtaining an LMI test for the Lyapunov inequalities with respect to the system (2) considering PWQ LF candidates.

A. Properties of the ramp function

Let $g \in \mathbb{R}^{n_g}$ and define the vector

$$\chi^\top(g) := \begin{bmatrix} 1 & \phi^\top(g) & (\phi(g) - g)^\top \end{bmatrix}.$$

The following two lemmas characterize the vector-valued ramp function ϕ defined in (3) (the proofs can be found in [6, Lemmas 1 and 3]):

Lemma 1. *[6] For any* $T \in \mathbb{D}^{n_g}$ *the following identity holds* $\forall g \in \mathbb{R}^{n_g}$

$$\chi^\top(g)\Psi(T)\chi(g) \equiv 0, \quad \Psi(T) := \begin{bmatrix} 0 & 0_{1,n_g} & 0_{1,n_g} \\ 0_{n_g,1} & 0_{n_g} & T \\ 0_{n_g,1} & T & 0_{n_g} \end{bmatrix}.$$

Lemma 2. *[6] For any* $M \in \mathbb{P}^{(1+2n_g)}$ *the following inequality holds* $\forall g \in \mathbb{R}^{n_g}$

$$\chi^\top(g) M \chi(g) \geq 0.$$

B. Nonnegativity of PWQ functions

Let $z \in \mathbb{R}^{n_z}$ and $g \in \mathbb{R}^{n_g}$, define the vector

$$\xi^\top(z, g) := \begin{bmatrix} 1 & z^\top & \phi^\top(g) & (\phi(g) - g)^\top \end{bmatrix},$$

and consider a PWQ function $h : \mathbb{R}^{n_z} \to \mathbb{R}$ defined as

$$h(z) = \xi^\top(z, g(z)) H \xi(z, g(z)), \tag{5a}$$

$$g(z) = C_g z + D_g \phi(g(z)) + e_g, \tag{5b}$$

with $H \in \mathbb{S}^{1+n_z+2n_g}$, $C_g \in \mathbb{R}^{n_g \times n_z}$, $D_g \in \mathbb{R}^{n_g \times n_g}$, and $e_g \in \mathbb{R}^{n_g}$. Assume that (5b) is well-posed (see Proposition 1). Note that V in (4) is a particular case of h in (5). Based on the properties of the ramp function presented in Lemmas 1 and 2, the following proposition states a sufficient condition for the positive semidefiniteness of a PWQ function.

Proposition 2. *If there exist $T \in \mathbb{D}^{n_g}$ and $M \in \mathbb{P}^{1+2n_g}$ such that*

$$h(z) + \chi^\top(g(z))(\Psi(T) - M)\chi(g(z)) \geq 0 \;\; \forall z \in \mathbb{R}^{n_z}, \quad (6)$$

then h as defined in (5) *is a positive semidefinite PWQ function.*

Proof. From the well-posedness of (5b) and Lemmas 1 and 2, (6) implies that $h(z) \geq \chi^\top(g(z))M\chi(g(z)) \geq 0 \;\; \forall z \in \mathbb{R}^{n_z}$. ■

The set of PWQ functions as in (5) for which (6) holds for some $T \in \mathbb{D}^{n_g}$ and $M \in \mathbb{P}^{1+2n_g}$ is denoted PWQ$_+$. The next proposition introduces a numerical test in the form of a matrix inequality to verify whether $h \in$ PWQ$_+$.

Proposition 3. *Let h be a PWQ function defined as in* (5)*. If there exist $T \in \mathbb{D}^{n_g}$ and $M \in \mathbb{P}^{(1+2n_g)}$ such that*

$$\Pi_\xi^\top (H + \Pi_\chi^\top(\Psi(T) - M)\Pi_\chi)\Pi_\xi \geq 0 \quad (7)$$

with

$$\Pi_\xi = \begin{bmatrix} 1 & 0_{1,n_z} & 0_{1,n_g} \\ 0_{n_z,1} & I_{n_z} & 0_{n_z,n_g} \\ 0_{n_g,1} & 0_{n_g,n_z} & I_{n_g} \\ -e_g & -C_g & I_{n_g} - D_g \end{bmatrix}, \Pi_\chi^\top = \begin{bmatrix} 1 & 0_{1,2n_g} \\ 0_{n_z,1} & 0_{n_z,2n_g} \\ 0_{2n_g,1} & I_{2n_g} \end{bmatrix},$$

then $h \in PWQ_+$.

