

Population games with replicator dynamics under event-triggered payoff provider and a demand response application

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Abstract—We consider a large population of decision makers that choose their evolutionary strategies based on simple pairwise imitation rules. We describe such a dynamic process by the replicator dynamics. Differently from the available literature, where the payoffs signals are assumed to be updated continuously, we consider a more realistic scenario where they are updated occasionally. Our main technical contribution is to devise two event-triggered communication schemes with asymptotic convergence guarantees to a Nash equilibrium. Finally, we show how our proposed approach is applicable as an efficient distributed demand response mechanism.

I. INTRODUCTION

Population games provide a framework to model the strategic behavior of large populations of decision-making agents [1]. Depending on the protocols that agents use to update their strategies, several evolutionary dynamics may arise. We focus on the so-called replicator dynamics (RD) [2], which comprise a class of imitative strategy-revision protocols, where agents repeatedly engage in random pairwise interactions and copy the strategy of their peers with a probability proportional to the difference of their perceived payoffs. Thus, under the RD, the agents are payoff-driven decision-makers that require bounded rationality levels [3]. As such, the RD have found relevance in large-scale control systems, with applications in wireless networks [4], road traffic congestion [5], subsidy design [6], and residential demand response [7], among others.

Under the considered framework, the payoffs perceived by the agents in general depend on the aggregate decision of the entire population. Therefore, to accurately compute/estimate their corresponding payoffs, the agents typically require access to non-local information. To avoid excessive inter-agent communication, which is costly for large-scale applications, one might introduce a high level entity, here referred to as the payoff provider, that observes the strategic distribution of the population and broadcasts the payoff signals to the agents [8]. Consequently, the payoff provider and the population form a closed-loop system. For instance, in congestion games,

the payoff provider would be the entity that provides non-local traffic information to the agents [5], whereas in demand response problems, the payoff provider would be the electric power utility that broadcasts cost signals on the aggregate demand of the system [7].

To analyze the stability properties of the population, the available literature models the payoff signal as the output of a continuous-time system, which can be either a static or a dynamic map [1], [3], [9]–[11]. However, applying such results to our framework would imply that the payoff provider has to continuously broadcast information to the agents, leading to high communication costs as well as practical implementation questions, e.g., how frequently should the information be broadcast so that the stability is not compromised? By exploiting the event-triggered control framework [12], over the last decade there has been an increasing interest in event-triggered communication methods for multi-agent systems [13], [14]. In fact, some recent works, e.g., [15]–[17], have considered event-triggered communications in games as well. The main advantage of event-triggered approaches is that they explicitly model that communications take place occasionally over time, and only when a given event occurs. As such, event-triggered methods can reduce the communication costs while still guaranteeing desirable stability properties for the closed-loop system.

Motivated by the previous discussion, in this paper, we formulate an event-triggered payoff provider that broadcasts the payoff signals only occasionally over time. Specifically, we devise two (Zeno-free) event-triggered mechanisms and we formally prove global asymptotic stability of the unique Nash equilibrium when the underlying payoff functions are Lipschitz continuous and strongly contractive. To the best of our knowledge, this is the first time that an event-triggered payoff provider has been formally analyzed for population games under the RD. Hybrid and event-triggered RD have been previously reported in [18] and [19], respectively. However, these works study the RD as continuous-time optimization dynamics for resource allocation, rather than as an evolutionary model for large populations of imitative agents. Nonetheless, if applied to our context, the results in [18] would still imply continuous communication from the payoff provider, as well as the global synchronization of the agents on a time-varying (time-scale) parameter, whilst the approach in [19] would limit the scope of application to quadratic strictly concave full-potential games with diagonal Hessian matrix. Instead, our framework enables non-continuous communication from the payoff provider, allows the agents to operate asynchronously, and it is applicable to

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more general (not-necessarily potential) population games. Finally, we implement our proposed approach as a distributed demand response scheme, improving upon [7]. The numerical results on large-scale test cases show that the proposed method not only significantly reduces the total number of broadcast operations, but also speeds up the convergence of the population game.

Notation: We use standard font for scalars, bold font for vectors and matrices, and calligraphic font for sets. Besides, all vectors are taken as columns by default. The set of real (integer) numbers is denoted by \mathbb{R} (\mathbb{Z}). The set of non-negative (strictly positive) real numbers is denoted by $\mathbb{R}_{\geq 0}$ ($\mathbb{R}_{> 0}$). A similar notation holds for integers, and $\mathbb{Z}_{\gg 1}$ denotes the integers much greater than 1. We denote the Euclidean norm by $\|\cdot\|_2$, and the Cartesian product by \prod . The operators $\text{col}(\cdot)$ and $\text{diag}(\cdot)$ create a column vector and a (block) diagonal matrix of the arguments, respectively. Given a vector $\mathbf{z} \in \mathbb{R}^m$, we let z_i denote its i -th element, and $\text{supp}(\mathbf{z}) = \{i \in \{1, 2, \dots, m\} : z_i > 0\}$ denote its support. Given a domain $\mathcal{D} \subseteq \mathbb{R}^m$ and an operator $T : \mathcal{D} \rightarrow \mathcal{D}$, $\text{fix}(T) := \{\mathbf{z} \in \mathcal{D} : \mathbf{z} = T(\mathbf{z})\}$ is the fixed point set of T .

