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Guaranteed Cost Boundary Control of the Semilinear Heat Equation

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Abstract— **We consider a 1D semilinear reaction-diffusion system with controlled heat flux at one of the boundaries. We design a finite-dimensional state-feedback controller guaranteeing that a given quadratic cost does not exceed a prescribed value for all nonlinearities with a predefined Lipschitz constant. To this end, we perform modal decomposition and truncate the highly damped (residue) modes. To deal with the nonlinearity that couples the residue and dominating modes, we combine the direct Lyapunov approach with the S-procedure and Parseval's identity. The truncation may lead to spillover: the ignored modes can deteriorate the overall system performance. Our main contribution is spillover avoidance via the L^2 separation of the residue. Namely, we calculate the L^2 input-to-state gains for the residue modes and add them to the control weight in the quadratic cost used to design a controller for the dominating modes. A numerical example demonstrates that the proposed idea drastically improves both the admissible Lipschitz constants and guaranteed cost bound compared to the recently introduced direct Lyapunov method.**

I. INTRODUCTION

Modal decomposition is a popular method of designing finite-dimensional controllers for partial differential equations (PDEs). Its idea is to design a controller for the dominating modes ignoring the highly damped residue modes [1], [2], [3], [4], [5]. A common problem with modal decomposition is spillover: the ignored modes can deteriorate the overall system performance [6], [7], [8]. Spillover can be studied qualitatively, where stability is guaranteed for a large enough number of considered modes, or quantitatively, where one specifies the exact number of required modes and provides performance guarantees. Qualitative results have been obtained using residual mode filters [9], [10], [11], spectral properties of linear operators [12], [13], [14], small-gain ideas [15], [16], and Lyapunov functionals [17], [18], [19]. Though some of these results can be used to estimate the required number of modes, the decay rate, or input-to-state gains, the resulting estimates may be quite conservative. Accurate quantitative results require a more careful residue analysis and were obtained using Lyapunov functionals in [8], [20], [21], [22], [23], [24]. The key step in the quantitative Lyapunov-based analysis is to use Young's inequality to split the cross terms between the control input and the residue modes (see Section V). This paper shows how to avoid restrictive Young's inequality and perform a more accurate analysis of the residue modes leading to a drastic performance improvement.

Our results are inspired by [25], where a finite-dimensional H_∞ controller was designed for the Euler–Bernoulli beam. The key idea is to consider the control signal as a disturbance in the truncated modes with the corresponding L^2 input-to-state gains. These gains are added to the control weight in the cost for the dominating modes and the optimal controller for the modified cost is designed. Since the modified cost accounts for the destabilizing effect of the control signal in the residue, spillover is avoided. This approach is inherently more accurate than those based on Young's inequality [8], [20], [21], [22], [23], [24]. To demonstrate this, we use it to design a finite-dimensional state-feedback boundary controller guaranteeing that a given quadratic cost does not exceed a prescribed value for all nonlinearities with a predefined Lipschitz constant. The proposed design method is simple: it only requires to solve a modified algebraic Riccati equation. A numerical example demonstrates that the L^2 separation method increases the admissible Lipschitz constant and reduces the upper bound on the cost by 90% compared to [23], where Young's inequality was used.

Notations: $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $|\cdot|$ is the Euclidean norm, $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ are the norm and scalar product in L^2. If P is a symmetric matrix, $P<0$ means that it is negative definite with the symmetric elements sometimes denoted by "$*$".

II. THE SEMILINEAR HEAT EQUATION

We consider the semilinear heat equation

$$z_t = z_{xx} + qz + f(\cdot, t, z(\cdot, t)), \tag{1a}$$

$$z_x(0,t) = 0, \quad z_x(\pi, t) = u(t) \tag{1b}$$

with state $z\colon [0,\pi]\times[0,\infty) \to \mathbb{R}$, control input $u\colon [0,\infty) \to \mathbb{R}$, and reaction coefficient $q>0$. A continuous $f:\mathbb{R}^3 \to \mathbb{R}$ satisfies $f(x,t,0) \equiv 0$ and the Lipschitz condition

$$\exists \sigma > 0\colon \quad |f(x,t,z_1) - f(x,t,z_2)| \le \sigma |z_1 - z_2|. \tag{2}$$

Note that if the coefficient in front of z_{xx} is not 1, or the spatial domain is not $[0,\pi]$, the equation can be transformed to the form (1) using the change of variables $\tilde{z}(x,t) = z(ax - x_0, bt)$ with suitable a, b, and x_0. Furthermore, the reaction term, qz, can be included in f increasing the Lipschitz constant σ. We keep it separated to obtain more accurate conditions.

Our objective is to design a finite-dimensional state-feedback control law that, for a given $r>0$, guarantees

$$J = \int_0^\infty \left[\|z(\cdot,t)\|^2 + ru^2(t)\right] dt \le \alpha \|z(\cdot,0)\|^2 \tag{3}$$

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with some $\alpha > 0$ (as small as possible) for system (1) with any $z(\cdot, 0) \in L^2(0, \pi)$ and any f satisfying (2). Note that (3) implies the asymptotic stability of (1) in the L^2 norm (see Remark 3). The key step in designing and analysing such a controller is the modal decomposition presented next.