Proof. Let $\nu^\top(z, g(z)) := \begin{bmatrix} 1 & z^\top & \phi^\top(g(z)) \end{bmatrix}$ and observe the following relations: $\chi(g(z)) = \Pi_\chi \xi(z, g(z))$ and $\xi(z, g(z)) = \Pi_\xi \nu(z, g(z))$. Thus, by pre- and post-multiplying (7) by $\nu^\top(z, g(z))$ and $\nu(z, g(z))$, respectively, and considering the aforementioned relations, we obtain (6). Thus, (7) implies that $h \in$ PWQ$_+$. ■

Remark 1. *Note that the matrix M in* (7) *must be copositive. A simple parameterization of a copositive matrix is to impose elementwise nonnegativity, as done in [6]. Other less conservative convex conditions for the copositivity of a symmetric matrix M, based on Sum-of-Squares (SoS) polynomials, are given in [12, p. 64]. For instance, a symmetric matrix $M \in \mathbb{S}^m$ is copositive if the polynomial in $p \in \mathbb{R}^m$*

$$\mathcal{P}_d(p) := \begin{bmatrix} p_{(1)}^2 \\ \vdots \\ p_{(m)}^2 \end{bmatrix}^\top M \begin{bmatrix} p_{(1)}^2 \\ \vdots \\ p_{(m)}^2 \end{bmatrix} \left(\sum_{i=1}^m p_{(i)}^2 \right)^d \quad (8)$$

admits an SoS decomposition for some $d \in \mathbb{N}_{\geq 0}$.

Remark 2. *Observe that since* (7) *has an affine dependence on H, T, and M, it can be written as an LMI whenever an LMI parameterization of copositivity is adopted (see Remark 1).*

C. Derivative of the ramp function

Let $x(t)$ be a trajectory of system (2). To compute the time derivative of $V(x(t))$ by using the chain rule, we can formally express

$$\dot{V}(x) = 2 \begin{bmatrix} x \\ \phi(\eta(x)) \end{bmatrix}^\top \begin{bmatrix} P_1 & P_2 \\ P_2^\top & P_3 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\phi}(\eta(x)) \end{bmatrix}, \quad (9)$$

where each term of $\dot{\phi}(\eta(x))$ corresponds to $\frac{d\phi_{(i)}(\eta)}{d\eta_{(i)}} \dot{\eta}_{(i)}$. However, since the derivative of the ramp is not defined whenever $\eta_{(i)}(x) = 0$, we may have $\dot{\phi}(\eta(x))$ not defined in the set $\mathcal{S} = \{x \in \mathbb{R}^n \mid \eta_{(i)}(x) = 0 \text{ for some } i \in \{1, \ldots, q\}\}$. To address this issue, we define a vector $\zeta(x)$ which is equal to the time derivative of ϕ *almost everywhere* (see [10] and [13]):

Definition 1 (Almost everywhere derivative of ϕ)**.** *The vector $\zeta(x) \in \mathbb{R}^q$, defined elementwise for $i \in \{1, \ldots, q\}$ as*

$$\zeta_{(i)}(x) := \begin{cases} 0 & \text{if } \eta_{(i)}(x) \leq 0 \\ (C_\eta \dot{x} + D_\eta \zeta(x))_{(i)} & \text{if } \eta_{(i)}(x) > 0, \end{cases} \quad (10)$$

is equal to the time derivative of $\phi(\eta(x))$ almost everywhere*, that is, in the set $\mathbb{R}^n \backslash \mathcal{S}$.*

From Definition 1, the following lemma regarding $\zeta(x)$ can be stated.