II. POPULATION GAMES WITH REPLICATOR DYNAMICS

A. Mathematical formulation

Let us consider a society with $P \in \mathbb{Z}_{\geq 1}$ populations of decision-making agents. Throughout, let $\mathcal{P} = \{1, 2, \dots, P\}$ be the set indexing the populations and, for each population $k \in \mathcal{P}$, let $N^k \in \mathbb{Z}_{\gg 1}$ be the total number of agents that belong to population k (where we assume that N^k is large and constant over time). Let $\mathcal{S}^k = \{1, 2, \dots, n^k\}$, with $n^k \in \mathbb{Z}_{\geq 2}$, be the set of decision strategies available to the agents of population k and, for each $i \in \mathcal{S}^k$, let $x_i^k \in [0, 1]$ denote the proportion of agents of population k choosing strategy i , i.e., $N^k x_i^k$ yields the total number of agents playing i in population k . Then, the strategic distribution of population k is given by $\mathbf{x}^k = \text{col}\left(\left(x_i^k\right)_{i \in \mathcal{S}^k}\right) \in \Delta^k := \left\{\mathbf{y} \in \mathbb{R}_{\geq 0}^{n^k} : \mathbf{1}_{n^k}^\top \mathbf{y} = 1\right\}$, whilst the strategic distribution of the entire society is given by $\mathbf{x} = \text{col}\left(\left(\mathbf{x}^k\right)_{k \in \mathcal{P}}\right) \in \Delta := \prod_{k \in \mathcal{P}} \Delta^k = \Delta^1 \times \Delta^2 \times \dots \times \Delta^P$.

Furthermore, we assume that each strategy $i \in \mathcal{S}^k$ is characterized by a fitness function $f_i^k : \Delta \rightarrow \mathbb{R}$. Namely, the value $f_i^k(\mathbf{x})$ is the payoff to be given to the agents of population $k \in \mathcal{P}$ playing strategy $i \in \mathcal{S}^k$ at the society's strategic distribution $\mathbf{x} \in \Delta$. Thus, the fitness functions determine the strategic environment for the society of decision-makers. Throughout, let $\mathbf{f}^k(\cdot) = \text{col}\left(f_1^k(\cdot), f_2^k(\cdot), \dots, f_{n^k}^k(\cdot)\right)$ be the fitness vector of population k , and let $\mathbf{f}(\cdot) = \text{col}\left(\mathbf{f}^1(\cdot), \mathbf{f}^2(\cdot), \dots, \mathbf{f}^P(\cdot)\right)$ be the overall fitness vector of the entire society. As such, a population game can be defined in normal form as the tuple $G = (\mathcal{P}, \Delta, \mathbf{f}(\cdot))$, which captures the set of populations \mathcal{P} , the set of strategic distributions Δ , and the overall fitness vector $\mathbf{f}(\cdot)$. Besides, we impose the following smoothness and monotonicity conditions on $\mathbf{f}(\cdot)$, which are also considered in [15]–[17] outside of the context of population games.

Standing Assumption 1: The overall fitness vector $\mathbf{f}(\cdot)$ is θ -Lipschitz continuous and μ -strongly contractive, i.e., there exist some $\theta, \mu \in \mathbb{R}_{> 0}$ such that, for every $\mathbf{x}, \mathbf{y} \in \Delta$, $\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\|_2 \leq \theta \|\mathbf{x} - \mathbf{y}\|_2$ and $(\mathbf{x} - \mathbf{y})^\top (\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})) \leq -\mu \|\mathbf{x} - \mathbf{y}\|_2^2$, respectively.

Similar to asynchronous gossip algorithms [20], in this paper we assume that agents communicate in a random pairwise fashion and at random instants of time. Moreover, as is often assumed in the replicator dynamics (RD) models reported in the literature, e.g., [1, Section 5.4], we consider the case where each agent can communicate with any other agent of its same population, i.e., there is an all-to-all interaction connectivity among agents of the same population. Nonetheless, inter-agent communications only occur in a pairwise fashion and at random sporadic instants of time. For the sake of clarity, we now proceed to formally describe the microscopic decision-making process followed by the agents.

Microscopic decision-making process: Let each agent be equipped with a Poisson alarm clock which provides (independent and identically distributed) strategy-revision opportunities according to an exponential distribution with rate $R \in \mathbb{R}_{> 0}$. Suppose that, at time $t \in \mathbb{R}_{\geq 0}$, an agent of population $k \in \mathcal{P}$ receives a revision opportunity. Then, this agent randomly and uniformly chooses a second agent from its population k . Without loss of generality, let the revising agent be playing an arbitrary strategy $i \in \mathcal{S}^k$, and let the second agent be playing an arbitrary strategy $j \in \mathcal{S}^k$. Throughout, we assume that agents employ an imitative revision protocol [1, Section 4.3.1], where the revising agent imitates the strategy of the second agent with probability

$$\rho_{ij}^k(\mathbf{x}(t)) = \frac{\max\{f_j^k(\mathbf{x}(t)) - f_i^k(\mathbf{x}(t)), 0\}}{R}, \quad (1)$$

assuming that

$$R \geq \max_{\mathbf{x} \in \Delta, k \in \mathcal{P}, i, j \in \mathcal{S}^k} \max\{f_j^k(\mathbf{x}) - f_i^k(\mathbf{x}), 0\}. \quad (2)$$

Here, to compute $\rho_{ij}^k(\mathbf{x}(t))$, it is assumed that the revising agent knows R and $f_i^k(\mathbf{x}(t))$, and that the second agent communicates $f_j^k(\mathbf{x}(t))$ to the revising agent. Since the probability of randomly choosing a second agent playing j is $x_j^k(t)$, the overall probability for the revising agent to switch its strategy from i to j is given by $x_j^k(t)\rho_{ij}^k(\mathbf{x}(t))$, while the probability for the revising agent to keep playing i is $1 - \sum_{j \in \mathcal{S}^k \setminus \{i\}} x_j^k(t)\rho_{ij}^k(\mathbf{x}(t))$. Under this framework, the (expected) instantaneous change in the proportion x_i^k is thus given by [1, Section 4.2]

$$\dot{x}_i^k(t) = x_i^k(t) \left(f_i^k(\mathbf{x}(t)) - \sum_{j \in \mathcal{S}^k} x_j^k(t) f_j^k(\mathbf{x}(t)) \right), \quad (3)$$

for all $i \in \mathcal{S}^k$ and all $k \in \mathcal{P}$. The dynamics in (3) are often referred to as the (mean) RD and if N^k is sufficiently large for all $k \in \mathcal{P}$, then (3) provides an arbitrarily accurate approximation of the temporal evolution of $\mathbf{x}(t)$ over any finite-time horizon [1, Chapter 10].

