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Impact on traffic of delayed information in navigation systems

Tommaso Toso, *Student Member, IEEE,* Alain Y. Kibangou, *Member, IEEE,* Paolo Frasca, *Senior Member, IEEE*

***Abstract*— Nowadays many car drivers resort to navigation apps to decide which route to take. To be efficient, these applications increasingly use real-time data rather than historical data. However, delay is unavoidable since data collection, communication, and processing are necessary before their usage in the App. For this purpose, we introduce a macroscopic dynamic traffic assignment model to describe the behaviour of drivers in choosing which route to follow to reach their destination. We assume that a part of the drivers follows the directions of a navigation App, whose directions are based on delayed traffic data. Through the stability analysis of the model, we show and quantify the excessive level of delay in traffic data that can be detrimental to the efficiency of the network by being responsible for the appearance of oscillating trajectories and unsatisfied demand.**

***Index Terms*— Delay systems, Traffic control, Transportation networks.**

I. Introduction

Navigation Apps are extremely popular nowadays among users of transportation networks, given the ease of access via smartphone and the propensity to use them due to the increasing number of congestion events over time. Considering the importance of the phenomenon and the obvious power of such Apps to influence drivers' decisions, thus traffic patterns, studying their impact is crucial. In this paper, the focus is on how the delay between data collection and the provision of route recommendations synthesized from them can bring potential losses in network efficiency and create unsatisfied traffic demand. Intuitively, excessive delay should lead to oscillations and unstable behaviors of the traffic patterns, as suggested by the theory of time-delay systems [7].

In the literature, almost all relevant works within the macroscopic dynamic routing framework present models in which the route choice is based on the current state of traffic and no delay is considered [2], [4], [5], [6]. It is worth to mention that in [1] the authors propose a framework that allows for dynamic routing of drivers based on delayed information, but they do not elaborate on the effects caused by the latter. In [9], [12], by using a microscopic traffic model, it was shown that information based on floating car data, which is intrinsically affected by delay, can lead to oscillating trajectories of the system, thus impacting the system efficiency.

In this work, we present a macroscopic dynamic traffic assignment model aiming to study traffic at an origin-destination pair connected by two alternative non intersecting homogeneous routes, where homogeneous means that the two routes have the same length and free-flow speed but can have different capacities. We assume that a given fraction of drivers relies on recommendations of a navigation App. The model describes traffic evolution on the two routes. The varying behavior of the app-informed drivers is captured by routing ratios determining drivers' choice. Routing ratios take into account both drivers' inclination to follow the App recommendations and, differently from previous work in literature, the delay affecting information, i.e., routing ratios depend on a retarded state of the system. Through some manipulations, we trace the system back to a scalar ordinary differential equation (ODE) in terms of the difference between the travel times on the two routes, from which, through its stability analysis, we are able to deduce implications from the traffic point of view.

Briefly, the main results we found imply that delay affecting information is detrimental in terms of the network efficiency, i.e., it induces instability (oscillating behavior) and unsatisfied demand. In addition, the system becomes more sensitive to smaller and smaller value of delay, as the demand, the fraction of app-informed drivers, and drivers' compliance to recommendations increase.

The paper is organized as follows. Section II provides the definition of the model and its main features. In Section III, the stability analysis of the system is carried out and an interpretation from a road traffic point of view is provided. Section IV contains numerical simulations before concluding the paper in Section V.

Notation: Let $\mathbb{R}^d$ ($\mathbb{R}^d_+$) be the space of real-valued (non-negative-valued) vectors of dimension d. We denote by $C([a,b],D)$ the set of continuous functions $\chi:[a,b]\to D$ with the norm $||\chi||_C=\max_{\omega\in[a,b]}|\chi(\omega)|$. Given a domain $\mathscr{D}\subseteq\mathbb{R}^d$ and an autonomous system of delay differential equation $\dot{y(t)}=g(y(t),y(t-\theta)),\ \theta>0$, with $g:D^2\to\mathbb{R}^d$, admitting a unique solution $x:\mathbb{R}_+\to\mathbb{R}^d$ with initial condition $x(\omega)=\chi(\omega),\ \omega\in[-\theta,0]$, we say that $\mathscr{X}\subseteq\mathscr{D}$ is an *invariant* region of the system if $\chi(\omega)\in\mathscr{X}$ implies that $x(t)\in\mathscr{X},\ \forall t\geq 0$.