Lemma 3. *For any N_1, N_2, and $N_3 \in \mathbb{D}^q$, the following identities hold $\forall x \in \mathbb{R}^n$*

$$\zeta^\top(x) N_1 (C_\eta \dot{x} + (D_\eta - I_m)\zeta(x)) \equiv 0, \quad (11a)$$

$$\phi^\top(\eta(x)) N_2 (C_\eta \dot{x} + (D_\eta - I_m)\zeta(x)) \equiv 0, \quad (11b)$$

$$(\phi(\eta(x)) - \eta(x)) N_3 \zeta(x) \equiv 0. \quad (11c)$$

Proof. The left-hand side of (11a) can be written as

$$\sum_{i=1}^q N_{1(i,i)} \zeta_{(i)}(x) (C_\eta \dot{x} + D_\eta \zeta(x) - \zeta(x))_{(i)}.$$

From the definition of $\zeta(x)$ in (10) we have that either $\zeta_{(i)}(x) = 0$ or $\zeta_{(i)}(x) = (C_\eta \dot{x} + D_\eta \zeta(x))_{(i)}$. Thus (11a) holds. Furthermore, the left-hand side of (11b) can be written as

$$\sum_{i=1}^q N_{2(i,i)} \phi_{(i)}(\eta(x)) (C_\eta \dot{x} + D_\eta \zeta(x) - \zeta(x))_{(i)}.$$

From the definition of $\zeta(x)$ in (10) we have that either $\phi_{(i)}(\eta(x)) = 0$ (if $\eta_{(i)}(x) \leq 0$) or $\zeta_{(i)}(x) = (C_\eta \dot{x} + D_\eta \zeta(x))_{(i)}$ (if $\eta_{(i)}(x) > 0$). Thus (11b) holds. Finally, the left-hand side of (11c) can be written as

$$\sum_{i=1}^q N_{3(i,i)} (\phi(\eta(x)) - \eta(x))_{(i)} \zeta_{(i)}(x)$$

and two possible cases arise for each term of the summation: if $\eta_{(i)}(x) \leq 0$, then $\zeta_{(i)}(x) = 0$; if $\eta_{(i)}(x) > 0$, then $(\phi(\eta(x)) - \eta(x))_{(i)} = 0$. Thus (11c) holds. ■

Remark 3. *For particular structures of* (4b) *other relations, similar to the ones presented in Lemma 3, can be obtained.*

To illustrate, consider the saturation nonlinearity with linear region $[-1,1]^{n_u}$

$$\begin{aligned} sat(Kx) &= Kx + \begin{bmatrix} -I_{n_u} & I_{n_u} \end{bmatrix} \phi(y(x)) \\ y(x) &= Cx + D\phi(y(x)) + e, \end{aligned} \tag{12}$$

where $C^\top = \begin{bmatrix} K^\top & -K^\top \end{bmatrix}$, $D = 0_{2n_u}$, *and* $e = -1_{2n_u}$, *with* $K \in \mathbb{R}^{n_u \times n}$ *[6]. The following lemma can be stated for this case.*

Lemma 4. *For the saturation nonlinearity* (12)*, if* $C_\eta = C$, $D_\eta = D$, *and* $e_\eta = e$ *(that is,* $\eta(x) = y(x)$*), then for any* $N_4 \in \mathbb{D}^{n_u}$ *the following identity holds* $\forall x \in \mathbb{R}^n$

$$\zeta^\top(x) \begin{bmatrix} 0_{n_u} & N_4 \\ N_4 & 0_{n_u} \end{bmatrix} \zeta(x) \equiv 0. \tag{13}$$