III. MODAL DECOMPOSITION OF THE SEMILINEAR HEAT EQUATION

The eigenvalues and eigenfunctions of the operator

$$\mathcal{A}\varphi = -\varphi'',\ D(\mathcal{A}) = \{\varphi \in H^2(0, \pi) \mid \varphi'(0) = 0 = \varphi'(\pi)\},$$

are

$$\lambda_n = n^2, \qquad n \in \mathbb{N}_0,$$
$$\varphi_n(x) = \begin{cases} 1/\sqrt{\pi}, & n = 0, \\ \sqrt{2/\pi} \cos nx, & n \in \mathbb{N}. \end{cases}$$

The eigenfunctions form an orthonormal basis of $L^2(0, \pi)$. Therefore, the state can be presented as the Fourier series

$$z(x,t) \overset{L^2}{=} \textstyle\sum_{n=0}^{\infty} z_n(t)\varphi_n(x), \qquad z_n(t) := \langle z(\cdot, t), \varphi_n\rangle.$$

The Fourier coefficients, z_n, satisfy

$$\begin{aligned}\dot{z}_n(t) &= \langle z_t(\cdot, t), \varphi_n\rangle \\ &\overset{(1a)}{=} \langle z_{xx}(\cdot, t), \varphi_n\rangle + q\langle z(\cdot, t), \varphi_n\rangle + \langle f(\cdot, t, z(\cdot, t)), \varphi_n\rangle.\end{aligned}$$

Since $\varphi_n \in D(\mathcal{A})$ and $\varphi_n'' = -\lambda_n\varphi_n$, integrating by parts twice, we obtain

$$\begin{aligned}&\langle z_{xx}(\cdot, t), \varphi_n\rangle \\ &\quad = z_x(\cdot, t)\varphi_n|_0^\pi - z(\cdot, t)\varphi_n'|_0^\pi + \textstyle\int_0^\pi z(x, t)\varphi_n''(x)\, dx \\ &\quad \overset{(1b)}{=} \varphi_n(\pi)u(t) - \lambda_n z_n(t).\end{aligned}$$

Therefore,

$$\dot{z}_n(t) = (q - \lambda_n)z_n(t) + f_n(t) + b_n u(t), \qquad n \in \mathbb{N}_0,$$

where

$$f_n(t) = \langle f(\cdot, t, z(\cdot, t)), \varphi_n\rangle, \qquad n \in \mathbb{N}_0,$$
$$b_n = \begin{cases} 1/\sqrt{\pi}, & n = 0, \\ (-1)^n\sqrt{2/\pi}, & n \in \mathbb{N}. \end{cases}$$

For any $\nu_f \geq 0$, Parseval's theorem and (2) imply

$$0 \leq \nu_f \left[\sigma^2 \textstyle\sum_{n=0}^{\infty} z_n^2 - \sum_{n=0}^{\infty} f_n^2\right]. \tag{4}$$

We will design a finite-dimensional state-feedback controller using the first $N + 1$ modes ($n = 0, 1, \ldots, N$). We take N such that

$$\lambda_{N+1} = (N + 1)^2 > q + \sigma. \tag{5}$$

This condition is very natural: we consider all the unstable modes in the control design. It guarantees that all the ρ_n, given in (14) below, are positive for some $\nu_f > 0$.

Separating the first $N + 1$ modes, we obtain

$$\dot{z}^N(t) = Az^N(t) + Bu(t) + F(t), \tag{6a}$$
$$\dot{z}_n(t) = (q - \lambda_n)z_n + f_n(t) + b_n u(t), \quad n > N, \tag{6b}$$

where

$$\begin{aligned}&z^N = [z_0, \ldots, z_N]^\top, \quad F = [f_0, \ldots, f_N]^\top, \\ &A = \operatorname{diag}\{q - \lambda_0, \ldots, q - \lambda_N\}, \\ &B = \sqrt{2/\pi}[1/\sqrt{2}, -1, 1, \ldots, (-1)^N]^\top.\end{aligned}$$

Since all the eigenvalues, λ_n, are different, the pair (A, B) is controllable, e.g., by the Hautus Lemma [26, Lemma 3.3.1]. Our objective is to find $K \in \mathbb{R}^{1\times(N+1)}$ such that

$$u(t) = -Kz^N(t) \tag{7}$$

guarantees (3) for any f subject to (2).