II. Model definition

A. Network geometry

Consider an origin-destination pair connected by two alternative non intersecting routes, see Figure 1. For each route $i=1,2$, the positive values C_i, B_i F_i, L_i represent

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Tommaso Toso and Paolo Frasca are with Univ. Grenoble Alpes, CNRS, Inria, Grenoble INP, GIPSA-lab, 38000 Grenoble, France (e-mail: tommaso.toso@gipsa-lab.grenoble-inp.fr; paolo.frasca@gipsa-lab.grenoble-inp.fr). Alain Kibangou is with Univ. Grenoble Alpes, CNRS, Inria, Grenoble INP, GIPSA-lab, 38000 Grenoble, France, and also with Univ. of Johannesburg (Auckland Park Campus), Johannesburg 2006, South Africa (e-mail: alain.kibangou@gipsa-lab.grenoble-inp.fr).

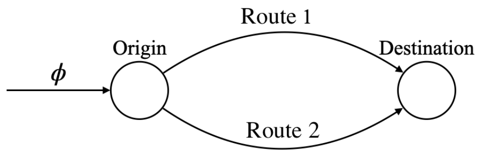

Fig. 1. Graph representation of the origin-destination pair considered.

the critical density (veh/km), the jam density (veh/km), the maximum capacity (veh/h) and the length (km), respectively. The demand is assumed to be a constant flow $\phi > 0$ that enters the network from the origin node and aims to reach the destination node through the two routes. We assume that it does not exceed the network capacity:

$$\phi < F_1 + F_2. \tag{1}$$

This assumption ensures that the network is well-dimensioned and able to withstand the traffic demand. The traffic density x_i on the i-th route evolves according to the following equation:

$$\dot{x}_i(t) = \frac{1}{L_i}\left(\min\{\phi R_i(x(t-\theta)), F_i\} - v_i x_i(t)\right), \tag{2}$$

for $i = 1,2$, where $R_i(x_i(t-\theta)) \geq 0$ and $v_i = F_i/C_i$ stand respectively for the routing ratio and the free-flow speed of the i-th route, while $x(t) = (x_1(t), x_2(t))^T$ represents the state of the network. We assume routing ratios to be Lipschitz continuous and to depend on a delayed state of the system, i.e., the demand splitting at the origin node at time t is determined by the state of the system at time $t-\theta$ for some nonnegative delay θ.

The minimum operator in (2) ensures that an inflow higher than the route capacity cannot enter it. Observe that the right-hand side of (2) is Lipschitz continuous. Hence, there exists an unique solution to (2) that continuously depends on the initial data for every initial condition $x(\omega)$, $\omega \in [-\theta, 0]$, where $x(\omega) \in C([-\theta,0],P)$ and $P := [0,C_1] \times [0,C_2]$ [7][Section 1.3.1].

Proposition 1: P is an invariant region for (2).

Proof: Since $v_i = F_i/C_i$, we can observe that $x_i(t) = C_i$ implies that the second term in (2) is equal to F_i, thus $\dot{x}_i(t) \leq 0$ for $i = 1,2$. Therefore trajectories cannot exit region P. ■

Remark 1 (Free-flow dynamics): Proposition 1 implies that, starting from a free-flow initial condition, the density dynamics will remain in free-flow for all times. This represents a limitation of the model at hand, in that it does not allow us to describe scenarios in which one of the two route becomes congested during the dynamics.

