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Interconnection Schemes in Modeling and Control

Pablo Borja, Joel Ferguson, and Arjan van der Schaft

***Abstract*— Interconnection schemes are ubiquitous in physical systems. For instance, in multi-domain systems consisting of interconnected subsystems from different physical domains. Furthermore, the interconnection of two or more systems has also been exploited to analyze and control dynamical systems, especially passive ones. To this end, the most common interconnection structure is the negative feedback interconnection. However, this approach is unsuitable to directly couple the states of the subsystems in the overall system's energy as customarily occurs in physical systems. This letter provides two interconnection approaches that overcome this issue. Notably, it is shown that these interconnection structures are suitable for decomposing passive systems into the interconnection of simpler passive subsystems. Moreover, these interconnections schemes allow the interpretation of some existing nonlinear control approaches as the interconnection of a passive plant with a passive controller. Additionally, the interpretation of the proposed interconnection structures is provided via bond graphs.**

I. Introduction

The behavior of complex systems can be simplified by decomposing them into simpler subsystems. This idea has had great success when dealing with physical systems, e.g., electrical circuits. However, this approach relies on understanding how the subsystems are interconnected. Therefore, identifying appropriate interconnection structures is essential in modeling, analyzing, and controlling dynamical systems, especially physical ones.

Probably the most illustrative example of the relevance of interconnection structures in the analysis and control of dynamical systems is given by the passivity theorem [1]. Loosely speaking, this theorem states that the interconnection of two passive systems—via a power-preserving interconnection scheme—yields another passive system. Such a result has been widely applied for control purposes, and it is the foundation of the passivity-based control (PBC) techniques.

Among the different PBC approaches, the control by interconnection (CbI) method directly considers the role of individual subsystems via power-preserving interconnections. In this control technique, the controller is designed as a passive system interconnected to the plant. Then, the energy of both systems are coupled through invariant functions, known as Casimir functions [2], which are used to shape the energy of the overall system and guarantee the stability of the desired equilibrium. An advantage of CbI over other nonlinear control techniques is the physical interpretation of the controller. For more details on CbI, we refer the reader to [3], [4], [5].

A broad range of physical systems composed of several subsystems—possibly from different physical domains—exhibit interconnections in their energy functions. However, negative feedback, the most common power-preserving interconnection scheme, cannot represent this phenomenon. We stress that a framework that permits interpreting energy couplings as the interaction of two or more physical subsystems is of utmost relevance in understanding, analyzing, and controlling complex physical systems. However, while some references exist, e.g., [6], [7], where alternative power-preserving interconnection structures are studied, this is a relatively unexplored topic.

This letter proposes two interconnection schemes that, in contrast to traditional interconnection methods, couple two or more (cyclo-)passive systems through the overall storage (energy) function. An essential difference between the proposed and conventional interconnection structures is that the subsystems are interconnected via the integral or derivative of their passive outputs, enabling the mentioned energy couplings. The proposed interconnections are paramount to enlarging the understanding of how the elements of physical systems are interconnected. In particular, they permit decomposing existing models into the interconnection of multiple simpler subsystems. Moreover, the proposed interconnection can be used for nonlinear control design and allow the physical interpretation of some existing PBC techniques—e.g., interconnection and damping assignment (IDA) PBC. The proposed approach greatly extends the applicability of CbI by offering versatile alternatives to the standard method based on Casimir functions. Another contribution of this work is the interpretation of the proposed interconnection schemes bond graphs [8], one of the most general modeling approaches.

The remainder of this letter is structured as follows: Section II is devoted to the preliminaries. Then, the main results of this paper, i.e., the new interconnection schemes, are presented in Section III. The bond graph interpretation of the interconnection schemes is provided in Section IV. Section V illustrates the role of the new interconnection schemes in the control and modeling of physical systems. Finally, concluding remarks and future research are discussed in Section VI.

Pablo Borja is with the School of Engineering, Computing and Mathematics, University of Plymouth, Plymouth, UK. pablo.borjarosales@plymouth.ac.uk

Joel Ferguson is with the School of Engineering, The University of Newcastle, Australia. joel.ferguson@newcastle.edu.au

Arjan van der Schaft is with the Jan C. Willems Center for Systems and Control, and the Bernoulli Institute for Mathematics, Computer Science, and Artificial Intelligence, University of Groningen, Groningen, The Netherlands. a.j.van.der.schaft@rug.nl

II. Preliminaries

A. Notation

The symbol I_n represents the $n \times n$ identity matrix; $\mathbf{0}$ is a vector or matrix whose entries are zeros; given $x \in \mathbb{R}^n$, $t \in \mathbb{R}$ and the mappings $f : \mathbb{R}^n \to \mathbb{R}^n$, $q : \mathbb{R} \to \mathbb{R}^m$ we adopt the notation $\frac{\partial f(x)}{\partial x} = \left[\frac{\partial f(x)}{\partial x_1} \ \ldots \ \frac{\partial f(x)}{\partial x_n}\right]^\top$, $\dot{q} = \frac{dq(t)}{dt}$. To simplify the notation, we omit the arguments of functions in proofs or when they are clear from the context.

B. Passive systems

Loosely speaking, a system is said to be passive if it cannot generate energy. In this paper, we restrict our attention to nonlinear systems of the form

$$\begin{aligned} \dot{x} &= f(x(t)) + g(x(t))u(t), \\ y &= h(x(t)) + j(x(t))u(t) \end{aligned} \tag{1}$$

where $t \in \mathbb{R}_{\geq 0}$ denotes time, $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ represents the states of the system; $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ denotes the input vector, with $m \leq n$; $f : \mathbb{R}^n \to \mathbb{R}^n$ is often referred as drift vector; $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ is the input matrix with rank m; $y : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ is the system's output; $h : \mathbb{R}^n \to \mathbb{R}^m$; $j : \mathbb{R}^n \to \mathbb{R}^{m\times m}$. The term $j(x(t))u(t)$ is referred to as the feedthrough term. Note that $j(x(t)) = \mathbf{0}$ implies that the relative degree of the output is equal to or greater than one. Otherwise, the relative degree of $y(t)$ equals zero.