In this paper, we focus on studying the convergence of the society to a Nash equilibrium (NE) of the underlying

population game, i.e., a strategic distribution where no agent can increase its fitness by unilaterally changing its strategy.

Definition 1 (Nash equilibria): Given a population game G , characterized by a fitness vector $\mathbf{f}(\cdot)$, the set of Nash equilibria of G is defined as $\text{NE}(\mathbf{f}) = \text{fix}(\arg \max_{\mathbf{x} \in \Delta} \mathbf{x}^\top \mathbf{f}(\cdot))$, or equivalently as $\text{NE}(\mathbf{f}) = \{\mathbf{x}^* \in \Delta : x_i^{k^*} > 0 \Rightarrow f_i^k(\mathbf{x}^*) = \max_{j \in \mathcal{S}^k} f_j^k(\mathbf{x}^*), \forall i \in \mathcal{S}^k, \forall k \in \mathcal{P}\}$.

Remark 1: [1, Thm. 5.4.13] Every $\mathbf{x}^* \in \text{NE}(\mathbf{f})$ is an equilibrium of the RD in (3), i.e., $\mathbf{x}(t) = \mathbf{x}^* \Rightarrow \dot{\mathbf{x}}(t) = \mathbf{0}_n$.

From Definition 1, it follows that the set $\text{NE}(\mathbf{f})$ coincides with the set of solutions of the variational inequality $\text{VI}(\Delta, -\mathbf{f}(\cdot))$. Thus, by Standing Assumption 1 it holds that there exists a unique $\mathbf{x}^* \in \text{NE}(\mathbf{f})$ [21, Thm. 2.3.3], and the fitness vector satisfies $(\mathbf{x}^* - \mathbf{x})^\top \mathbf{f}(\mathbf{x}) > 0$, for all $\mathbf{x} \in \Delta \setminus \{\mathbf{x}^*\}$. Hence, \mathbf{x}^* is asymptotically stable under the RD in (3) [1, Thm. 7.2.4]. On the other hand, we remark that $\text{NE}(\mathbf{f})$ is invariant under positive scales of the fitness vector, i.e., $\text{NE}(\mathbf{f}) = \text{NE}(\alpha \mathbf{f})$, $\forall \alpha \in \mathbb{R}_{>0}$. Consequently, by appropriately designing the fitness vector one can always guarantee the condition in (2) for any given R , without changing the set of Nash equilibria. Finally, we impose the following technical condition for our stability analyses.

Standing Assumption 2: For every $k \in \mathcal{P}$ it holds that $\mathbf{f}^k(\mathbf{x}^*) \in \text{span}(\mathbf{1}_{n^k})$, where $\mathbf{x}^* \in \text{NE}(\mathbf{f})$.

Standing Assumption 2 means that at the NE all the strategies within each population yield the same fitness value. A sufficient yet not necessary condition to satisfy Standing Assumption 2 is for the NE to belong to the relative interior of Δ , i.e., $\mathbf{x}^* \in \Delta \cap \mathbb{R}_{>0}^n$. Given that the RD suffers from extinction, i.e., $x_i^k(\tilde{t}) = 0 \Rightarrow x_i^k(t) = 0, \forall t \geq \tilde{t}$, in many applications one might enforce the presence of an interior NE to ensure that no strategy goes extinct in the long term.

B. Problem statement

According to the microscopic decision-making process described in Section II-A, to evaluate $\varrho_{ij}^k(\mathbf{x}(t))$ in (1) for a strategy-revision executed at time t , both the revising agent and the randomly chosen agent must know the fitness value of their selected strategies at time t . However, as fitness functions may depend on the strategic distribution of the entire society, letting agents compute their own fitness values would require for every agent to repeatedly have full-decision information regarding the strategic selections of all the $\sum_{k \in \mathcal{P}} N^k$ society members. To avoid excessive information-exchange among agents, we assume the existence of a payoff provider [7], [8] that operates as follows. Every time an agent updates its strategy, it informs its new selected strategy to the payoff provider, and the payoff provider then broadcasts the updated fitness values to the society. Namely, the following three steps are repeated in order: i) the payoff provider broadcasts the fitness values to the society; ii) an agent receives a revision opportunity and compares their fitness values with a second randomly selected agent; iii) the revising agent imitates the strategy of the second agent with probability $\varrho_{ij}^k(\mathbf{x}(t))$ and then informs the payoff provider about its strategy-update. Nonetheless, although the proposed

framework reduces the information-exchange among agents, it still requires for the payoff provider to broadcast the fitness values every time that an agent updates its strategy. Since on average the society receives $\sum_{k \in \mathcal{P}} N^k R dt$ revision opportunities over every (small) time interval of length dt , the considered framework still implies high communication demands for large societies, rendering the approach nonviable for many practical applications.

III. AN EVENT-TRIGGERED PAYOFF PROVIDER

To overcome the aforementioned issues and reduce the requirements on the communication capabilities of the payoff provider, yet still guaranteeing asymptotic stability of $\text{NE}(\mathbf{f})$ under the RD, we formulate an event-triggered payoff provider, which broadcasts the fitness values at a rate (possibly aperiodic) that is independent of the size of the society. Our proposed event-triggered scheme is as follows. Let $(t_\ell)_{\ell \in \mathbb{Z}_{\geq 0}}$ denote the sequence of event times and define

$$\hat{\mathbf{x}}(t) = \mathbf{x}(t_\ell), \quad \forall t \in [t_\ell, t_{\ell+1}), \quad (4a)$$