B. Modeling the routing ratios

To take into account the influence of route recommendations on the demand splitting, routing ratios should depend on travel time, which is typically modeled as a strictly increasing functions of road density [1], [2], [10]. In this paper, we model travel time as follows:

$$\tau_i(x_i(t)) = a_i \frac{x_i(t)}{B_i} + \frac{L_i}{v_i}, \quad a_i > 0, \tag{3}$$

$i = 1,2$. The choice of affine travel time functions is largely used in the dynamic traffic assignment literature, especially when considering a free-flow regime [8], [14]. The constant term in (3) corresponds to the free-flow travel time, while the coefficient a_i is such that $a_i C_i / B_i$ is the time added to the route travel time when the route density reaches the critical density C_i [8]. We model routing ratios assuming that a fraction $\alpha \in [0,1]$ (which we will refer to as *penetration rate*) of the user demand is influenced by the App recommendations, whereas the remaining fraction gets split according to given ratios $r^0 = (r_1^0, r_2^0)$, where $r_i^0 \geq 0$, $r_1^0 + r_2^0 = 1$. Therefore, we define routing ratios as follows:

$$\begin{aligned} R_1(d(t-\theta)) &:= (1-\alpha) r_1^0 + \alpha \frac{1}{1 + \frac{r_2^0}{r_1^0} \exp\left(-\frac{1}{\eta} d(t-\theta)\right)}, \\ R_2(d(t-\theta)) &= 1 - R_1(d(t-\theta)), \end{aligned} \tag{4}$$

and

$$d(t) := \tau_2(x_2(t)) - \tau_1(x_1(t))$$

denotes the difference between the traveling times. The second term, describing the behaviour of the app-informed drivers, takes the form of the *logit dynamics* [3], [4], [5] and is a function of the delayed state of the system: $1/\eta > 0$ is the so called *drivers' compliance*, which quantifies the tendency of drivers to actually follow the App recommendations. Indeed, when $1/\eta \to 0$, i.e., drivers' compliance is very low, drivers do not really exploit information and the demand splitting stays close to r^0. On the contrary, when $1/\eta \to +\infty$, all drivers tend to take the shortest travel time route. Notice from (4) that the fraction of user demand directed towards each route is non negative, i.e., $0 \leq R_i(d(t-\theta)) \leq 1$, $\forall x(t-\theta) \in [0,B_1] \times [0,B_2]$, $i = 1,2$, and route recommendations direct an higher quantity of demand on the shortest travel time route (according to delayed data), i.e., they advise to take the route with the shortest travel time.

C. Unsatisfied demand

When on route i the fraction of user demand ϕR_i is less than the capacity F_i, then the first term of (2) is not saturated and the demand is able to enter the route freely. Instead, when ϕR_i is more than F_i, then the first term of (2) is saturated and part of the demand is unable to enter the route. We refer to the latter situation as unsatisfied demand. Then, each route is characterized by two possible *modes* that can be made explicit by rewriting (2) as follows:

$$\dot{x}_i(t) = \begin{cases} \dfrac{\phi R_i(d(t-\theta)) - v_i x_i(t)}{L_i}, & \text{if } R_i(d(t-\theta)) \leq \frac{F_i}{\phi}, \\ \dfrac{F_i - v_i x_i(t)}{L_i}, & \text{if } R_i(d(t-\theta)) > \frac{F_i}{\phi}, \end{cases} \tag{5}$$

where the second equation is associated to the unsatisfied demand regime. Notice that (1) implies that the two routes cannot present unsatisfied demand simultaneously.

D. Reducing the model to a 1-dimensional dynamics

We make the following assumption of homogeneity of the two routes.

Assumption 1 (Free-flow speeds and route lengths): The free-flow speed and the route length are the same for both routes:

$$v_i = v, \quad L_i = L, \quad i = 1,2. \tag{6}$$

From now on, we will work on this special case. Assumption 1 refers to a scenario in which the two routes are of similar length and subject to the same speed limit but can have different capacities, e.g., two different routes in a urban road network.