The following definition formalizes the concept of passive system.

Definition 1 ((Cyclo-)Passive system): A system of the form (1) is said to be passive if there exists a nonnegative function $S : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$—referred to as the storage function—such that

$$S(x(t)) \leq S(x(0)) + \int_0^t u^\top(\tau)y(\tau)d\tau \quad \forall\ t \in \mathbb{R}_{\geq 0}. \tag{2}$$

If (2) holds, $y(t)$ is referred to as the passive output. Moreover, (1) is said to be cyclo-passive if (2) holds, but the nonnegative condition on the storage function is relaxed, i.e., $S : \mathbb{R}^n \to \mathbb{R}$.

The supply rate $u^\top(t)y(t)$ represents the power inflow into the system. Moreover, assuming differentiability of $S(x(t))$, (2) can be rewritten as[1]

$$\dot{S} \leq u^\top y. \tag{3}$$

We refer the reader to [1] and [9] for a detailed exposition of (cyclo-)passivity and (cyclo-)passive systems.

Passive outputs play an essential role in the interconnection of dynamical systems. However, multiple passive outputs may exist corresponding to the same storage function—see [10] for a characterization of the passive outputs. Below, we introduce an assumption that ensures the existence of $h(x)$ and $j(x)$ such that (3) holds.

Assumption 1: Considering (1), there exists a storage function $S(x) \in \mathbb{R}$ such that

$$\left(\frac{\partial S(x)}{\partial x}\right)^\top f(x) \leq 0.$$

[1] Henceforth, we omit the argument t to simplify the notation.

Note that if (1) satisfies Assumption 1, then a straightforward selection of $h(x)$ and $j(x)$ is given by

$$h(x) = g^\top(x)\frac{\partial S(x)}{\partial x}, \quad j(x) = \mathbf{0},$$

yielding

$$y = g^\top(x)\frac{\partial S(x)}{\partial x}, \tag{4}$$

While (4) is the most common passive output in the literature, other selections of $h(x)$ and $j(x)$ generate other passive outputs that can be interesting for interconnection purposes. See, for instance, [4].

C. Interconnection of passive systems

Consider two (cyclo-)passive systems

$$\Sigma_i : \begin{cases} \dot{x}_i = f_i(x_i) + g_i(x_i)u_i \\ y_i = h_i(x_i) + j_i(x_i)u_i \end{cases} \qquad i \in \{1, 2\}, \tag{5}$$

where the corresponding storage functions are $S_i(x_i)$.

Customarily, the interconnection of the systems (5) is carried out through an interconnection subsystem of the form

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & I_n \\ -I_n & \mathbf{0} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \tag{6}$$

where $v_1, v_2 \in \mathbb{R}^m$ are external signals (e.g., control inputs). Hence, the interconnected system takes the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1) + y_2 \\ f_2(x_2) - y_1 \end{bmatrix} + \begin{bmatrix} g_1(x_1)v_1 \\ g_2(x_2)v_2 \end{bmatrix}. \tag{7}$$

Considering the definitions

$$x_{\mathtt{T}} := [x_1^\top \ x_2^\top]^\top, \ y_{\mathtt{T}} := [y_1^\top \ y_2^\top]^\top, \ v_{\mathtt{T}} := [v_1^\top \ v_2^\top]^\top,$$

the interconnected system (7) is (cyclo-)passive with passive output $y_{\mathtt{T}}$ and storage function

$$S_{\mathtt{T}}(x_{\mathtt{T}}) = S_1(x_1) + S_2(x_2) \tag{8}$$

In particular, it follows from (6), (7), and (8) that

$$\dot{S}_{\mathtt{T}} \leq v_1^\top y_1 + v_2^\top y_2 = v_{\mathtt{T}}^\top y_{\mathtt{T}}. \tag{9}$$

Remark 1: The interconnection scheme (6) yields a storage function (8) that only contains terms depending on x_1 or x_2, and no terms depending on both of them.

Remark 2: Customarily, the signals v_i are external control inputs used to inject damping into the corresponding subsystem. The most common damping injection terms are of the form $v_i = -K_{\mathtt{d}_i}y_i$, where $K_{\mathtt{d}_i} \in \mathbb{R}^{m\times m}$ are positive semi-definite matrices. Thus,

$$\dot{S}_{\mathtt{T}} \leq v_1^\top y_1 + v_2^\top y_2 = -y_1^\top K_{\mathtt{p}_1}y_1 - y_2^\top K_{\mathtt{p}_2}y_2 \leq 0.$$

III. Interconnection through Coupled Storage Functions

Dynamical systems, including physical ones, are customarily coupled through their storage functions, which are often related to energy. However, as mentioned in Remark 1, this cannot be achieved via (6). To explain the interconnection of two systems through the interconnected storage function, this section proposes two alternative interconnection schemes.

A. Interconnection through the integration of passive outputs

The following assumption is necessary to present the interconnection on the passive outputs through integral terms.

Assumption 2: Given the (cyclo-)passive systems (5), there exist differentiable functions $\gamma_1, \gamma_2 : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\dot{\gamma}_1 = \left(\tfrac{\partial \gamma_1(x_1)}{\partial x_1}\right)^\top (f_1(x_1) + g_1(x_1)u_1) = y_1$$
$$\dot{\gamma}_2 = \left(\tfrac{\partial \gamma_2(x_2)}{\partial x_2}\right)^\top (f_2(x_2) + g_2(x_2)u_2) = y_2.$$

In [11], the authors provide sufficient conditions to guarantee that Assumption 2 holds for port-Hamiltonian systems.