$$\mathbf{e}(t) = \hat{\mathbf{x}}(t) - \mathbf{x}(t). \quad (4b)$$

Namely, $\hat{\mathbf{x}}(t)$ takes the value of $\mathbf{x}(t_\ell)$ when the ℓ -th event occurs, and $\hat{\mathbf{x}}(t)$ is held constant in between events, while $\mathbf{e}(t)$ denotes the error of $\hat{\mathbf{x}}(t)$ with respect to $\mathbf{x}(t)$. We assume that the payoff provider can only broadcast the fitness values at the event times. Thus, at time t_ℓ , the payoff provider computes and broadcasts $\mathbf{f}(\mathbf{x}(t_\ell))$, and for any $t \in [t_\ell, t_{\ell+1})$, the agents update their strategies based on the (constant) fitness vector $\mathbf{f}(\hat{\mathbf{x}}(t))$. As such, for an agent of population k revising its strategy at time t , the probability to switch from strategy i to j is $x_j^k(t) \varrho_{ij}^k(\hat{\mathbf{x}}(t))$ (i.e., agents operate under the microscopic dynamics of Section II-A, but with $\varrho_{ij}^k(\cdot)$ in (1) evaluated at $\hat{\mathbf{x}}(t)$ instead of $\mathbf{x}(t)$). Consequently, under our event-triggered scheme, the RD in (3) become

$$\dot{x}_i^k(t) = x_i^k(t) \left(f_i^k(\hat{\mathbf{x}}(t)) - \sum_{j \in \mathcal{S}^k} x_j^k(t) f_j^k(\hat{\mathbf{x}}(t)) \right), \quad (5)$$

for all $i \in \mathcal{S}^k$ and all $k \in \mathcal{P}$, where $\hat{\mathbf{x}}(t)$ is as in (4a). Clearly, if $\hat{\mathbf{x}}(t) = \mathbf{x}(t)$ for all t , then the RD in (5) recover the RD in (3). Besides, we recall that when an agent updates its strategy, it informs its new strategy to the payoff provider (but not to the other agents). Thus, the payoff provider always knows $\mathbf{x}(t)$ regardless of the event times.

We remark that the RD in (5) resemble the event-triggered control scheme [12], where a nonlinear system (the RD in (3)) is subject to an event-triggered control input (the fitness vector). Therefore, motivated by the analytical framework in [12], we now state our main technical results. To this end, let $\Delta_{\mathbf{x}^{k^*}}^k = \{\mathbf{x}^k \in \Delta^k : \text{supp}(\mathbf{x}^{k^*}) \subseteq \text{supp}(\mathbf{x}^k)\}$ be the set of strategic distributions in Δ^k whose support contains the support of \mathbf{x}^{k^*} , and let $\Delta_{\mathbf{x}^*} = \prod_{k \in \mathcal{P}} \Delta_{\mathbf{x}^{k^*}}^k$. In addition, note that (5) can be rewritten compactly for each population $k \in \mathcal{P}$ as $\dot{\mathbf{x}}^k(t) = \mathbf{g}^k(\mathbf{x}^k(t), \hat{\mathbf{x}}(t))$, and for the entire society as $\dot{\mathbf{x}}(t) = \mathbf{g}(\mathbf{x}(t), \hat{\mathbf{x}}(t))$, where $\mathbf{g}^k(\mathbf{x}^k, \hat{\mathbf{x}}) = \text{diag}(\mathbf{x}^k) (\mathbf{I}_{n^k} - \mathbf{1}_{n^k} \mathbf{x}^{k\top}) \mathbf{f}^k(\hat{\mathbf{x}})$ and

$\mathbf{g}(\mathbf{x}, \hat{\mathbf{x}}) = \text{diag}(\mathbf{x})(\mathbf{I}_n - \mathbf{M}(\mathbf{x}))\mathbf{f}(\hat{\mathbf{x}})$, with $\mathbf{M}(\mathbf{x}) = \text{diag}\left(\left(\mathbf{1}_{n^k} \mathbf{x}^{k\top}\right)_{k \in \mathcal{P}}\right) \in \mathbb{R}^{n \times n}$.

Theorem 1: Consider the RD in (5) under an event-triggered payoff provider with event times specified by $t_{\ell+1} = \min\{t' \geq t_\ell + \tau : \|\mathbf{e}(t')\|_2 \geq \tau \|\mathbf{g}(\mathbf{x}(t'), \hat{\mathbf{x}}(t'))\|_2\}$, with $\tau = 2\mu/(\theta(\theta + \mu))$, and $\hat{\mathbf{x}}(t), \mathbf{e}(t)$ are as in (4). The unique $\mathbf{x}^* \in \text{NE}(\mathbf{f})$ is globally asymptotically stable from every $\mathbf{x}(0) \in \Delta_{\mathbf{x}^*}$. Besides, the proposed trigger is free from Zeno behavior with minimum inter-event time $\tau_{\min} = \tau$. ■ Theorem 1 reveals that employing the periodic trigger given by $t_{\ell+1} = t_\ell + \tau$ is enough to guarantee the global asymptotic stability of \mathbf{x}^* within $\Delta_{\mathbf{x}^*}$. Hence, by following such a periodic sequence, the payoff provider does not need to evaluate any state-dependent event-triggering condition at the cost of possibly more overall broadcast communications than the event-triggered counterpart.

As formally stated next, when the payoff provider knows \mathbf{x}^* in advance, we can obtain a different triggering condition that yields less frequent triggers than that of Theorem 1.

Proposition 1: Consider the RD in (5) under an event-triggered payoff provider with event times specified by $t_{\ell+1} = \min\{t' \geq t_\ell + \tau : \|\mathbf{e}(t')\|_2 \geq (\mu/\theta) \|\mathbf{x}^* - \mathbf{x}(t')\|_2\}$, with $\tau = 2\mu/(\theta(\theta + \mu))$, and $\mathbf{e}(t)$ is as in (4b). The unique $\mathbf{x}^* \in \text{NE}(\mathbf{f})$ is globally asymptotically stable from every $\mathbf{x}(0) \in \Delta_{\mathbf{x}^*}$. Besides, the proposed trigger is free from Zeno behavior with minimum inter-event time $\tau_{\min} = \tau$. ■

Finally, we remark that both of the proposed event-triggered payoff providers will perpetually trigger (at the periodic rate $t_{\ell+1} = t_\ell + \tau$) even after $\mathbf{x}(t) = \mathbf{x}^*$. Nevertheless, to overcome unnecessary broadcast of information, the payoff provider might simply avoid to broadcast the fitness vector whenever $\mathbf{f}(t_{\ell+1}) = \mathbf{f}(t_\ell)$, because that would not change the fitness values perceived by the agents.