Now, let us multiply $\dot{x}_i(t)$ by a_i/B_i, $i = 1,2$ in (2), and then subtract the first equation from the the second one. Thanks to Assumption 1, we get the following scalar delay-differential equation in $d(t)$:

$$\dot{d}(t) = -\frac{v}{L}d(t) + \rho(d(t-\theta)), \tag{7}$$

where

$$\begin{aligned}\rho(d(t-\theta)) := \frac{1}{L}\Bigg(&\frac{a_2}{B_2}\min\left(F_2, \phi(1-R_1(d(t-\theta)))\right)\\ &- \frac{a_1}{B_1}\min\left(F_1, \phi R_1(d(t-\theta))\right)\Bigg)\end{aligned} \tag{8}$$

is a globally Lipschitz continuous function with Lipschitz constant $K = \frac{\alpha\phi}{4\eta L}\left(\frac{a_2}{B_2} + \frac{a_1}{B_1}\right)$. Recalling Remark 1, the state space of (7) is given by $[-a_1C_1/B_1, a_2C_2/B_2]$.

Observe that because of the one-to-one correspondence between $R_1(d(t))$ and $d(t)$, any conclusion about the stability of the trajectories of $d(t)$ will be valid for the stability of the trajectories of $R_1(d(t))$, as well.

III. Stability analysis

This section contains an analysis of the stability of (7). First, we will show that (7) admits a single equilibrium point. Then, we provide sufficient conditions for the global asymptotic stability of (7) $\forall\theta \geq 0$, using a Lyapunov approach. Finally, by means of a local analysis, we give sufficient conditions for the system to become unstable when the delay θ is sufficiently large. Particular emphasis is placed on the link between the above conditions and the parameters ϕ, α and $1/\eta$.

A. Uniqueness of the equilibrium point

Proposition 2: The dynamics (7) admits a unique equilibrium point.

Proof: Define $g(d) := L\rho(d)/v$. We see that d^* is an equilibrium of (7) if and only if is a fixed point of $g(d)$. Now, $g(d)$ is a continuous and strictly decreasing function in d. Hence, it admits a unique fixed point, i.e., (7) has a unique equilibrium point d^*. ∎

B. Delay-independent global asymptotic stability

We now provide a condition for the delay-independent global asymptotic stability (GAS) of (7).

Theorem 1 (Delay-independent GAS): The unique equilibrium point d^* of the dynamics (7) is globally asymptotically stable for all $\theta \geq 0$ if $K < v/L$.

Proof: For convenience, consider the dynamics obtained by shifting (7) so that d^* corresponds to the origin of the system., so take $u(t) = d(t) - d^*$ and

$$\dot{u}(t) = -\frac{v}{L}u(t) - \sigma(u(t-\theta)), \tag{9}$$

where $\sigma(u(t)) := \rho(u(t)+d^*) - \rho(d^*)$. Clearly, the asymptotic properties of (7) coincides with those of (9). Define now the following Lyapunov functional:

$$V(t) := \frac{1}{2}u^2(t) + \frac{v}{2L}\int_{t-\theta}^{t} u^2(s)ds. \tag{10}$$

First of all, notice that

$$\frac{1}{2}|u(t)|^2 \leq V(u(t)) \leq \frac{1}{2}\left(1 + \frac{v\theta}{L}\right)\max_{s\in[t-\theta,t]}|u(s)|^2.$$

We find that

$$\begin{aligned}\dot{V}(t) &= u(t)\dot{u}(t) + \frac{v}{2L}\left(u^2(t) - u^2(t-\theta)\right) \leq \\ &\leq -\frac{v}{2L}u^2(t) - \frac{v}{2L}u^2(t-\theta) + K|u(t)||u(t-\theta)| = \\ &= -\begin{pmatrix}|u(t)| & |u(t-\theta)|\end{pmatrix}\begin{pmatrix}\frac{v}{2L} & -\frac{K}{2}\\ -\frac{K}{2} & \frac{v}{2L}\end{pmatrix}\begin{pmatrix}|u(t)|\\ |u(t-\theta)|\end{pmatrix}.\end{aligned}$$