The following proposition establishes an interconnection scheme through the signals $\gamma_1(x_1)$ and $\gamma_2(x_2)$. We omit the arguments of γ_1 and γ_2 to simplify the notation.

Proposition 1: Consider the subsystems (5) satisfying Assumptions 1 and 2 and a differentiable function $\Phi_\mathtt{T} : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$. The interconnection structure

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -I_n & \mathbf{0} \\ \mathbf{0} & -I_n \end{bmatrix} \begin{bmatrix} \frac{\partial \Phi_\mathtt{T}(\gamma_1,\gamma_2)}{\partial \gamma_1} \\ \frac{\partial \Phi_\mathtt{T}(\gamma_1,\gamma_2)}{\partial \gamma_2} \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{10}$$

ensures that the interconnected system is (cyclo-)passive with a storage function

$$S_\mathtt{T}(x_\mathtt{T}) = S_1(x_1) + S_2(x_2) + \Phi_\mathtt{T}(\gamma_1, \gamma_2) \tag{11}$$

and input-output port $(v_\mathtt{T}, y_\mathtt{T})$.

Proof: The time derivative of (11) is given by

$$\begin{aligned} \dot{S}_\mathtt{T} &\leq u_1^\top y_1 + u_2^\top y_2 + \left(\tfrac{\partial \Phi_\mathtt{T}}{\partial \gamma_1}\right)^\top \dot{\gamma}_1 + \left(\tfrac{\partial \Phi_\mathtt{T}}{\partial \gamma_2}\right)^\top \dot{\gamma}_2 \\ &= \left(u_1 + \tfrac{\partial \Phi_\mathtt{T}}{\partial \gamma_1}\right)^\top y_1 + \left(u_2 + \tfrac{\partial \Phi_\mathtt{T}}{\partial \gamma_2}\right)^\top y_2 \\ &= v_\mathtt{T}^\top y_\mathtt{T}. \end{aligned}$$

■

The interconnection structure (10) can be used for control design by considering the plant as one of the subsystems and the controller as the other. In that case, $\Phi_\mathtt{T}(\gamma_1, \gamma_2)$ must be designed such that $S_\mathtt{T}(x_\mathtt{T})$, given in (11), is positive definite with respect to the desired equilibrium. Note that the states of the subsystems are coupled through $\Phi_\mathtt{T}(\gamma_1, \gamma_2)$. Hence, no further Casimir functions are needed to relate the states of both subsystems.

Remark 3: The expression (10) generalizes the interconnection of two input-output Hamiltonian systems with dissipation, where $\Phi_\mathtt{T}(\gamma_1, \gamma_2) = -\gamma_1^\top \gamma_2$ (see [7]).

B. Interconnection through the differentiation of passive outputs

To present the results of this section, we introduce a particular representation of the element-wise differentiation of a matrix. To this end, we consider a differentiable matrix $\Psi_\mathtt{T} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{m \times m}$. Hence, the element-wise derivative of $\Psi_\mathtt{T}(x_\mathtt{T})$ can be expressed as follows

$$\dot{\Psi}_\mathtt{T} = \beta_1(x_\mathtt{T}, \dot{x}_1) + \beta_2(x_\mathtt{T}, \dot{x}_2) \tag{12}$$

with

$$\beta_i(x_\mathtt{T}, \dot{x}_i) := \sum_{j=1}^{n} \frac{\partial \Psi_\mathtt{T}(x_\mathtt{T})}{\partial (x_i)_j} (\dot{x}_i)_j, \quad i \in \{1, 2\},$$

where $(x_i)_j$ denotes the jth element of x_i. Given two (cyclo-)passive subsystems of the form (5), the following proposition proposes an interconnection scheme through the derivative of their passive outputs.

Proposition 2: Consider the subsystems (5) satisfying Assumption 1 and an element-wise differentiable matrix $\Psi_\mathtt{T} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{m \times m}$. The interconnection structure

$$\begin{aligned} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} &= \begin{bmatrix} \mathbf{0} & -\Psi_\mathtt{T}(x_\mathtt{T}) \\ -\Psi_\mathtt{T}^\top(x_\mathtt{T}) & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} \\ &\quad + \begin{bmatrix} \mathbf{0} & -\beta_2(x_\mathtt{T}, \dot{x}_\mathtt{T}) \\ -\beta_1^\top(x_\mathtt{T}, \dot{x}_\mathtt{T}) & \mathbf{0} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \end{aligned} \tag{13}$$

ensures that the interconnected system is (cyclo-)passive with a storage function

$$S_\mathtt{T}(x_\mathtt{T}) = S_1(x_1) + S_2(x_2) + y_1^\top \Psi_\mathtt{T}(x_\mathtt{T}) y_2. \tag{14}$$

Proof: The time derivative of (14) is given by

$$\begin{aligned} \dot{S}_\mathtt{T} &\leq u_1^\top y_1 + u_2^\top y_2 + \dot{y}_1^\top \Psi_\mathtt{T} y_2 + \dot{y}_2^\top \Psi_\mathtt{T}^\top y_1 + y_1^\top \dot{\Psi}_\mathtt{T} y_2 \\ &= (u_1 + \Psi_\mathtt{T} \dot{y}_2 + \beta_2 y_2)^\top y_1 + \left(u_2 + \Psi_\mathtt{T}^\top \dot{y}_1 + \beta_1^\top y_1\right)^\top y_2 \\ &= v_\mathtt{T}^\top y_\mathtt{T}. \end{aligned}$$