IV. GUARANTEED COST CONTROL VIA THE L^2 RESIDUE SEPARATION

Using Parseval's identity, $\|z(\cdot, t)\|^2 = \sum_{n=0}^{\infty} z_n^2(t)$, the objective (3) can be expressed in terms of the Fourier coefficients, $z_n(t)$. Our main idea is to decompose it as

$$\begin{aligned}&J = \textstyle\int_0^\infty \left[\sum_{n=0}^N z_n^2(t) + (r + \bar{\rho}_N)u^2(t)\right] dt \\ &\quad + \textstyle\int_0^\infty \sum_{n=N+1}^\infty \left[z_n^2(t) - \rho_n u^2(t)\right] dt \leq \alpha \sum_{n=0}^\infty z_n^2(0),\end{aligned}$$

where

$$\bar{\rho}_N = \textstyle\sum_{n=N+1}^\infty \rho_n. \tag{8}$$

Given this decomposition, (3) holds if

$$\int_0^\infty \left[\sum_{n=0}^N z_n^2(t) + (r + \bar{\rho}_N)u^2(t)\right] dt \leq \alpha \sum_{n=0}^N z_n^2(0), \tag{9a}$$
$$\int_0^\infty \left[z_n^2(t) - \rho_n u^2(t)\right] dt \leq \alpha z_n^2(0), \quad n > N. \tag{9b}$$

Our intuition is that, since the modes with $n > N$ are ignored in the controller design, $u(t)$ should be viewed as a disturbance in these modes. Condition (9b) means that the square of the L^2 gain from $u(t)$ to $z_n(t)$ for systems (6b) is not greater than ρ_n. Then $\bar{\rho}_N$, given in (8), reflects the combined L^2 gain from $u(t)$ to the residue modes. By adding it in (9a), we guarantee that the control designed for the dominating modes will not lead to spillover.

The remainder of this section ensures (9). First, we assume that $\alpha > 0$ is fixed and find the minimum ρ_n guaranteeing (9b). Then, we calculate their sum, $\bar{\rho}_N$, as in (8). Finally, we find $u(t)$ guaranteeing (9a). This idea is inspired by [25], where the H_∞ problem was solved for a linear beam PDE.

The main difference compared to [25] is that the conditions in (9) cannot be fulfilled independently due to the nonlinearity f in the heat equation (1). When calculating ρ_n guaranteeing (9b), one needs to use the conditions on f_n, which are given in (4) and cannot be separated for each n. We overcome this difficulty using the Lyapunov functional

$$V(z(\cdot, t)) = (z^N(t))^\top Pz^N(t) + \textstyle\sum_{n=N+1}^\infty z_n^2(t) \tag{10}$$

with $0 < P \in \mathbb{R}^{(N+1)\times(N+1)}$. By Parseval's theorem,

$$c_1\|z(\cdot, t)\|^2 \leq V(z(\cdot, t)) \leq c_2\|z(\cdot, t)\|^2,$$

where $c_1 = \min\{1, \lambda_{\min}(P)\}$ and $c_2 = \max\{1, \lambda_{\max}(P)\}$. In what follows, we find the conditions guaranteeing

$$\dot{V} + \eta\left[\|z(\cdot,t)\|^2 + ru^2(t)\right] \le 0 \tag{11}$$

for $\eta > 0$. Integrating the above from 0 to ∞ in t, we obtain

$$\begin{aligned}\int_0^\infty \left[\|z(\cdot,t)\|^2 + ru^2(t)\right] dt &\le \eta^{-1}(V(z(\cdot,0)) - V(z(\cdot,t)))\\ &\le \eta^{-1}V(z(\cdot,0)) \le \eta^{-1}c_2\|z(\cdot,0)\|^2.\end{aligned}$$

Note that there is not much benefit in considering $\sum_{n=N+1}^\infty p_n z_n^2(t)$ as the second term of V since p_n has to converge to a constant for V to be "sandwiched" between the L^2 norms of the state.

For (10) to be differentiable, we consider $z(\cdot,0) \in H^1$ and the corresponding classical solution. In this case, we derive conditions guaranteeing (3). Since H^1 is dense in L^2, (3) remains true (by continuous extension) for $z(\cdot,0) \in L^2$.

To guarantee (11), we calculate the time derivative of V along the trajectories of (6) and add (4) to compensate f (i.e., apply the S-procedure):

$$\begin{aligned}
&\dot{V} + \eta\|z(\cdot,t)\|^2 + \eta ru^2(t) \overset{(6)}{=} 2(z^N)^\top P[Az^N + Bu + F]\\
&\quad + 2\sum_{n=N+1}^\infty z_n[(q-\lambda_n)z_n + f_n + b_n u]\\
&\quad + \eta|z^N|^2 + \eta\sum_{n=N+1}^\infty z_n^2 + \eta ru^2\\
&\overset{(4)}{\le} (z^N)^\top[PA + A^\top P + \eta I + \nu_f\sigma^2 I]z^N\\
&\quad + 2(z^N)^\top PBu + 2(z^N)^\top PF - \nu_f F^\top F\\
&\quad + 2\sum_{n=N+1}^\infty z_n[(q-\lambda_n)z_n + f_n + b_n u] + \eta\sum_{n=N+1}^\infty z_n^2\\
&\quad + \nu_f\sum_{n=N+1}^\infty(\sigma^2 z_n^2 - f_n^2) + \left(\eta r + \bar{\rho}_N - \sum_{n=N+1}^\infty \rho_n\right)u^2.
\end{aligned} \tag{12}$$