If the matrix defining the quadratic form above is positive definite, then there exists $\gamma > 0$ such that $\dot{V}(t) < -\gamma|u(t)|^2$ and global asymptotic stability of (7) comes from [7][Theorem 3.1]. The matrix is positive definite if and only if $v/L > K$. ∎

The inequality $K < v/L$ can be rewritten equivalently as follows:

$$\phi < \frac{4\eta v B_1 B_2}{\alpha(a_2B_1 + a_1B_2)} =: \Phi. \tag{11}$$

Then, some considerations can be made about Theorem 1. First of all, the delay-independent condition (11) for the GAS of (7) is expressed in terms of an upper bound on the user demand. This upper bound on the user demand depends on both the penetration rate α and drivers' compliance $1/\eta$. Specifically, the upper bound in (11) is decreasing in α and $1/\eta$. Therefore, we can say that if enough drivers receive the route recommendations and follow them accurately, then the system becomes sensitive to delay, in the sense that delays can destabilise it.

C. Instability and oscillations for large demand and delay

When the user demand exceeds the upper bound in (11), i.e., $v/L < K$, it might be that stability is lost for sufficiently large values of delay. In the following, we investigate this possibility by performing a local stability analysis around the unique equilibrium point d^* of the system to deduce sufficient conditions for d^* to be unstable.

We will focus on a relevant subset of the parameter set, in which the following three facts hold.

1) There is no unsatisfied demand when the penetration rate is zero, which is equivalent to

$$\phi r_i^0 < F_i, \quad i = 1,2. \tag{12}$$

2) There is no unsatisfied demand at equilibrium, which is equivalent to

$$1-\frac{F_2}{\phi} < R_1(d^*) < \frac{F_1}{\phi}. \quad (13)$$

3) The user demand ϕ and the penetration rate α are large enough to allow unsatisfied demand to emerge on one of the two routes. This requirement will be satisfied by the following condition:

$$\phi > F_i,\ \alpha > \underline{\alpha}_i := \frac{F_i - \phi r_i^0}{\phi(1-r_i^0)}, \quad i=1,2. \quad (14)$$

Assumption 2: Conditions (12), (13) and (14) are satisfied.

Theorem 2 (Local stability): Suppose Assumption 2 holds. Then, the following assertions hold true for dynamics (7):

1) if $|\rho'(d^*)| < \frac{v}{L}$, then d^* is asymptotically stable for all $\theta \geq 0$.
2) if $\rho'(d^*) < -\frac{v}{L}$, then d^* is asymptotically stable for $\theta < \theta^*$ and unstable for $\theta > \theta^*$, where

$$\theta^* := \frac{1}{\sqrt{(\rho'(d^*))^2 - \frac{v^2}{L^2}}} \arccos\left(\frac{v}{L\rho'(d^*)}\right), \quad (15)$$

undergoing a Hopf bifurcation at $d = d^*$ when $\theta = \theta^*$.

Proof: Assumption 2 ensures that no terms of (8) are saturated at equilibrium. Then, (7) can be rewritten as follows:

$$\dot{d}(t) = -\frac{v}{L}d(t) + \frac{\phi}{L}\left(\frac{a_2}{B_2} - \left(\frac{a_1}{B_1}+\frac{a_2}{B_2}\right)R_1(d(t-\theta))\right). \quad (16)$$

Since the inequalities in (13) are strict, we are able to find a neighborhood $I \subset \mathbb{R}$ of d^* where (7) takes the form (16), with a differentiable right-hand side. Therefore, we are able to perform a local stability analysis within I by analysing the linearization of (16) in I. The statement follows after applying [13][Theorem 2.3] to (7). ∎