IV. A DEMAND RESPONSE APPLICATION

Let us consider a large society of consumers (agents) engaged in a demand response (DR) program. The goal of the DR program is to shave the aggregate demand by at least C kW, and the multiple populations in \mathcal{P} characterize different types of DR agents according to their allowed power commitment levels. Specifically, each DR agent of population k must choose a power commitment strategy from the set $\mathcal{S}^k = \{1, 2, \dots, n^k\}$, where each strategy in \mathcal{S}^k corresponds to an individual power demand reduction of r_i^k kW. Let $\mathbf{r}^k = \text{col}((r_i^k)_{i \in \mathcal{S}^k}) \in \mathbb{R}_{\geq 0}^{n^k}$, for all $k \in \mathcal{P}$. Moreover, let $\mathcal{A}^k = \{1, 2, \dots, N^k\}$ be the set indexing the agents of population k , and let $\mathbf{s}_a^k \in \{0, 1\}^{n^k}$ represent the selected strategy of agent $a \in \mathcal{A}^k$ (i.e., if agent a chooses strategy $i \in \mathcal{S}^k$, then \mathbf{s}_a^k is the i -th column of the $n^k \times n^k$ identity matrix). Hence, the power committed by the a -th agent is given by $\mathbf{r}^{k\top} \mathbf{s}_a^k$, and the total power committed by the entire society is given by $\sum_{k \in \mathcal{P}} \sum_{a \in \mathcal{A}^k} \mathbf{r}^{k\top} \mathbf{s}_a^k$.

To encourage the participation in the DR program, the energy power utility (EPU) provides monetary incentives to the agents based on their power demand reductions. Let $P_{i,i}^k \in \mathbb{R}_{>0}$ be the monetary incentive for a power demand

reduction of r_i^k kW, and let $\mathbf{P}^k = \text{diag}\left(\left(P_{i,i}^k\right)_{i \in \mathcal{S}^k}\right) \in \mathbb{R}^{n^k \times n^k}$. The goal of the DR program is given by

$$\begin{aligned} \min_{\{\mathbf{s}_a^k\}_{a \in \mathcal{A}^k, k \in \mathcal{P}}} & \sum_{k \in \mathcal{P}} \left(\sum_{a \in \mathcal{A}^k} \mathbf{s}_a^k \right)^\top \mathbf{P}^k \left(\sum_{a \in \mathcal{A}^k} \mathbf{s}_a^k \right) \\ \text{s.t.} & \sum_{k \in \mathcal{P}} \sum_{a \in \mathcal{A}^k} \mathbf{r}^{k\top} \mathbf{s}_a^k \geq C. \end{aligned} \quad (6)$$

The optimization problem in (6) is an integer optimization problem, which regards the minimization of the overall monetary incentive subject to the desired shaving of the demand. To solve (6) in a distributed fashion, we employ the framework of population games and RD. Given that \mathbf{x}^k describes the proportions of agents choosing the multiple strategies in population $k \in \mathcal{P}$, it follows that $N^k \mathbf{x}^k = \sum_{a \in \mathcal{A}^k} \mathbf{s}_a^k$. Thus, (6) can be rewritten as

$$\min_{\mathbf{x} \in \Delta} \sum_{k \in \mathcal{P}} (N^k)^2 \mathbf{x}^{k\top} \mathbf{P}^k \mathbf{x}^k \quad \text{s.t.} \quad \sum_{k \in \mathcal{P}} N^k \mathbf{r}^{k\top} \mathbf{x}^k \geq C. \quad (7)$$

Note that (7) is no longer an integer optimization problem, as $\mathbf{x} \in \Delta \subset \mathbb{R}^n$, and its optimal solution \mathbf{x}^* can be computed with standard quadratic optimization solvers. Since the EPU knows the DR program parameters (C and $\{N^k, \mathbf{P}^k, \mathbf{r}^k\}_{k \in \mathcal{P}}$), it is reasonable to assume that the EPU knows \mathbf{x}^* . As such, a population game with $\{\mathbf{x}^*\} = \text{NE}(\mathbf{f})$ can be designed by the EPU, which acts as a payoff provider, and the consumers can then employ the (microscopic) imitative revision protocols of Section II-A to solve (6) in a (resilient [7]) distributed fashion. As in [7], it is assumed that the EPU can measure in real-time the proportion of power demand reductions ($\mathbf{x}(t)$) at the distribution substation level.

To establish a population game G whose unique NE matches \mathbf{x}^* , the EPU sets $f_i^k(\mathbf{x}) = \alpha (x_i^{k*} - x_i^k)$, for all $i \in \mathcal{S}^k$ and all $k \in \mathcal{P}$, where $\alpha \in \mathbb{R}_{>0}$ is a gain parameter to be set in brief. Therefore, the resulting fitness vector is $\mathbf{f}(\mathbf{x}) = \alpha (\mathbf{x}^* - \mathbf{x})$, and it immediately holds that $\mathbf{x}^* \in \text{NE}(\mathbf{f})$ (c.f., Definition 1). In addition, $\mathbf{f}(\cdot)$ satisfies Standing Assumption 1 with $\theta = \mu = \alpha$, and satisfies Standing Assumption 2 as $\mathbf{f}(\mathbf{x}^*) = \mathbf{0}_n \in \text{span}(\mathbf{1}_n)$. Throughout, we assume that the Poisson alarm clocks of the agents are characterized by $R = 1 \text{ s}^{-1}$, and so we set $\alpha = 1/2$ to satisfy (2). Hence, $\tau_{\min} = 1/\alpha = 2 \text{ s}$ (the time units of τ_{\min} are given by the units of R in the Poisson clocks).