The last term equals ηru^2 in view of (8). We substitute $u = -Kz^N$ for the first $N+1$ modes and keep it as u for the remaining modes:

$$\begin{aligned}\dot{V}+\eta\|z(\cdot,t)\|^2+\eta ru^2(t) &\le \begin{bmatrix} z^N \\ F\end{bmatrix}^\top \left[\begin{array}{c|c}\Phi_{11} & P\\ \hline P & -\nu_f I\end{array}\right]\begin{bmatrix} z^N \\ F\end{bmatrix}\\ &+ \sum_{n=N+1}^\infty \begin{bmatrix} z_n \\ f_n \\ u\end{bmatrix}^\top \Phi_n \begin{bmatrix} z_n \\ f_n \\ u\end{bmatrix},\end{aligned} \tag{13}$$

where

$$\begin{aligned}\Phi_{11} &= P(A-BK) + (A-BK)^\top P\\ &\quad + (\eta r + \bar{\rho}_N)K^\top K + (\eta + \nu_f\sigma^2)I,\\
\Phi_n &= \left[\begin{array}{c|c|c} 2(q-\lambda_n) + \eta + \nu_f\sigma^2 & 1 & b_n\\ \hline 1 & -\nu_f & 0\\ \hline b_n & 0 & -\rho_n\end{array}\right].\end{aligned}$$

In what follows, we find the minimum $\rho_n > 0$ guaranteeing $\Phi_n \le 0$ (Section IV-A) and design $K \in \mathbb{R}^{1\times(N+1)}$ guaranteeing that the first term in the right-hand side of (13) is negative (Section IV-B).

A. L^2 gain calculation for the residue

By the Schur complement lemma, $\Phi_n \le 0$ follows from

$$2(q-\lambda_n) + \eta + \nu_f\sigma^2 + \nu_f^{-1} + b_n^2\rho_n^{-1} = 0, \quad n > N.$$

Solving this equation, we obtain

$$\rho_n = \frac{b_n^2}{2(\lambda_n - q) - \eta - \nu_f\sigma^2 - \nu_f^{-1}} = \frac{\pi^{-1}}{\lambda_n - d} \tag{14}$$

with $d = q + \frac{\eta}{2} + \frac{\nu_f\sigma^2}{2} + \frac{1}{2\nu_f}$. Since $\lambda_n = n^2$ increases monotonically, $\rho_n > 0$ for any $n > N$ if and only if

$$\lambda_{N+1} - q - \frac{\eta}{2} - \frac{\nu_f\sigma^2}{2} - \frac{1}{2\nu_f} > 0$$

with fixed q and σ. This holds when $\nu_f \in (\nu_-, \nu_+)$ with

$$\nu_\pm = \frac{\lambda_{N+1} - q - \frac{\eta}{2} \pm \sqrt{(\lambda_{N+1} - q - \frac{\eta}{2})^2 - \sigma^2}}{\sigma^2}. \tag{15}$$

For this set to be non-empty, we need

$$0 < \eta < 2(\lambda_{N+1} - q - \sigma), \tag{16}$$

which is feasible in view of (5).

For the ρ_n given in (14), the series in (8) can be calculated explicitly, e.g., using the Mittag-Leffler expansion for the cotangent [29, Section 7.10]:

$$\pi\cot\pi z = \frac{1}{z} + 2\sum_{n=1}^\infty \frac{z}{z^2 - n^2}.$$

Substituting $z = \sqrt{d}$ and reorganizing the terms, we obtain

$$\begin{aligned}\bar{\rho}_N = \textstyle\sum_{n=N+1}^\infty \rho_n &= \tfrac{1}{\pi}\textstyle\sum_{n=N+1}^\infty \tfrac{1}{n^2 - d}\\ &= \tfrac{1}{\pi}\left[\textstyle\sum_{n=0}^N \tfrac{1}{d-n^2} - \tfrac{1+\pi\sqrt{d}\cot\pi\sqrt{d}}{2d}\right].\end{aligned}$$

The expression with the cotangent is not defined when d is a square of an integer, but the limit will always exist. Since the series converges, $\bar{\rho}_N \to 0$ as $N \to \infty$, meaning that when more modes are considered in the control design, the destabilizing effect of the control imposed by the residue is reduced.

B. Controller design for the first $N+1$ modes

By the Schur complement lemma,

$$\left[\begin{array}{c|c}\Phi_{11} & P\\ \hline P & -\nu_f I\end{array}\right] \le 0 \tag{17}$$

follows from the algebraic Riccati equation (ARE)

$$\begin{aligned}P(A-BK) + (A-BK)^\top P + (\eta r + \bar{\rho}_N)K^\top K&\\ + (\eta + \nu_f\sigma^2)I + \nu_f^{-1}P^2 &= 0.\end{aligned}$$

Following the H_∞ conventions, we are looking for the controller gain in the form

$$K = \mu B^\top P, \qquad \mu > 0.$$

Substituting and reorganizing the terms, we obtain

$$\begin{aligned}PA + A^\top P + P(\nu_f^{-1}I - (2\mu - \eta r\mu^2 - \bar{\rho}_N\mu^2)BB^\top)P&\\ + (\eta + \nu_f\sigma^2)I &= 0.\end{aligned}$$