This result provides a necessary and sufficient condition for instability. However, it involves several conditions that cannot easily be tested because the equilibrium d^* is not known in closed form. In order to obtain more readable and testable conditions, we begin by deriving a sufficient condition to replace (13), always assuming to be in a sufficiently small neighborhood of d^* where (7) reads as (16), as in the proof of Theorem 2. To this purpose, one can readily verify that condition (13) is equivalent to

$$d_{FC} < d^* < d_{CF}, \quad (17)$$

where

$$d_{CF} := \eta \log\left(\frac{r_2^0}{r_1^0}\frac{\gamma_1}{\alpha\phi - \gamma_1}\right),\ d_{FC} := \eta \log\left(\frac{r_2^0}{r_1^0}\frac{\alpha\phi - \gamma_2}{\gamma_2}\right),$$

with $\gamma_i := F_i - (1-\alpha)\phi r_i^0,\ i=1,2$. Next, we derive the sufficient condition.

Lemma 1: Given $\phi > 0$ satisfying (12) and (14). Then, (17) is satisfied for all $\alpha > \max\{\underline{\alpha}_1, \underline{\alpha}_2\}$ and $1/\eta > 0$, if the following condition hold:

$$r_1^0 < \frac{a_2 B_1}{a_1 B_2 + a_2 B_1} < \frac{F_1}{\phi} \quad \text{or} \quad r_2^0 < \frac{a_1 B_2}{a_1 B_2 + a_2 B_1} < \frac{F_2}{\phi}. \quad (18)$$

Proof: See Appendix. ∎

We now make the following assumption, testable on the system parameters.

Assumption 3: Conditions (12), (14) and (18) are satisfied.

Finally, we combine Theorem 2 with the following lower and upper bounds on the absolute value of $\rho'(d)$.

Lemma 2: Suppose that Assumption 2 holds. Then, the following inequalities hold:

$$Q < |\rho'(d^*)| < K, \quad Q := \min(|\rho'(d_{CF})|, |\rho'(d_{FC})|) \quad (19)$$

where

$$\begin{aligned} |\rho'(d_{CF})| &= \frac{\phi}{\eta L}\left(\frac{a_1}{B_1}+\frac{a_2}{B_2}\right)\gamma_1\left(1-\frac{\gamma_1}{\alpha}\right), \\ |\rho'(d_{CF})| &= \frac{\phi}{\eta L}\left(\frac{a_1}{B_1}+\frac{a_2}{B_2}\right)\gamma_2\left(1-\frac{\gamma_2}{\alpha}\right). \end{aligned} \quad (20)$$

Proof: See Appendix. ∎

We thus get the following result only involving testable conditions.

Corollary 1: Under Assumption 3, the following assertions hold:

1) if $K < \frac{v}{L}$, then d^* is asymptotically stable for any $\theta \geq 0$.
2) if $\frac{v}{L} < Q$, then the second assertion of Theorem 2 holds and the critical delay value satisfies

$$\theta^* < \theta_Q^* := \frac{1}{\sqrt{Q^2 - \frac{v^2}{L^2}}} \arccos\left(-\frac{v}{LQ}\right). \quad (21)$$

Proof: The first assertion follows trivially from the first assertion of Theorem 1 and the second inequality in (19). Similarly, the second assertion follows directly from the second assertion of Theorem 2, the first inequality in (19) and the fact that θ^* is an increasing function of $\rho'(d)$ when $\rho'(d) < -v/L$. ∎

The first assertion of Corollary 1 is in fact a special case of Theorem 1, which stated that $K < v/L$ implies the delay-independent *global* asymptotic stability of d^*. Instead, the second assertion provides a sufficient condition for the unique equilibrium point d^* to be unstable and an upper bound for the critical delay θ^*, which are explicitly written as functions of the system parameters.

We can easily deduce from (20) that Q is increasing in α and $1/\eta$ and therefore θ_Q^* is decreasing α and $1/\eta$. This fact provides us with some indications about the qualitative behavior of θ^* with respect to the above mentioned parameters, suggesting that increases in the penetration rate and in drivers' compliance reduce the delay threshold after which the system equilibrium is sure to loose its stability. One can also verify that θ^* is decreasing in ϕ, even though the relevance of this observation is tempered by the fact that too large demand can lead outside the set of assumptions under consideration.