■

The interconnection approach (13) poses a problem for implementation as $u_\mathtt{T}$ is a function of $\dot{y}_i$ and $\dot{x}_\mathtt{T}$, both of which could contain $u_\mathtt{T}$. This results in an implicit expression of the control law, which requires resolution. To this end, we introduce the following mappings

$$f_\mathtt{T}(x_\mathtt{T}) := \begin{bmatrix} f_1(x_1) \\ f_2(x_2) \end{bmatrix}, \qquad g_\mathtt{T}(x_\mathtt{T}) := \begin{bmatrix} g_1(x_1) \\ g_2(x_2) \end{bmatrix}$$
$$\Theta(x_\mathtt{T}) := \begin{bmatrix} \mathbf{0} & \theta_1(x_\mathtt{T}) \\ \theta_2(x_\mathtt{T}) & \mathbf{0} \end{bmatrix},$$

where

$$\theta_1(x_\mathtt{T}) = \Psi(x_\mathtt{T}) \left(\frac{\partial y_2}{\partial x_2}\right)^\top + \sum_{i=1}^{n} \frac{\partial \Psi(x_\mathtt{T})}{\partial x_{2_i}} y_2 e_i^\top$$
$$\theta_2(x_\mathtt{T}) = \Psi^\top(x_\mathtt{T}) \left(\frac{\partial y_1}{\partial x_1}\right)^\top + \sum_{i=1}^{n} \frac{\partial \Psi^\top(x_\mathtt{T})}{\partial x_{1_i}} y_1 e_i^\top.$$

The following assumptions characterize the class of systems such that the control laws given in (13) can be resolved in an explicit form.

Assumption 3: The passive output of the subsystems (5) do not contain feedthrough terms, i.e., $j_i(x_i) = \mathbf{0}$. Consequently, $y_i = h_i(x_i) = g_i^\top(x_i) \frac{\partial S_i(x_i)}{\partial x_i}$.

Assumption 4: $\det(I_{2m} + \Theta(x_\mathtt{T}) g_\mathtt{T}(x_\mathtt{T})) \neq 0$.

Note that (13) can be rewritten as

$$u_\mathtt{T} = -\Theta(x_\mathtt{T}) (f_\mathtt{T}(x_\mathtt{T}) + g_\mathtt{T}(x_\mathtt{T}) u_\mathtt{T}) + v_\mathtt{T}. \tag{15}$$

Accordingly, Assumptions 3 and 4 ensure that (15) can be expressed as

$$u_\mathtt{T} = -(I_{2m} + \Theta(x_\mathtt{T}) g_\mathtt{T}(x_\mathtt{T}))^{-1} (\Theta(x_\mathtt{T}) f_\mathtt{T}(x_\mathtt{T}) + v_\mathtt{T}). \tag{16}$$

As the control input $u_\mathtt{T}$ is a function of only $x_\mathtt{T}$ and $v_\mathtt{T}$, it can be directly implemented.

C. Discussion

To ease the presentation of the interconnection structures in this section, we have considered only two subsystems with ports of the same dimension. However, we stress that the interconnection of N (cyclo-)passive subsystems is possible following the same rationale. Moreover, if the ports (u_i, y_i) have different dimensions, i.e., $u_1, y_1 \in \mathbb{R}^{m_1}$ and $u_2, y_2 \in \mathbb{R}^{m_2}$, it is possible to adapt (10) and (13). For (10), we can propose constant matrices to match the dimensions—e.g., $\gamma_1(x_1)$, $\Gamma\gamma_2(x_2)$, where $\Gamma \in \mathbb{R}^{m_1 \times m_2}$. For (13), we consider $\Psi_{\mathtt{T}} : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{m_1 \times m_2}$.

A PID structure can be obtained through the interconnection schemes (6), (10), and (13). In particular, suppose that Σ_1 corresponds to the plant to be controlled, satisfying Assumptions 1 and 2, and Σ_2 to the controller to be designed. Consider the controller state

$$x_2 = \begin{bmatrix} K_{\mathtt{p}}\gamma_1 \\ y_1 \end{bmatrix},$$

where $K_{\mathtt{p}} \in \mathbb{R}^{m \times m}$ is positive definite, and

$$\Sigma_2 : \begin{cases} \dot{x}_2 = \begin{bmatrix} K_{\mathtt{p}} y_1 \\ u_2 \end{bmatrix} \\ y_2 = K_{\mathtt{p}} y_1. \end{cases}$$

Σ_2 satisfies Assumptions 1 and 2 with $S_2(x_2) = \frac{1}{2} y_1^\top K_{\mathtt{p}} y_1$ and $\gamma_2 = K_{\mathtt{p}}\gamma_1$, respectively. Consider $\Psi_{\mathtt{T}} = K_{\mathtt{d}} K_{\mathtt{p}}^{-1}$ and

$$\Phi_{\mathtt{T}}(\gamma_1, \gamma_2) = \gamma_1^\top K_{\mathtt{i}} K_{\mathtt{p}}^{-1} \gamma_2 = \gamma_1^\top K_{\mathtt{i}} \gamma_1,$$

where the gain matrices $K_{\mathtt{i}}, K_{\mathtt{d}} \in \mathbb{R}^{m \times m}$ are positive definite. This selection guarantees that Assumptions 3 and 4 hold—because $\Theta = \mathbf{0}$. Thus, by interconnecting the plant Σ_1 with the dynamical controller Σ_2 via (6), (10), and (13), with $v_i = \mathbf{0}$, the control input to the plant is given by

$$\begin{aligned} u_1 &= -y_2 - K_{\mathtt{i}} K_{\mathtt{p}}^{-1} \gamma_2 - K_{\mathtt{i}} K_{\mathtt{p}}^{-1} \dot{y}_2 \\ &= -K_{\mathtt{p}} y_1 - K_{\mathtt{i}} \gamma_1 - K_{\mathtt{d}} \dot{y}_1 \\ &= -K_{\mathtt{p}} y_1 - K_{\mathtt{i}} \int y_1(t) dt - K_{\mathtt{d}} \dot{y}_1. \end{aligned} \tag{17}$$

The controller (17) is known as PID-PBC in the literature. We refer the reader to [12] for further details.