The maximum of $2\mu - \eta r\mu^2 - \bar{\rho}_N\mu^2$ is $(\eta r + \bar{\rho}_N)^{-1}$ achieved at $\mu = (\eta r + \bar{\rho}_N)^{-1}$. Substituting this μ, we obtain

$$PA + A^\top P + P(\nu_f^{-1} I - (\eta r + \bar{\rho}_N)^{-1} BB^\top)P + (\eta + \nu_f\sigma^2)I = 0. \quad (18)$$

This algebraic Riccati equation solves the H_∞ full-information control problem for (6a) with the disturbance $F(t)$ and objective (see, e.g., [30])

$$\int_0^\infty \big[(\eta + \nu_f\sigma^2)|z^N(t)|^2 + (\eta r + \bar{\rho}_N)u^2(t) - \nu_f|F(t)|^2\big]dt \le 0. \quad (19)$$

The resulting control law is

$$u = -(\eta r + \bar{\rho}_N)^{-1} B^\top P z^N. \quad (20)$$

In Section IV-A, we selected ρ_n that guarantees $\Phi_n \le 0$ in the second term of (13). In this section, we found P that guarantees (17), i.e., the first term in (13) is negative. Therefore, (11) is true, which implies the following result.

Theorem 1 (Guaranteed cost): Consider the semilinear heat equation (1) subject to (2). Let $N \in \mathbb{N}$ satisfy (5), $\eta > 0$ satisfy (16), and $\nu_f \in (\nu_-, \nu_+)$ with $\nu_\pm$ defined in (15). If there exists $0 < P \in \mathbb{R}^{(N+1)\times(N+1)}$ solving (18), then the state-feedback control law (20) guarantees that (3) holds with $\alpha = \eta^{-1}c_2 = \eta^{-1}\max\{1, \lambda_{\max}(P)\}$.

Remark 1 (Number of modes and the Lipschitz bound): When N grows, the maximum admissible Lipschitz constant, σ, cannot decrease. Indeed, $\Phi_{N+1} \le 0$ guarantees that $V_{N+1} = z_{N+1}^2$ satisfies

$$\dot{V}_{N+1} + (\eta + \nu_f\sigma^2)z_{N+1}^2 - \rho_{N+1}u^2 - \nu_f f_n^2 \le 0.$$

Integrating this from 0 to ∞ and taking $z_{N+1} = 0$, we obtain

$$\int_0^\infty \left[(\eta + \nu_f\sigma^2)z_{N+1}^2 - \rho_{N+1}u^2 - \nu_f f_n^2\right] dt \le 0.$$

By adding this to (19), we find that (19) holds when N is replaced with $N+1$. By [30, Theorem 6.3.6], (18) must be feasible for $N+1$.

Remark 2 (LMI formulation): By the Schur complement lemma, (17) is equivalent to

$$\begin{bmatrix} \tilde{\Phi}_{11} & P & (\eta r + \bar{\rho}_N)K^\top & (\eta + \nu_f\sigma^2)I \\ * & -\nu_f I & 0 & 0 \\ * & * & -(\eta r + \bar{\rho}_N) & 0 \\ * & * & * & -(\eta + \nu_f\sigma^2)I \end{bmatrix} \le 0$$

with $\tilde{\Phi}_{11} = P(A - BK) + (A - BK)^\top P$. Multiplying by $\operatorname{diag}\{P^{-1}, I, 1, I\}$ from left and right, and denoting $\bar{P} = P^{-1}$, $Y = KP^{-1}$, we obtain

$$\begin{bmatrix} \bar{\Phi}_{11} & I & (r + \bar{\rho}_N)Y^\top & (1 + \nu_f\sigma^2)\bar{P} \\ * & -\nu_f I & 0 & 0 \\ * & * & -(r + \bar{\rho}_N) & 0 \\ * & * & * & -(1 + \nu_f\sigma^2)I \end{bmatrix} \le 0 \quad (21)$$

with $\bar{\Phi}_{11} = A\bar{P} + \bar{P}A - BY - (BY)^\top$. Therefore, instead of solving (18), one can solve (21) and take $K = Y\bar{P}^{-1}$. LMIs (21) take more time to solve compared to (18) because the number of decision variables is higher and the solvers tailored for solving (18) are more efficient than the universal LMI solvers. However, the LMIs (21) are useful if the results of this paper are extended to the delayed input case.