As illustration, we first consider a fixed instance of the problem with $P = 2$, $N^1 = N^2 = 10^4$, $n^1 = 3$, $n^2 = 2$, $\mathbf{r}^1 = [0.001, 0.01, 0.1]^\top$, $\mathbf{r}^2 = [0.1, 1]^\top$, $P_{i,i}^k = r_i^k, \forall i, k$, and $C = 5 \cdot 10^3$. Under such parameters, the solution of (7) is $\mathbf{x}^* = [0.261, 0.364, 0.375, 0.602, 0.398]^\top$. We compare the performance of the proposed event-triggered payoff provider against a continuously-triggered (CT) payoff provider that broadcasts the fitness vector after every strategy-update. Besides, we simulate both the expected evolution under the ODEs in (3) and (5) and the actual evolution of the microscopic decision-making processes of $2 \cdot 10^4$ agents. We simulate the dynamics under 500 different initial strategic distributions $\mathbf{x}(0)$ randomly sampled from $\Delta_{\mathbf{x}^*}$, and, for all cases, we measure the normalized distance to the NE

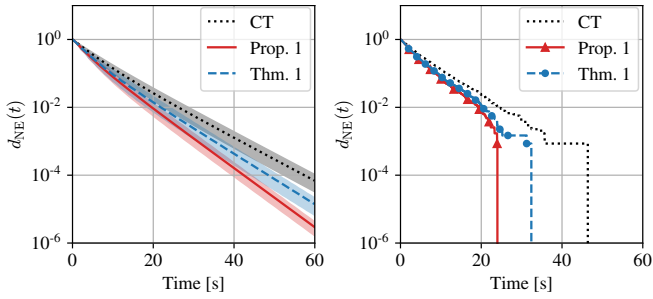


Fig. 1. Numerical results for the considered (fixed) instance of the DR problem. (Left) The expected strategic evolution under the ODEs in (3) and (5), averaged over 500 random initial conditions. (Right) A particular simulation of the actual evolution of the microscopic dynamics of $2 \cdot 10^4$ agents. Here, the markers depict the corresponding event times.

given by $d_{NE}(t) = \|\mathbf{x}^* - \mathbf{x}(t)\|_2 / \|\mathbf{x}^* - \mathbf{x}(0)\|_2$. Figure 1 (left) shows the expected convergence of the event-triggered RD in (5) under the triggering schemes of Theorem 1 and Proposition 1. These dynamics converge faster than their continuously-triggered counterpart. This counterintuitive phenomenon can be (informally) explained by comparing the right hand sides of (3) and (5). Note that while the fitness values in (3) get closer to zero as $\mathbf{x}(t) \rightarrow \mathbf{x}^*$, the fitness values in (5) remain constant in between event times. Thus, $\|\dot{\mathbf{x}}(t)\|_2$ is larger under (5) than under (3) as $\mathbf{x}(t) \rightarrow \mathbf{x}^*$, which implies that the RD in (5) moves faster near \mathbf{x}^* than the RD in (3). Figure 1 (right) shows that similar results hold for the actual microscopic dynamics, and in this case convergence is achieved in finite time.

Next, we simulate the microscopic dynamics for 50 randomized instances of the DR problem, with $P \sim \mathcal{U}[1, 3]$, $N^k \sim \mathcal{U}[5 \cdot 10^3, 10^4]$, $n^k \sim \mathcal{U}[2, 4]$, $r_i^k \sim \mathcal{U}[0.001, 1]$, $P_{i,i}^k = r_i^k$, and C uniformly sampled from the maximum and minimum attainable demand reduction capacities. For simplicity, we only consider interior Nash equilibria where $x_i^{k*} \geq 0.05$ (note that, for boundary solutions, the EPU might simply remove the unused strategies from the DR program). Figure 2 shows that the event-triggered scheme requires significantly (i.e., two order of magnitude) less broadcasting operations than its continuously-triggered counterpart while maintaining comparable convergence time. Besides, the trigger of Proposition 1 slightly outperforms the one of Theorem 1, at the expense of requiring the knowledge of \mathbf{x}^* .

V. CONCLUDING REMARKS AND FUTURE DIRECTIONS

In population games under the replicator evolutionary dynamics, convergence to a Nash equilibrium can be guaranteed even with event-based communication between the payoff provider and the agents. In our numerical simulations on a large-scale demand response problem, the proposed event-triggered schemes significantly improve communication efficiency. Future work should extend the framework to other evolutionary dynamics models and large-scale applications.

APPENDIX

We first prove Proposition 1 and then Theorem 1.

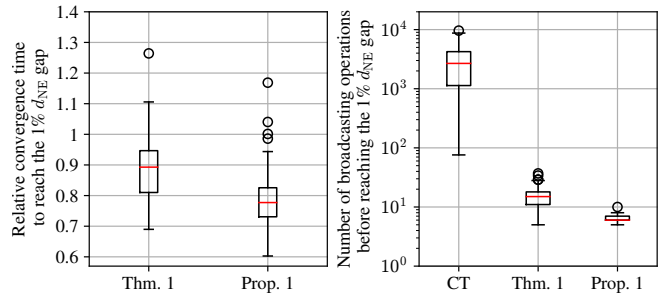


Fig. 2. Performance comparison under 50 randomized instances of the DR program. Relative convergence time refers to the quotient of the convergence time of the event-triggered scheme over the convergence time of the CT one. A relative time below 1 implies faster convergence than the CT approach.

A. Proof of Proposition 1

We can observe that the proposed trigger is free from Zeno behavior as $t_{\ell+1} - t_\ell \geq \tau > 0$ follows immediately from the definition of the triggering condition. Moreover, if $\|\mathbf{e}(t)\|_2 = \|\mathbf{x}^* - \mathbf{x}(t)\|_2 = 0$, then the event will trigger periodically throughout $[t, \infty)$ with $t_{\ell+1} = t_\ell + \tau$, implying that $\tau_{\min} = \tau$.