Below, we list some additional observations:

- The interconnection schemes (6), (10), and (13) can be combined. This is illustrated in Section V.
- The external inputs v_i can be exploited to inject damping into the subsystems or modify (independently) their storage functions. This is shown in Section V.
- The interconnection approach (10) is not hampered by the dissipation obstacle.
- The energy of the interconnected system can be shaped through $\Phi_{\mathtt{T}}(\gamma_1, \gamma_2)$ and $\Upsilon(x_{\mathtt{T}}, y_{\mathtt{T}})$ without solving partial differential equations.

IV. Bond Graph Interpretation of the Interconnection Structures

This section presents a graphical interpretation of the interconnection schemes introduced in Section III. The bong graph formalism is used for this presentation, and the interested reader is referred to [8] for an introduction.

The standard feedback interconnection (6) can be interpreted as an interconnection of the two passive subsystems via a gyrator element, as shown in Fig. 1. In this case, the storage function of the interconnected system is the sum of the sub-system storage functions, resulting in (8) and the passivity inequality (9).

The passive interconnection via a storage function (10) can be interpreted as shown in Fig. 2. The added storage function $\Phi_{\mathtt{T}}(\gamma_1, \gamma_2)$ forms a 2-port capacitor element

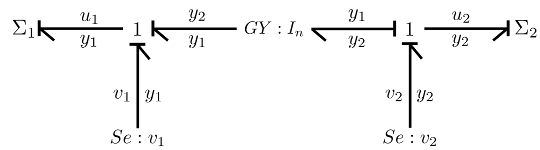


Fig. 1. A typical feedback structure for passive systems.

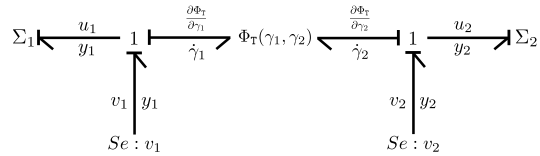


Fig. 2. Bond-graph representation of the interconnection (10).

with states γ_1 and γ_2. From Assumption 2, γ_1 and γ_2 can be identified with functions of the states x_1 and x_2, allowing the energy $\Phi_{\mathtt{T}}(\gamma_1, \gamma_2)$ to be described as a function of plant states. The total energy of the interconnected system is the sum of all energy-storing components, resulting in (11). The function $\Phi_{\mathtt{T}}(\gamma_1, \gamma_2)$ is a user-defined function to shape the energy of the interconnected system.

The passive interconnection (13) introduces an auxiliary storage function of the form

$$\Upsilon(x_{\mathtt{T}}, y_{\mathtt{T}}) = y_1^\top \Psi_{\mathtt{T}}(x_1, x_2) y_2, \tag{18}$$

which is present in the interconnected storage function (14). This scheme has a graphical interpretation, shown in Fig. 3, where the added storage function is a 3-port element. To understand the bond graph representation, recall the definitions (18) and (12) to express the time derivative of the added storage function as

$$\begin{aligned} \dot{\Upsilon} &= \left(\frac{\partial \Upsilon}{\partial x_1}\right)^\top \dot{x}_1 + \left(\frac{\partial \Upsilon}{\partial x_2}\right)^\top \dot{x}_2 + \left(\frac{\partial \Upsilon}{\partial y_1}\right)^\top \dot{y}_1 + \left(\frac{\partial \Upsilon}{\partial y_2}\right)^\top \dot{y}_2 \\ &= (\beta_2 y_2)^\top y_1 + (\beta_1^\top y_1)^\top y_2 + y_2^\top \Psi_{\mathtt{T}}^\top \dot{y}_1 + y_1^\top \Psi_{\mathtt{T}} \dot{y}_2. \end{aligned}$$

From this expression, it can be seen that $\mathcal{D}$ represents a power-preserving Dirac structure that satisfies the relationships

$$y_1^\top \dot{\Psi}_{\mathtt{T}} y_2 = \left(\frac{\partial \Upsilon}{\partial x_{\mathtt{T}}}\right)^\top \dot{x}_{\mathtt{T}} = (\beta_2 y_2)^\top y_1 + (\beta_1^\top y_1)^\top y_2.$$

The transformer elements enforce the remaining relationships

$$\left(\frac{\partial \Upsilon}{\partial y_1}\right)^\top \dot{y}_1 = y_2^\top \Psi_{\mathtt{T}}^\top \dot{y}_1, \quad \left(\frac{\partial \Upsilon}{\partial y_2}\right)^\top \dot{y}_2 = y_1^\top \Psi_{\mathtt{T}} \dot{y}_2.$$

Note that the causal strokes on the transformer elements indicate that $\dot{y}_i$ is the output from the storage function, resulting in a causality conflict. As discussed in Section III-B, this causality conflict can be resolved under Assumptions 3 and 4. The total storage function for the interconnected system is just the sum of the subsystem energies, resulting in (14). The function $\Upsilon(x_{\mathtt{T}}, y_{\mathtt{T}})$ can be chosen to shape the energy of the interconnected system.