Proof. Note that $\forall x \in \mathbb{R}^n$, $\zeta(x) \in \mathbb{R}^{2n_u}$ and the left-hand side of (13) can be rewritten as

$$2 \sum_{i=1}^{n_u} N_{4(i,i)} \zeta_{(i)}(x) \zeta_{(i+n_u)}(x),$$

from where two possible cases arise for each term of the summation. If $y_{(i)}(x) \leq 0$, then $\zeta_{(i)}(x) = 0$ since $\eta(x) = y(x)$. On the other hand, it follows from the saturation structure that if $y_{(i)}(x) > 0$ then $y_{(i+n_u)}(x) < 0$, since a simultaneous positive and negative saturation is impossible. This implies that $\zeta_{(i+n_u)}(x) = 0$ since $\eta(x) = y(x)$. Thus (13) holds. ■

IV. Global Stability Analysis

We provide now conditions to assess the GES of the origin of CPWA systems considering PWQ LF candidates based on the results of Section III. The following lemma introduces an expression that is equal to the time derivative of V given by (4) almost everywhere.

Lemma 5. *For any diagonal matrices* N_1, N_2, *and* $N_3 \in \mathbb{D}^q$, *let*

$$\begin{aligned} \dot{V}_\zeta(x) &:= 2 \begin{bmatrix} x \\ \phi(\eta(x)) \end{bmatrix}^\top \begin{bmatrix} P_1 & P_2 \\ P_2^\top & P_3 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \zeta(x) \end{bmatrix} \\ &\quad + 2\zeta^\top(x) N_1 (C_\eta \dot{x} + (D_\eta - I_m)\zeta(x)) \\ &\quad + 2\phi^\top(\eta(x)) N_2 (C_\eta \dot{x} + (D_\eta - I_m)\zeta(x)) \\ &\quad + 2(\phi(\eta(x)) - \eta(x))^\top N_3 \zeta(x). \end{aligned} \tag{14}$$

Then, $\dot{V}_\zeta(x) = \dot{V}(x) \ \ \forall x \in \mathbb{R}^n \backslash \mathcal{S}$.

Proof. According to Lemma 3 the second, third, and fourth terms of the right-hand side of (14) are identically zero. By Definition 1 one has that $\zeta(x) = \dot{\phi}(\eta(x)) \ \forall x \in \mathbb{R}^n \backslash \mathcal{S}$. Therefore $\dot{V}_\zeta(x) = \dot{V}(x)$ almost everywhere. ■

By observing that V in (4) is a PWQ function in x while $\dot{V}_\zeta$ in (14) is a PWQ function in x and $\zeta(x)$, the following result can be stated.

Theorem 1. *If there exist* $P_1 \in \mathbb{S}^n$, $P_2 \in \mathbb{R}^{n \times q}$, $P_3 \in \mathbb{S}^q$, $N_1 \in \mathbb{D}^q$, $N_2 \in \mathbb{D}^q$, $N_3 \in \mathbb{D}^q$, *and positive scalars* ϵ_1, ϵ_2, *and* γ *such that*

$$h_1(x) := V(x) - \epsilon_1 x^\top x \in \mathit{PWQ}_+, \tag{15a}$$

$$h_2(x) := -V(x) + \epsilon_2 x^\top x \in \mathit{PWQ}_+, \tag{15b}$$

$$h_3(x, \zeta(x)) := -\dot{V}_\zeta(x) - 2\gamma V(x) \in \mathit{PWQ}_+, \tag{15c}$$

then the origin of the CPWA system (2) *is globally exponentially stable with decay rate* γ.

Proof. From (15a) and (15b) it follows that

$$\epsilon_1 \|x\|_2^2 \leq V(x) \leq \epsilon_2 \|x\|_2^2,$$

which ensures that V is positive definite, radially unbounded and defined for all $x \in \mathbb{R}^n$ while (15c) implies that

$$\dot{V}_\zeta(x) \leq -2\gamma V(x).$$

The remainder of the proof is based on [17, pp 97–99]. Thanks to the continuity of the ramp function and the well-posedness of (2b) and (4b), we have that $\dot{x}$ in (2) is continuous in $\mathbb{R}^n$ and V in (4) is locally Lipschitz, which implies that $\dot{V}(x)$ is defined for almost all t. Since $\dot{V}_\zeta(x) = \dot{V}(x)$ almost everywhere, by (15c) we have that $\dot{V}(x) \leq -2\gamma V(x)$ for almost all t. Combined with the fact that V is radially unbounded and lower bounded by a class-$\mathcal{K}_\infty$ function thanks to (15a) and (15b), we conclude that the trajectories converge exponentially to the origin with decay rate γ. ■