From [1, Thm. 5.4.7], $\mathbf{x}(0) \in \Delta_{\mathbf{x}^*} \Rightarrow \mathbf{x}(t) \in \Delta_{\mathbf{x}^*}, \forall t \geq 0$. Thus, let us consider the Lyapunov function candidate given by $V(\mathbf{x}) = \sum_{k \in \mathcal{P}} \sum_{i \in \text{supp}(\mathbf{x}^{k*})} x_i^{k*} \log(x_i^{k*}/x_i^k)$. From [1, Thm. 7.2.4], $V(\cdot)$ is a valid Lyapunov function candidate. Moreover, $V(\mathbf{x}) \rightarrow \infty$ as $\mathbf{x} \rightarrow \Delta \setminus \Delta_{\mathbf{x}^*}$, and so $V(\cdot)$ is radially unbounded with respect to $\Delta_{\mathbf{x}^*}$. In addition,

$$\begin{aligned} \dot{V}(t) &= - \sum_{k \in \mathcal{P}} \mathbf{x}^{k* \top} (\mathbf{I}_{n^k} - \mathbf{1}_{n^k} \mathbf{x}^{k \top}(t)) \mathbf{f}^k(\hat{\mathbf{x}}(t)) \\ &= -\mathbf{d}(t)^\top \mathbf{f}(\hat{\mathbf{x}}(t)), \quad [\text{with } \mathbf{d}(t) = \mathbf{x}^* - \mathbf{x}(t)] \\ &= -\mathbf{d}(t)^\top \mathbf{f}(\mathbf{x}(t)) + \mathbf{d}(t)^\top (\mathbf{f}(\mathbf{x}(t)) - \mathbf{f}(\hat{\mathbf{x}}(t))) \\ &\leq -\mathbf{d}(t)^\top \mathbf{f}(\mathbf{x}(t)) + \theta \|\mathbf{d}(t)\|_2 \|\mathbf{x}(t) - \hat{\mathbf{x}}(t)\|_2 \\ &= -(\mathbf{x}^* - \mathbf{x}(t))^\top \mathbf{f}(\mathbf{x}(t)) + \theta \|\mathbf{x}^* - \mathbf{x}(t)\|_2 \|\mathbf{e}(t)\|_2 \\ &\leq -\mu \|\mathbf{x}^* - \mathbf{x}(t)\|_2^2 + \theta \|\mathbf{x}^* - \mathbf{x}(t)\|_2 \|\mathbf{e}(t)\|_2, \end{aligned}$$

where the last inequality follows from Standing Assumption 1 in conjunction with the fact that $(\mathbf{x}^* - \mathbf{x})^\top \mathbf{f}(\mathbf{x}^*) \geq 0$, for all $\mathbf{x} \in \Delta$ (see Definition 1). Now, we observe that if $\|\mathbf{e}(t)\|_2 < (\mu/\theta) \|\mathbf{x}^* - \mathbf{x}(t)\|_2$, then $\dot{V}(t) < 0$, for all $\mathbf{x}(t) \neq \mathbf{x}^*$. Furthermore, the proposed trigger guarantees that

$$\|\mathbf{e}(t)\|_2 < \frac{\mu}{\theta} \|\mathbf{x}^* - \mathbf{x}(t)\|_2, \quad \forall t \in [t_\ell + \tau, t_{\ell+1}) : \mathbf{x}(t) \neq \mathbf{x}^*. \quad (8)$$

Therefore, to prove the global asymptotic stability of \mathbf{x}^* within $\Delta_{\mathbf{x}^*}$, it suffices to show that

$$\|\mathbf{e}(t)\|_2 < \frac{\mu}{\theta} \|\mathbf{x}^* - \mathbf{x}(t)\|_2, \quad \forall t \in [t_\ell, t_\ell + \tau) : \mathbf{x}(t_\ell) \neq \mathbf{x}^*. \quad (9)$$

Observe that if $\mathbf{x}(t_\ell) = \mathbf{x}^*$, then $\hat{\mathbf{x}}(t_\ell) = \mathbf{x}^*$ and $\dot{\mathbf{x}}(t) = \mathbf{g}(\mathbf{x}^*, \mathbf{x}^*) = \mathbf{0}_n$, for all $t \geq t_\ell$ (see Remark 1), implying that the RD in (5) have converged to \mathbf{x}^* . We now proceed to prove that (9) indeed holds.

First, from Standing Assumption 2: $\mathbf{f}^k(\mathbf{x}^*) \in \text{span}(\mathbf{1}_{n^k})$, for all $k \in \mathcal{P}$. Thus, $(\mathbf{I}_{n^k} - \mathbf{1}_{n^k} \mathbf{x}^{k \top}) \mathbf{f}^k(\mathbf{x}^*) = \mathbf{0}_{n^k}$, for all $\mathbf{x}^k \in \Delta^k$, and $(\mathbf{I}_n - \mathbf{M}(\mathbf{x})) \mathbf{f}(\mathbf{x}^*) = \mathbf{0}_n$, for all $\mathbf{x} \in \Delta$. In

consequence,

$$\dot{\mathbf{x}}(t) = \text{diag}(\mathbf{x}(t)) (\mathbf{I}_n - \mathbf{M}(\mathbf{x}(t))) (\mathbf{f}(\hat{\mathbf{x}}(t)) - \mathbf{f}(\mathbf{x}^*)). \quad (10)$$

Thus, $\|\dot{\mathbf{x}}(t)\|_2 \leq \theta \|\mathbf{Z}(t)\|_2 (\|\mathbf{e}(t)\|_2 + \|\mathbf{x}(t) - \mathbf{x}^*\|_2)$, where $\mathbf{Z}(t) = \text{diag}(\mathbf{x}(t)) (\mathbf{I}_n - \mathbf{M}(\mathbf{x}(t)))$. Here, notice that $\mathbf{Z}(t)$ is an $n \times n$ block diagonal matrix whose k -th block is given by $\mathbf{Z}^k(t) = \text{diag}(\mathbf{x}^k(t)) - \mathbf{x}^k(t)\mathbf{x}^{k\top}(t)$. Hence, the elements of the k -th block are given by $Z_{i,i}^k(t) = x_i^k(t)(1 - x_i^k(t))$ and $Z_{i,j}^k(t) = -x_i^k(t)x_j^k(t)$, for all $i \neq j$. Therefore, for every $\mathbf{x}(t) \in \Delta$, the matrix $\mathbf{Z}(t)$ is symmetric and diagonally dominant with non-negative diagonal elements, i.e., $\mathbf{Z}(t)$ is positive semi-definite. Consequently, by the Gershgorin Circle Theorem, $\|\mathbf{Z}(t)\|_2 \leq 1/2$, and so

$$\|\dot{\mathbf{x}}(t)\|_2 \leq \frac{\theta}{2} (\|\mathbf{e}(t)\|_2 + \|\mathbf{x}^* - \mathbf{x}(t)\|_2). \quad (11)$$