V. Examples

A. Magnetic levitation system

Consider the system depicted in Fig. 4(a), which consists of a metallic ball suspended in a magnetic field created

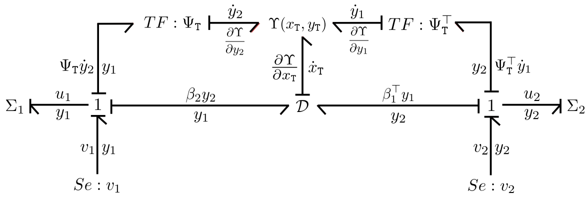
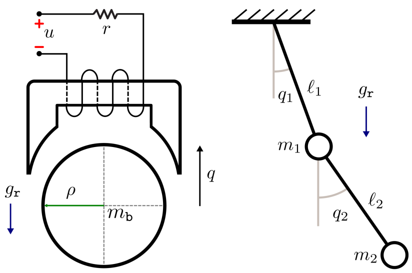


Fig. 3. Bond-graph representation of the interconnection (13).



(a) (b)

Fig. 4. (a) Magnetic levitation system (b) Double pendulum system

by an electromagnet. We consider the model proposed in [13], which is given by

$$\begin{bmatrix}\dot{q}\\ \dot{p}\\ \dot{\varphi}\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & -r\end{bmatrix}\begin{bmatrix}\frac{\partial H(q,p,\varphi)}{\partial q}\\ \frac{\partial H(q,p,\varphi)}{\partial p}\\ \frac{\partial H(q,p,\varphi)}{\partial \varphi}\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix} u \tag{19}$$

$$H(q,p,\varphi) = m_{\mathtt{b}} g_{\mathtt{r}} q + \tfrac{1}{2m_{\mathtt{b}}} p^2 + \tfrac{1}{2k}\varphi^2 (\rho - q)$$

where q, p, and φ are the state variables representing the position of the ball's center of mass, its momenta, and the magnetic flux, respectively; $m_{\mathtt{b}}$ denotes the mass of the ball; $g_{\mathtt{r}}$ is the constant of gravitational acceleration; ρ is the radius of the ball such that $q = \rho$ implies that the ball is touching the electromagnet; k is a constant parameter depending on the number of turns in the coil. Using the result of Proposition 1, we show that the system (19) is the result of interconnecting a mechanical and an electrical system through the integration of their passive outputs, i.e., both systems are interconnected via the interconnection scheme (10). In particular, we consider the mechanical subsystem

$$\Sigma_{\mathtt{m}} : \begin{cases} \begin{bmatrix}\dot{q}\\ \dot{p}\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}\frac{\partial H_{\mathtt{m}}(q,p)}{\partial q}\\ \frac{\partial H_{\mathtt{m}}(q,p)}{\partial p}\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} u_{\mathtt{m}}\\ H_{\mathtt{m}}(q,p) = \frac{1}{2m_{\mathtt{b}}} p^2 + m_{\mathtt{b}} g_{\mathtt{r}} q\\ u_{\mathtt{m}} = \frac{1}{2k}\varphi^2, \end{cases} \tag{20}$$

and the electrical subsystem

$$\Sigma_{\mathtt{e}} : \begin{cases} \dot{\varphi} = -r\frac{\partial H_{\mathtt{e}}(\varphi)}{\partial \varphi} + r u_{\mathtt{e}}\\ H_{\mathtt{e}}(\varphi) = \frac{\rho}{2k}\varphi^2\\ u_{\mathtt{e}} = \frac{1}{k}\varphi q + \frac{1}{r} u. \end{cases} \tag{21}$$

The time derivatives of the energy functions satisfy

$$\dot{H}_{\mathtt{m}} = \dot{q} u_{\mathtt{m}}, \qquad \dot{H}_{\mathtt{e}} \leq \dot{\varphi} u_{\mathtt{e}}.$$

Accordingly, the subsystems $\Sigma_{\mathtt{m}}$ and $\Sigma_{\mathtt{e}}$ satisfy Assumptions 1 and 2, with

$$y_{\mathtt{m}} = \frac{1}{m_{\mathtt{b}}} p = \dot{q}; \quad y_{\mathtt{e}} = \dot{\varphi}; \quad \gamma_{\mathtt{m}} = q; \quad \gamma_{\mathtt{e}} = \varphi.$$

Thus, the system (19) is the result of interconnecting $\Sigma_{\mathtt{m}}$ with $\Sigma_{\mathtt{e}}$, via the interconnection scheme (10), where $\Phi_{\mathtt{T}}(q,\varphi) = -\frac{1}{2k}\gamma_{\mathtt{e}}^2 \gamma_{\mathtt{m}} = -\frac{1}{2k}\varphi^2 q$. Note that, the energy function $H(q,p,\varphi)$ given in (19) satisfies $H(q,p,\varphi) = H_{\mathtt{m}}(q,p) + H_{\mathtt{e}}(\varphi) + \Phi_{\mathtt{T}}(q,\varphi)$.

In [13], the authors provide a controller derived from the IDA-PBC approach to stabilize (19) at the desired equilibrium $(q,p,\varphi) = (q_\star, 0, \sqrt{2k m_{\mathtt{b}} g_{\mathtt{r}}})$, with $q_\star \in (\rho, \infty)$. A critical remark is that the interconnection pattern should be modified. Below, we explain, in the context of this paper, the controller reported in [13].