As discussed in Remark 2, constraints such as (15) can be cast as LMIs whenever an LMI relaxation for matrix copositivity is adopted. Let us first observe that h_i in (15), for $i \in \{1,2,3\}$, are PWQ functions and can be written as in (5)

$$\begin{aligned} h_i(z_i) &= \xi^\top(z_i, g_i(z_i)) H_i \xi(z_i, g_i(z_i)), \\ g_i(z_i) &= C_{gi} z_i + D_{gi} \phi(g_i(z_i)) + e_{gi}, \end{aligned}$$

with $z_1 = z_2 = x$, $z_3^\top = \begin{bmatrix} x^\top & \zeta^\top(x) \end{bmatrix}$,

$$\begin{aligned} &H_1 = \operatorname{diag}\left(0, \begin{bmatrix} P_1 - \epsilon_1 I_n & P_2 \\ P_2^\top & P_3 \end{bmatrix}, 0_q\right), \\ &H_2 = \operatorname{diag}\left(0, \begin{bmatrix} -P_1 + \epsilon_2 I_n & -P_2 \\ -P_2^\top & -P_3 \end{bmatrix}, 0_q\right), \ H_3 \text{ as in (16)}, \\ &C_{g1} = C_{g2} = C_\eta, \ D_{g1} = D_{g2} = D_\eta, \ e_{g1} = e_{g2} = e_\eta, \\ &C_{g3} = \begin{bmatrix} C & 0_{m,q} \\ C_\eta & 0_q \end{bmatrix}, \ D_{g3} = \begin{bmatrix} D & 0_{m,q} \\ 0_{q,m} & D_\eta \end{bmatrix}, \ e_{g3} = \begin{bmatrix} e \\ e_\eta \end{bmatrix}. \end{aligned}$$

Thus, by using Proposition 3, (15) can be cast as LMIs for a fixed γ and an LMI relaxation for matrix copositivity (see Remarks 1 and 2). If we consider elementwise nonnegativity for matrix copositivity, (15) becomes an LMI with size $3 + 3n + m + 4q$ and the number of decision variables grows quadratically with respect to n, m, and q.

V. Numerical Examples

This section presents two numerical examples to illustrate the application of the results of Theorem 1. First, an example regarding a linear system subject to input saturation, which is a particular case of CPWA system, is considered, and the results obtained with Theorem 1 are compared with other methods in the literature. The second example consists

$$
H_3 = -\mathrm{He}\left\{\begin{bmatrix}
0 & 0_{1,n} & 0_{1,q} & 0_{1,m} & 0_{1,q} & 0_{1,m} & 0_{1,q} \\
0_{n,1} & P_1A+\gamma P_1 & P_2 & P_1B & \gamma P_2 & 0_{n,m} & 0_{n,q} \\
0_{q,1} & N_1C_\eta A & N_1(D_\eta - I_q) & N_1C_\eta B & 0_q & 0_{q,m} & 0_q \\
0_{m,1} & 0_{m,n} & 0_{m,q} & 0_m & 0_{m,q} & 0_m & 0_{m,q} \\
0_{q,1} & \gamma P_2^\top + P_2^\top A + N_2C_\eta A & P_3 + N_2(D_\eta - I_q) & P_2^\top B + N_2C_\eta B & \gamma P_3 & 0_{q,m} & 0_q \\
0_{m,1} & 0_{m,n} & 0_{m,q} & 0_m & 0_{m,q} & 0_m & 0_{m,q} \\
0_{q,1} & 0_{q,n} & N_3 & 0_{q,m} & 0_q & 0_{q,m} & 0_q
\end{bmatrix}\right\} \tag{16}
$$

of a scalar CPWA system, where the potential benefits of considering a different partition for the LF candidates are highlighted. For both examples, once a value of γ is fixed and an LMI relaxation for matrix copositivity is considered, the resulting LMIs associated with (15) are solved with MOSEK [1]. A line search based on solving SDP problems allows the computation of the maximum estimate of the decay rate γ, which will be used to compare the proposed conditions with other methods.