Next, let $\mathbf{x}(t_\ell) \neq \mathbf{x}^*$ and $y(t) = \|\mathbf{e}(t)\|_2 / \|\mathbf{x}^* - \mathbf{x}(t)\|_2$, for all $t \in [t_\ell, t_{\ell+1})$. Notice that $y(t_\ell) = 0$ is well-defined and

$$\begin{aligned} \frac{dy(t)}{dt} &\leq \frac{\|\dot{\mathbf{e}}(t)\|_2}{\|\mathbf{x}^* - \mathbf{x}(t)\|_2} + \frac{\|\mathbf{e}(t)\|_2 \|\dot{\mathbf{x}}(t)\|_2}{\|\mathbf{x}^* - \mathbf{x}(t)\|_2^2} \\ &= \frac{\|\dot{\mathbf{x}}(t)\|_2}{\|\mathbf{x}^* - \mathbf{x}(t)\|_2} (1 + y(t)), \end{aligned}$$

where the equality uses the fact that $\|\dot{\mathbf{x}}(t)\|_2 = \|\dot{\mathbf{e}}(t)\|_2$, for all $t \in [t_\ell, t_{\ell+1})$. Using (11) one can further conclude that $dy(t)/dt \leq (\theta/2)(1 + y(t))^2$, and hence, by the Comparison Lemma [22, Lemma 3.4] it holds that $y(t) \leq \phi(t)$, where $\phi(t) = \theta(t - t_\ell)/(2 - \theta(t - t_\ell))$ is the solution to $d\phi(t)/dt = (\theta/2)(1 + \phi(t))^2$, for all $t \in [t_\ell, t_\ell + 2/\theta)$ and with $\phi(t_\ell) = 0$. Therefore, for every $\sigma \in [0, 1)$, it holds that

$$y(t_\ell + \sigma\tau) \leq \frac{\theta\sigma\tau}{2 - \theta\sigma\tau} = \frac{\sigma\mu}{\theta + (1 - \sigma)\mu} < \frac{\mu}{\theta}.$$

Consequently, $\|\mathbf{e}(t)\|_2 / \|\mathbf{x}^* - \mathbf{x}(t)\|_2 = y(t) < \mu/\theta$, for all $t \in [t_\ell, t_\ell + \tau) \subset [t_\ell, t_\ell + 2/\theta)$, and thus (9) holds.

B. Proof of Theorem 1

As in the proof of Proposition 1, it is straightforward to show that the proposed trigger is free from Zeno behavior with $\tau_{\min} = \tau$.

Now, let $\mathbf{g}(t) \triangleq \mathbf{g}(\mathbf{x}(t), \hat{\mathbf{x}}(t))$. The trigger guarantees that $\|\mathbf{e}(t)\|_2 < \tau \|\mathbf{g}(t)\|_2$, for all $t \in [t_\ell + \tau, t_{\ell+1}) : \mathbf{g}(t) \neq \mathbf{0}_n$. From (11) in the proof of Proposition 1, it follows that $\|\mathbf{g}(t)\|_2 \leq (\theta/2)(\|\mathbf{e}(t)\|_2 + \|\mathbf{x}^* - \mathbf{x}(t)\|_2)$, which implies that the proposed trigger ensures that

$$\|\mathbf{e}(t)\|_2 < \frac{\mu}{\theta} \|\mathbf{x}^* - \mathbf{x}(t)\|_2, \quad \forall t \in [t_\ell + \tau, t_{\ell+1}) : \mathbf{g}(t) \neq \mathbf{0}_n. \quad (12)$$

Observe that if $\mathbf{g}(\tilde{t}) = \mathbf{0}_n$ at some arbitrary $\tilde{t} \in [t_\ell + \tau, t_{\ell+1})$, then $t_{\ell+1} = \tilde{t}$ and $\|\mathbf{e}(\tilde{t})\|_2 = \|\mathbf{g}(\mathbf{x}(\tilde{t}), \hat{\mathbf{x}}(\tilde{t}))\|_2 = 0$.

On the other hand, from [1, Thm. 5.4.13] it follows that $\zeta \in \Delta_{\mathbf{x}^*} \Rightarrow (\mathbf{g}(\zeta, \zeta) = \mathbf{0}_n \Leftrightarrow \zeta = \mathbf{x}^*)$. Hence, since $\mathbf{x}(t) \in \Delta_{\mathbf{x}^*}, \forall t \geq 0$, and $\|\mathbf{e}(t)\|_2 = 0 \Rightarrow \hat{\mathbf{x}}(t) = \mathbf{x}(t)$, it holds that

$$\|\mathbf{e}(t)\|_2 = 0 \quad \Rightarrow \quad (\|\mathbf{g}(\mathbf{x}(t), \hat{\mathbf{x}}(t))\|_2 = 0 \Leftrightarrow \mathbf{x}(t) = \mathbf{x}^*).$$

Therefore, if $\|\mathbf{e}(t)\|_2 = \|\mathbf{g}(t)\|_2 = 0$, then $\mathbf{x}(t) = \mathbf{x}^*$ and the RD in (5) have converged to \mathbf{x}^* . By marshalling all of these

facts, we conclude that by guaranteeing (12), the proposed trigger also guarantees (8). As such, the global asymptotic stability of \mathbf{x}^* within $\Delta_{\mathbf{x}^*}$ follows directly from the proof of Proposition 1 by recalling that (9) holds.

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