C. Exponential stability

The above L^2-separation idea can be extended to guarantee the exponential stability of (1) under (7) with a given decay rate $\delta > 0$. To this end, one needs (cf. (11))

$$\dot{V} + 2\delta V \le 0.$$

Then, the calculations (12) are modified in a straightforward way: $\eta = 0$ and q should be replaced by $q + \delta$. The corresponding L^2 gain for the residue is given by (14) with

$$d = q + \delta + \frac{\nu_f\sigma^2}{2} + \frac{1}{2\nu_f}. \quad (22)$$

Furthermore, the ARE (18) becomes

$$PA_\delta + A_\delta^\top P + P(\nu_f^{-1} I - \bar{\rho}_N^{-1} BB^\top)P + \nu_f\sigma^2 I = 0$$

with $A_\delta = A + \delta I$. Dividing by ν_f and defining $P_\nu = P/\nu_f > 0$, we obtain

$$P_\nu A_\delta + A_\delta^\top P_\nu + P_\nu(I - \kappa_N^{-1} BB^\top)P_\nu + \sigma^2 I = 0 \quad (23)$$

with $\kappa_N = (\bar{\rho}_N\nu_f)^{-1}$. Clearly, κ_N should be minimized to improve feasibility. Similarly to the previous section, $\rho_n > 0$ if and only if $\nu_f \in (\nu_-, \nu_+)$ with $\nu_\pm$ given by (15) with $\eta/2$ replaced by δ. Therefore, we take

$$\kappa_N = \min_{\nu_f \in (\nu_-, \nu_+)} \frac{\bar{\rho}_N(\nu_f)}{\nu_f}. \quad (24)$$

Summarizing, we have the following result.

Theorem 2 (Exponential stability): Consider the semilinear heat equation (1) subject to (2). Let $\delta > 0$ be a desired decay rate. For any given $N \in \mathbb{N}$ satisfying (cf. (5))

$$\lambda_{N+1} = (N+1)^2 > q + \sigma + \delta, \quad (25)$$

take d as in (22) and κ_N as in (24). Let $0 < P_\nu \in \mathbb{R}^{(N+1)\times(N+1)}$ be the solution of (23). Then the state-feedback control law

$$u = -\kappa_N^{-1} B^\top P_\nu z^N$$

makes (1) globally exponentially stable in the L^2 norm with the decay rate δ.

Remark 3: Note that (11) implies that $\dot{V} \le -2\delta V$ with $\delta = \frac{\eta}{2c_2} > 0$, which guarantees the exponential stability of (1), (7) in the L^2 norm.

V. GUARANTEED COST CONTROL VIA YOUNG'S INEQUALITY

To compare our results with the approach in [23], we extend it to system (1). Recall that for V defined in (10),

$$\dot{V} + \eta[\|z(\cdot,t)\|^2 + ru^2(t)] \overset{(6),(7)}{=} 2(z^N)^\top P[(A - BK)z^N + F] + 2\sum_{n=N+1}^{\infty} z_n[(q - \lambda_n)z_n + f_n - b_n K z^N] + \eta|z^N|^2 + \eta\sum_{n=N+1}^{\infty} z_n^2 + \eta r|Kz^N|^2. \quad (26)$$

Young's inequality gives

$$\sum_{n=N+1}^{\infty} 2z_n f_n \le \sum_{n=N+1}^{\infty} \nu_f^{-1} z_n^2 + \nu_f \sum_{n=0}^{\infty} f_n^2 - \nu_f |F|^2 \overset{(4)}{\le} (\nu_f^{-1} + \nu_f \sigma^2) \sum_{n=N+1}^{\infty} z_n^2 + \nu_f \sigma^2 |z^N|^2 - \nu_f |F|^2$$

and

$$-\sum_{n=N+1}^{\infty} 2z_n b_n K z^N \le \sum_{n=N+1}^{\infty} \nu_2 \lambda_n z_n^2 + \sum_{n=N+1}^{\infty} \frac{|b_n K z^N|^2}{\nu_2 \lambda_n} = \sum_{n=N+1}^{\infty} \nu_2 \lambda_n z_n^2 + \frac{\chi_N}{\nu_2} |K z^N|^2, \tag{27}$$

where $\nu_f > 0$, $\nu_2 > 0$, and

$$\chi_N := \sum_{n=N+1}^{\infty} \frac{2}{\pi \lambda_n} = \frac{2}{\pi} \left[\frac{\pi^2}{6} - \sum_{n=1}^{N} \frac{1}{n^2} \right].$$

Using these in (26), we obtain

$$\dot{V} + \eta[\|z(\cdot,t)\|^2 + ru^2(t)] \le \begin{bmatrix} z^N \\ F \end{bmatrix}^\top \left[\begin{array}{c|c} \Psi_{11} & P \\ \hline P & -\nu_f I \end{array} \right] \begin{bmatrix} z^N \\ F \end{bmatrix} + \sum_{n=N+1}^{\infty} (-2\lambda_n + 2q + \eta + \nu_f^{-1} + \nu_f \sigma^2 + \nu_2 \lambda_n) z_n^2,$$

where

$$\Psi_{11} = P(A - BK) + (A - BK)^\top P + (\eta r + \chi_N \nu_2^{-1}) K^\top K + (\nu_f \sigma^2 + \eta) I.$$