A. Numerical Example 1 - Saturation

Consider the following example from [13], consisting of a third-order single-input continuous-time linear system controlled by a saturating static state-feedback written in the ramp-based implicit representation (2) with

$$
A = \begin{bmatrix} -(2+\alpha) & -2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ B = \begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},
$$
$$
C = \begin{bmatrix} -\alpha & 0 & 0 \\ \alpha & 0 & 0 \end{bmatrix},\ D = 0_2,\ \text{and } e = \begin{bmatrix} -1 \\ -1 \end{bmatrix},
$$

where $0 \leq \alpha \leq 100$ is a parameter of the system. The goal is to estimate the decay rate of the trajectories for a fixed value of α, which can be carried out by employing the conditions of Theorem 1 and a line search to maximize γ. We choose $C_\eta = C$, $D_\eta = D$, and $e_\eta = e$ (*i.e.* the partition of the LF candidate is the same as the partition of the system dynamics) and we fix $\epsilon_1 = 1 \times 10^{-6}$, while ϵ_2 remains a decision variable. The obtained results are shown in Figure 1, where the cyan line represents the estimate of the decay rate as a function of α.

The results obtained can be further improved by considering Lemma 4 to include the quadratic term in the left-hand side of (13) in the definition of $\dot{V}_\varsigma(x)$ in (14). Applying again the conditions of Theorem 1 results in the estimates of the decay rate shown by the dashed blue line in Figure 1. The improvement is highlighted by the fact that the GES of the origin is certified for all values of parameter α considered. The performance obtained in this numerical example was also compared with three other methods: a global sector condition based on quadratic LFs [16, global version of Proposition 3.6], a method for asymmetric saturation based on PWQ LFs [10, global version of Theorem 5], and a method for PWA systems [9, Theorem 1]. The estimates of γ for [16] and [10] are depicted in Figure 1 by the orange and red lines, respectively. For [9], the GES of the origin could not be certified when choosing the matrices defining

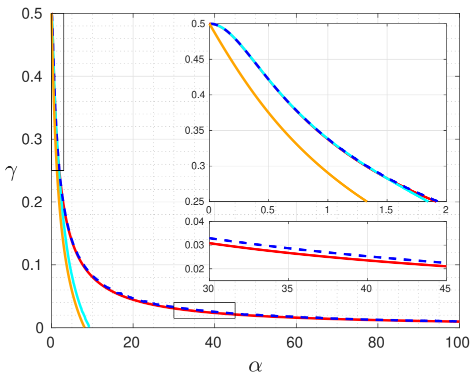

Fig. 1. Performance of different methods for Example 1, [16] (solid orange), [10] (solid red), Theorem 1 (solid cyan), and Theorem 1 with Lemma 4 (dashed blue).

the dynamics to be the continuity matrices (see [9, eq. (3)]). The estimates of the decay rate obtained by the conditions in [10] are similar to the values obtained from the conditions of Theorem 1. This illustrates the applicability of the PWA framework proposed for particular cases of interest, such as saturation.