Note that the external input only affects the electrical system. Hence, we can modify the energy and, consequently, the passive output of $\Sigma_{\mathtt{e}}$, but $u_{\mathtt{m}}$ cannot be changed.[2] Moreover, we assume that $u_{\mathtt{m}} = \frac{1}{2k}\varphi^2$ corresponds to an interconnection scheme of the form (6), (10), or (13), considering the new energy function and passive output for the electrical subsystem. Note that $\frac{1}{2k}\varphi^2$ is integrable. Thus, we can propose

$$\begin{aligned} H_{\mathtt{de}}(\varphi) &= \textstyle\int_0^t \frac{1}{2k}\varphi^2(\tau) = \frac{1}{6k}\varphi^3(t)\\ \implies \quad y_{\mathtt{de}} &= \frac{\partial H_{\mathtt{de}}(\varphi)}{\partial \varphi} = \frac{1}{2k}\varphi^2 = u_{\mathtt{m}}. \end{aligned}$$

This leads to an interconnection scheme of the form (6). Consequently, the new electrical system is given by

$$\dot{\varphi} = -r\frac{\partial H_{\mathtt{de}}(\varphi)}{\partial \varphi} - y_{\mathtt{m}} + v_{\mathtt{de}} = -\frac{r}{2k}\varphi^2 - \frac{1}{m_{\mathtt{b}}} p + v_{\mathtt{de}}. \tag{22}$$

Equating (21) with the dynamics of φ given in (22), we obtain the control law

$$u = \frac{r}{k}\varphi(\rho - q) - \frac{r}{2k}\varphi^2 - \frac{1}{m_{\mathtt{b}}} p + v_{\mathtt{de}}, \tag{23}$$

yielding the interconnected system

$$\begin{aligned} \dot{q} &= \tfrac{\partial H_{\mathtt{T}}(x_{\mathtt{T}})}{\partial p}\\ \dot{p} &= -\tfrac{\partial H_{\mathtt{T}}(x_{\mathtt{T}})}{\partial q} + \tfrac{\partial H_{\mathtt{T}}(x_{\mathtt{T}})}{\partial \varphi}\\ \dot{\varphi} &= -\tfrac{\partial H_{\mathtt{T}}(x_{\mathtt{T}})}{\partial p} - r\tfrac{\partial H_{\mathtt{T}}(x_{\mathtt{T}})}{\partial \varphi} + v_{\mathtt{de}}, \end{aligned} \tag{24}$$

with $H_{\mathtt{T}}(q,p,\varphi) = H_{\mathtt{m}}(q,p) + H_{\mathtt{de}}(\varphi)$. Some simple computations show that $\dot{H}_{\mathtt{T}} \leq y_{\mathtt{d}} v_{\mathtt{de}}$, where $y_{\mathtt{d}} := \frac{1}{r}(\dot{q} + \dot{\varphi})$. Hence, $\gamma_{\mathtt{d}}(q,\varphi) = \frac{1}{r}(\dot{q} + \dot{\varphi})$ satisfies that $\dot{\gamma}_{\mathtt{d}} = y_{\mathtt{d}}$. Thus,

$$H_{\mathtt{d}}(q,p,\varphi) = H_{\mathtt{T}}(q,p,\varphi) + \Phi_{\mathtt{d}}(\gamma_{\mathtt{d}})$$

satisfies

$$\dot{H}_{\mathtt{d}} \leq v_{\mathtt{de}} y_{\mathtt{d}} + \frac{\partial \Phi_{\mathtt{d}}(\gamma_d)}{\partial \gamma_d}\dot{\gamma}_d = \left(v_{\mathtt{de}} + \frac{\partial \Phi_{\mathtt{d}}(\gamma_d)}{\partial \gamma_d}\right) y_{\mathtt{d}}.$$

Thus, the control law

$$v_{\mathtt{de}} = -\frac{\partial \Phi_{\mathtt{d}}(\gamma_{\mathtt{d}})}{\partial \gamma_{\mathtt{d}}} \tag{25}$$

ensures that $\dot{H}_{\mathtt{d}} \leq 0$. To guarantee that $H_{\mathtt{d}}(q,p,\varphi)$ qualifies as a Lyapunov function to prove the stability of desired equilibrium, it remains to design $\Phi_{\mathtt{d}}(\gamma_d)$ such that $H_{\mathtt{d}}(q,p,\varphi)$ is positive definite with respect to the desired equilibrium. To this end, we propose

$$\begin{aligned} \Phi_{\mathtt{d}}(\gamma_{\mathtt{d}}) &= \tfrac{1}{2}K_{\mathtt{i}}\tilde{\gamma}_{\mathtt{d}}^2 - r m_{\mathtt{b}} g_{\mathtt{r}} \gamma_{\mathtt{d}}(q,\varphi) + \kappa\\ &= \tfrac{1}{2}K_{\mathtt{i}}(\tilde{\varphi} + \tilde{q})^2 - m_{\mathtt{b}} g_{\mathtt{r}}(q + \varphi) + \kappa \end{aligned} \tag{26}$$

where $K_{\mathtt{i}} > 0$ and

$$\tilde{\gamma}_{\mathtt{d}} := \gamma_{\mathtt{d}}(q,\varphi) - \gamma_{\mathtt{d}}(q_\star, \sqrt{2k m_{\mathtt{b}} g_{\mathtt{r}}}); \qquad \tilde{q} := q - q_\star$$

$$\kappa := \tfrac{2}{3} m_{\mathtt{b}} g_{\mathtt{r}} \sqrt{2k m_{\mathtt{b}} g_{\mathtt{r}}}; \quad \tilde{\varphi} := \varphi - \sqrt{2k m_{\mathtt{b}} g_{\mathtt{r}}}.$$

[2]This restriction is equivalent to the matching equation in IDA-PBC.