Clearly, $\dot{V} + \eta[\|z(\cdot,t)\|^2 + ru^2(t)] < 0$ follows from

$$\Psi := \left[\begin{array}{c|c} \Psi_{11} & P \\ \hline P & -\nu_f I \end{array} \right] < 0, \tag{28a}$$

$$-2\lambda_{N+1} + 2q + \eta + \nu_f^{-1} + \nu_f \sigma^2 + \nu_2 \lambda_{N+1} < 0. \tag{28b}$$

By the Schur complement lemma, (28a) is equivalent to

$$\begin{bmatrix} \Xi & P & K^\top & rK^\top & \sigma I & I \\ * & -\nu_f I & 0 & 0 & 0 & 0 \\ * & * & -\frac{\nu_2}{\chi_N} & 0 & 0 & 0 \\ * & * & * & -\frac{r}{\eta} & 0 & 0 \\ * & * & * & * & -\nu_f^{-1} I & 0 \\ * & * & * & * & * & -\frac{1}{\eta} I \end{bmatrix} < 0,$$

where $\Xi = P(A - BK) + (A - BK)^\top P$. Multiplying this by $\operatorname{diag}\{P^{-1}, \nu_f^{-1} I, 1, I, I\}$ from left and right, and introducing

$$\bar{P} = P^{-1}, \quad Y = KP^{-1}, \quad \nu_1 = \nu_f^{-1}, \quad \eta_1 = \eta^{-1},$$

we obtain

$$\begin{bmatrix} \tilde{\Xi} & \nu_1 & Y^\top & rY^\top & \sigma \bar{P} & \bar{P} \\ * & -\nu_1 I & 0 & 0 & 0 & 0 \\ * & * & -\frac{\nu_2}{\chi_N} & 0 & 0 & 0 \\ * & * & * & -\eta_1 r & 0 & 0 \\ * & * & * & * & -\nu_1 I & 0 \\ * & * & * & * & * & -\eta_1 I \end{bmatrix} < 0 \tag{29}$$

with $\tilde{\Xi} = A\bar{P} + \bar{P}A^\top - BY - Y^\top B^\top$. By the Schur complement lemma, (28b) is equivalent to

$$\begin{bmatrix} -2\lambda_{N+1} + 2q + \nu_1 + \nu_2 \lambda_{N+1} & \sigma & 1 \\ * & -\nu_1 & 0 \\ * & * & -\eta_1 \end{bmatrix} < 0. \tag{30}$$

Note that (29) and (30) are LMIs that depend on $\bar{P}$, Y, ν_1, ν_2, and η_1. If (29) and (30) hold, the controller gain is $K = Y\bar{P}^{-1}$. Summarizing, we have the following result.

Theorem 3 (Guaranteed cost): Consider the semilinear heat equation (1) subject to (2). Let $N \in \mathbb{N}$ satisfy (5). If there exist $0 < \bar{P} \in \mathbb{R}^{(N+1)\times(N+1)}$, $Y \in \mathbb{R}^{1\times(N+1)}$, and scalars $\nu_1 > 0$, $\nu_2 > 0$, and $\eta_1 > 0$ such that (29) and (30) hold, then the state-feedback control law (7) with $K = Y\bar{P}^{-1}$ guarantees (3) with $\alpha = \eta^{-1} c_2 = \eta_1 \max\{1, \lambda_{\max}(\bar{P}^{-1})\}$ (equivalently, minimum $\alpha > 0$ such that $\alpha \ge \eta_1$ and $\bar{P} \ge \eta_1 \alpha^{-1} I$).

Similarly to Section IV-C, for a given decay rate $\delta > 0$, we let $N \in \mathbb{N}$ satisfy (25) and arrive at $\dot{V} + 2\delta V \le 0$ provided

$$\begin{gathered} \begin{bmatrix} A_\delta \bar{P} + \bar{P} A_\delta^\top - BY - Y^\top B^\top & \nu_1 & Y^\top & \sigma \bar{P} \\ * & -\nu_1 I & 0 & 0 \\ * & * & -\frac{\nu_2}{\chi_N} & 0 \\ * & * & * & -\nu_1 I \end{bmatrix} < 0, \\ \begin{bmatrix} -2\lambda_{N+1} + 2q + 2\delta + \nu_1 + \nu_2 \lambda_{N+1} & \sigma \\ * & -\nu_1 \end{bmatrix} < 0. \end{gathered} \tag{31}$$

Summarizing, we have the following conditions for the exponential stability.

Theorem 4 (Exponential stability): Consider the semilinear heat equation (1) subject to (2). Let $\delta > 0$ be a desired decay rate. For a given $N \in \mathbb{N}$ satisfying (25), let there exist $0 < \bar{P} \in \mathbb{R}^{(N+1)\times(N+1)}$, $Y \in \mathbb{R}^{1\times(N+1)}$, and scalars $\nu_1 > 0$ and $\nu_2 > 0$ satisfying (31). Then, the state-feedback control law (7) with $K = Y\bar{P}^{-1}$ makes (1) globally exponentially stable in the L^2 norm with the decay rate δ.