B. Numerical Example 2 - Scalar CPWA System

Consider a scalar CPWA system given by

$$
\dot{x} = \begin{cases} -10x & \text{if } x \in \Gamma_1 = (-\infty,\ 1) \\ -x - 9 & \text{if } x \in \Gamma_2 = [1,\ 3) \\ -10x + 18 & \text{if } x \in \Gamma_3 = [3,\ \infty), \end{cases} \tag{17}
$$

with $x \in \mathbb{R}$. This system can be represented in the ramp-based implicit representation (2) with

$$
A = -10,\ B^\top = \begin{bmatrix} 9 \\ -9 \end{bmatrix},\ C = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ D = 0_2,\ \text{and } e = \begin{bmatrix} -1 \\ -3 \end{bmatrix}.
$$

Thanks to the simplicity of this example, it is possible to demonstrate the following two facts: *i)* The origin is GES with decay rate $\gamma = 10$; *ii)* If a quadratic LF is used to assess the GES of the origin of the system, the maximum decay rate estimate is $\gamma = 4$.

Indeed, if we verify the stability using Theorem 1 and a quadratic LF candidate (that is, with $P_2 = 0_{1,2}$ and $P_3 = 0_2$ in (4)), we obtain $\gamma = 4$ as the estimate of the decay rate. On the other hand, if we consider a PWQ LF candidate with the same partition as the system dynamics, that is, $C_\eta = C$, $D_\eta = D$, and $e_\eta = e$, we obtain $\gamma = 5.9993$. We refine

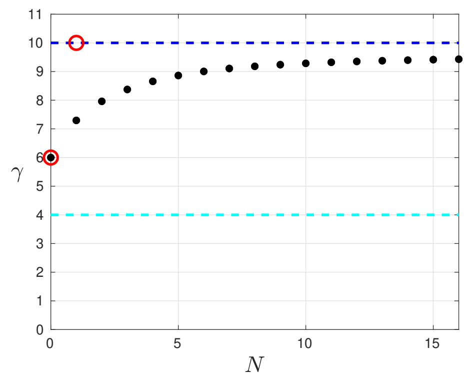

Fig. 2. Estimate of γ as a function of N for different copositivity relaxations: elementwise nonnegativity (black dots) and SoS decomposition (8) with $d = 1$ (red circles). The blue and cyan dashed lines represent, respectively, the global decay rate and its estimate with a quadratic LF.

the LF partition by adding N equally spaced sub-sets in the set Γ_2 of the state space and then applying Theorem 1 with a line search for estimating the decay rate γ. Figure 2 shows the estimate of the decay rate γ as a function of the refinement parameter N for two convex relaxations for matrix copositivity, namely elementwise nonnegativity and (8) with $d = 1$. Observe that, by increasing N and considering elementwise nonnegativity for copositivity (black dots in Figure 2), we obtain better estimates for γ, although without achieving $\gamma = 10$ for the considered values for N. On the other hand, a single subdivision in the set Γ_2 ($N = 1$) and an SoS-based relaxation for copositivity ((8) with $d = 1$) allows to estimate $\gamma = 9.9996$, which is equal to the actual global decay rate of the origin within the tolerance of the line search algorithm for γ. This example illustrates the potential benefits of considering a partition of the LF candidates different from the one of the system dynamics and of using SoS relaxations for ensuring the copositivty of the slack matrix M in the application of Proposition 3.

VI. Conclusion

This work addressed the global exponential stability analysis of the origin of continuous-time Continuous Piecewise Affine (CPWA) systems using a ramp-based implicit representation. Piecewise Quadratic (PWQ) Lyapunov Functions (LF) were considered to certify the stability of the origin. Moreover, a convex relaxation for ensuring copositivity of matrices based on Sum-of-Squares constraints was employed for the test of positive semidefiniteness of PWQ functions. The use of the ramp-based implicit representation presents three main advantages to the stability analysis problem. First, it inherently considers PWQ LFs that are continuous across the set boundaries, withdrawing the need for additional constraints (potentially conservative) to impose continuity. Second, the growth of the dimensions of the LMI constraints with respect to the number of sets of the state space partition is reduced when compared to methods such as [9]. Finally, it also allows to consider uncertainties in the partition in a straightforward manner [2]. Future work includes extending the results to discontinuous PWA systems, where sliding modes (in the sense of Filippov solutions) may exist.

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