Some straightforward computations—omitted due to space constraints—show that the gradient and Hessian of $H_{\mathtt{d}}(q,p,\varphi)$ are zero and positive definite, respectively, at the desired equilibrium. Hence, $H_{\mathtt{d}}(q,p,\varphi)$ has a strict minimum at the desired equilibrium. This together with the fact that $H_{\mathtt{d}}(q_\star, 0, \sqrt{2km_{\mathtt{b}}g_{\mathtt{r}}}) = 0$, implies that $H_{\mathtt{d}}(q,p,\varphi)$ is positive definite with respect to the desired equilibrium and qualifies as a Lyapunov function to prove its stability. Consequently, combining (23), (25), and (26), we obtain that the stabilizing controller is

$$u = \frac{r}{k}\varphi(\rho - q) - \frac{r}{2k}\varphi^2 - \frac{1}{m_{\mathtt{b}}}p - K_{\mathtt{i}}\left(\tilde{\varphi} + \tilde{q}\right) + rm_{\mathtt{b}}g_{\mathtt{r}},$$

which is the controller provided in [13], with $\alpha = 1$.

B. Double pendulum

Consider the double pendulum depicted in Fig. 4(b). This system can be modeled via the Euler-Lagrange formalism, yielding the following dynamics

$$\mathcal{M}(q_{\mathtt{T}})\ddot{q}_{\mathtt{T}} + \mathcal{C}(q_{\mathtt{T}}, \dot{q}_{\mathtt{T}})\dot{q}_{\mathtt{T}} + \mathcal{G}(q_{\mathtt{T}}) = \mathbf{0}, \tag{27}$$

with $q_{\mathtt{T}} = [q_1 \ q_2]^\top$ and

$$\mathcal{M}(q_{\mathtt{T}}) = \begin{bmatrix} (m_1 + m_2)\,\ell_1^2 & m_2\ell_1\ell_2\cos(q_1 - q_2) \\ m_2\ell_1\ell_2\cos(q_1 - q_2) & m_2\ell_2^2 \end{bmatrix}$$

$$\mathcal{C}(q_{\mathtt{T}}, \dot{q}_{\mathtt{T}}) = \begin{bmatrix} 0 & m_2\ell_1\ell_2\dot{q}_2\sin(q_1 - q_2) \\ -m_2\ell_1\ell_2\dot{q}_1\sin(q_1 - q_2) & 0 \end{bmatrix}$$

$$\mathcal{G}(q_{\mathtt{T}}) = \begin{bmatrix} (m_1 + m_2)\,g_{\mathtt{r}}\ell_1\sin(q_1) \\ m_2 g_{\mathtt{r}}\ell_2\sin(q_2) \end{bmatrix}.$$

where q_i, m_i, and ℓ_i denote the angular position, mass, and length associated with the ith pendulum, respectively; $g_{\mathtt{r}}$ is the constant of gravitational acceleration.

Based on the results of Propositions 1 and 2, the double pendulum system can be interpreted as the interconnection of two simple pendulums with dynamics

$$m_i\ell_i^2\ddot{q}_i + m_i g_{\mathtt{r}}\ell_i\sin(q_i) = u_i; \quad i \in \{1, 2\}.$$

Note that each subsystem is passive with a storage function

$$S_i(q_i, \dot{q}_i) = \frac{1}{2}m_i\ell_i^2\dot{q}_i^2 + m_i g_{\mathtt{r}}\ell_i\left(1 - \cos(q_i)\right)$$

and passive output $y_i = \dot{q}_i$. Hence, Assumptions 1 and 2 are satisfied. Furthermore, by combining the interconnection schemes (10) and (13), with

$$\begin{aligned} \Phi_{\mathtt{T}}(q_{\mathtt{T}}) &= m_2 g_{\mathtt{r}}\ell_1\cos(q_1), & v_1 &= -m_2\dot{y}_1, \\ \Psi_{\mathtt{T}}(q_{\mathtt{T}}) &= m_2\ell_1\ell_2\cos(q_1 - q_2), & v_2 &= 0, \end{aligned}$$

we obtain

$$\begin{aligned} u_1 &= -m_2 g_{\mathtt{r}}\ell_1\sin(q_1) - m_2\ell_1\ell_2\cos(q_1 - q_2)\ddot{q}_2 \\ &\quad - m_2\ell_1\ell_2\sin(q_1 - q_2)\dot{q}_2 + m_2\ddot{q}_1 \\ u_2 &= -m_2\ell_1\ell_2\cos(q_1 - q_2)\ddot{q}_1 + m_2\ell_1\ell_2\sin(q_1 - q_2)\dot{q}_1 \\ S_{\mathtt{T}}(q_{\mathtt{T}}, \dot{q}_{\mathtt{T}}) &= S_1(q_1, \dot{q}_1) + S_2(q_2, \dot{q}_2) + \tfrac{1}{2}m_2\dot{q}_1^2 + \Phi_{\mathtt{T}}(q_{\mathtt{T}}) \\ &\quad + \dot{q}_1\Psi_{\mathtt{T}}(q_{\mathtt{T}})\dot{q}_2. \end{aligned}$$

Consequently, the interconnected system reduces to (27). Note that v_1 is not an interconnection term. However, it can be derived from the passive output of the first pendulum.

VI. Concluding Remarks

This paper proposed two new interconnection schemes that couple the subsystems through the overall energy function, while ensuring that the interconnected system is (cyclo-)passive. This permits the decomposition of physical systems into the interconnection of simpler subsystems. In contrast to traditional interconnection strategies, the proposed structures consider the integral and derivative of the passive output, which often have a physical interpretation (e.g., position and acceleration for a mechanical system). Moreover, the proposed interconnection approaches are suitable for interpreting some existing nonlinear controllers—as shown in the magnetic levitation system example—while endowing them with a physical interpretation. In particular, the reported results open new possibilities to CbI, offering an alternative to Casimir functions.

The design of a constructive control methodology based on the proposed interconnections schemes and exploring the interconnection of non-affine systems are suggested future research directions.

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