Remark 4: The proofs of Theorems 3 and 4 use Young's inequality in (27) to separate the control input from the residue. Our approach circumvents conservative Young's inequality by leveraging the L^2-gain ideas: the cross-terms $z_n b_n u$ in (13) are compensated by $-\rho_n u^2$ with ρ_n later added to the control weight in the cost. This leads to a drastic improvement compared to [23], which is demonstrated by an example in the next section.

VI. Example

As an example, we consider (1) with $q = 1$ or $q = 5$, and the nonlinearity that makes the open-loop system unstable. First, let $\delta = 10^{-2}$ be the desired decay rate. To compare Theorems 2 and 4, we perform linear search over $\sigma > 0$ to find the maximum Lipschitz constant preserving the feasibility of (23) with κ_N from (24) and the LMIs (29), (30), respectively. The maximum σ for $N \in \{1, \dots, 6\}$ are given in Table I. The residue separation method developed in this paper always leads to a larger Lipschitz constant compared to the approach based on Young's inequality. In particular, for $q = 5$ and $N = 2$, the Lipschitz constant increases by 26%.

TABLE I

The maximum admissible σ.

	$N =$	1	2	3	4	5	6
Thm 2	$q = 1$	0.3763	0.4099	0.4195	0.4235	0.4256	0.4268
Thm 4	$q = 1$	0.3564	0.4003	0.4137	0.4196	0.4228	0.4247
Thm 2	$q = 5$	–	0.0841	0.1119	0.1217	0.1263	0.1289
Thm 4	$q = 5$	–	0.0667	0.1006	0.1142	0.1209	0.1248

Next, we consider the guaranteed cost control (3) and try to minimize $\alpha > 0$. We consider $q = 1$ with $\sigma = 0.35$ and $q = 5$ with $\sigma = 0.06$, and take $r = 0.1$. For Theorem 1, we perform linear search over $\alpha > 0$ and make a grid of

$\eta \in (0, 2(\lambda_{N+1} - q - \sigma))$ and $\nu_f \in (\nu_-, \nu_+)$ with ν_-, ν_+ defined in (15) to find minimum $\alpha \geq \eta^{-1} \max\{1, \lambda_{\max}(P)\}$ preserving the feasibility of (18) with $P > 0$. For comparison of the results with Theorem 3, we solve LMIs (29) and (30) with the constraints $\alpha \geq \eta_1$ and $\bar{P} \geq \eta_1 \alpha^{-1} I$ to find the minimum value of $\alpha > 0$ that preserves the feasibility. The minimum α for $N \in \{1, \ldots, 6\}$ are given in Table II. For the same Lipschitz constant, the residue separation method always leads to a smaller $\alpha > 0$ compared to the approach based on Young's inequality. In particular, for $q = 5$ and $N = 2$, the value of α is reduced by 90%.

TABLE II

THE MINIMUM VALUE OF α.

	$(q,\sigma)\backslash N$	1	2	3	4	5	6
Thm 1	$(1, 0.35)$	2261.4	521.6	425.9	402.4	393.8	390.1
Thm 3	$(1, 0.35)$	25419.2	699.9	480.8	428.2	408.4	399.1
Thm 1	$(5, 0.06)$	–	27408	9634	8371	8061	7945
Thm 3	$(5, 0.06)$	–	266031	12824	9340	8511	8198

In simulations, we consider $q = 1$, $f(z) = 0.35 \sin(z)$, and $N = 6$. Using the grid search, we find $\eta = 8.8$ and $\nu_f = 496$. Solving (18), we obtain the control gain

$$\begin{aligned} K &= (\eta r + \bar{\rho}_N)^{-1} B^\top P \\ &= [153.96, 80.19, -2.60, -0.05, 0.50, -0.52, 0.46]. \end{aligned}$$

Fig. 1 shows $J(t) = \int_0^t \left[\|z(\cdot,s)\|^2 + ru^2(s)\right] ds$ for the initial condition $z(x,0) = \sum_{n=0}^{N} \phi_n(x) z_n(0)$ with

$$\begin{aligned} &z^N(0) = [z_0(0), \ldots, z_N(0)]^\top \\ &= [-86.13, -50.73, 2.69, -0.83, 0.33, -0.16, 0.08]^\top \times 10^2, \end{aligned}$$

which was chosen to satisfy $Pz^N(0) = \lambda_{\max}(P) z^N(0)$ and $|z^N(0)| = 1$. The theoretical upper bound on J from Table II is $\alpha \|z(\cdot,0)\|^2 = 390.1$.

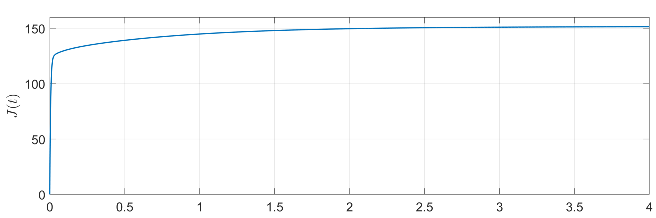

Fig. 1. Evolution of $J(t) = \int_0^t \left[\|z(\cdot,s)\|^2 + ru^2(s)\right] ds$.

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