Overall, the results provided in this section are consistent with those presented in Section III-B: increases in ϕ, α and $1/\eta$ negatively affect the system stability.

Remark 2: Assumptions 2 and 3 have two interesting features. First, they select a set of parameters large enough to include realistic traffic scenarios, as will be demonstrated in the next section. Second, they focus on a very interesting situation in which, despite not having unsatisfied demand both at equilibrium and in the absence of informed drivers, the destabilising effect of delays might cause demand dissatisfaction on one of the two routes, as we are going to show in the next section.

IV. Numerical Example and Discussion

Consider the network in Figure 1 characterized by the following parameters:

$$\begin{gathered} F_1 = 1200 \text{ veh/h},\ F_2 = 600 \text{ veh/h}, \\ C_1 = 24 \text{ veh/km},\ C_2 = 12 \text{ veh/km}, \\ B_1 = 120 \text{ veh/km},\ B_2 = 60 \text{ veh/km}, \\ a_1 = a_2 = 0.1 \text{ h (6 min)},\ r_1^0 = 0.66,\ r_2^0 = 0.34, \end{gathered}$$

and assume that the network is subject to a constant user demand of $\phi = 1750$ veh/km, the length of the two routes is $L = 1.5$ km and the average free-flow speed is 50 km/h. The parameters of the two routes were chosen to represent a two-lane urban route and a one-lane urban route. Consider also two possible values of α, 0.33 and 0.66, and two values of $1/\eta$, 100 and 200. We observe that these parameters satisfy Assumption 2, meaning there is no unsatisfied demand both at equilibrium and in absence of informed drivers.

Finally, consider two realistic [15], [16] values of θ, 1 and 8 minutes. The numerical simulations in Figure 2 show the system behavior for these values of delay in three different cases:

- $\alpha = 0.33,\ 1/\eta = 100$: in this case, $K \approx 24.06$, which is less than $v/L \approx 33.33$, and $\phi < \Phi \approx 2666$ veh/h. Hence, Theorem 1 holds and, as shown in the plots in the first column of Figure 2, the delay increase does not alter the stability of the equilibrium point of the system.
- $\alpha = 0.66,\ 1/\eta = 100$: in this case, $K \approx 48.13$ and $\phi > \Phi \approx 1333$ veh/h. Hence, Theorem 1 is no longer applicable. Moreover, $Q \approx 40.50$ and exceeds $v/L \approx 33.33$. Hence, by Corollary 1, sufficiently high delays can destabilise the system. Indeed, since $\theta_Q^* \approx$ 6 min and 24 s, for $\theta = 8$ min the equilibrium point is unstable and the trajectory is oscillating.
- $\alpha = 0.33,\ 1/\eta = 200$: similarly to the previous case, the decrease of noise destabilises the equilibrium point when the system is affected by a delay of 8 minutes. Consistently with Corollary 1, in this case $K \approx 48.13$ and $Q \approx 37.23$, which are both greater than $v/L = 33.33$, $\phi > \Phi \approx 1333$ veh/h and $\theta_Q^* \approx 7$ min and 42 s.

In the second and third cases, the excessive delay affecting recommendations destabilises the equilibrium point of the system and causes trajectories to oscillate, i.e., the incoming informed drivers split between the two routes in an unsteady and periodic way. Moreover, this oscillating behavior causes periodic demand dissatisfaction, thus negatively affecting the network performance.

Remark 3 (Effects of unsatisfied demand on oscillations): The saturation of the supply functions, i.e., the occurrence of unsatisfied demand, is not the cause of oscillations, considering that, as shown in Figure 2, oscillations can arise starting from initial conditions with no supply saturation. Instead, the only cause of oscillations is information delay.

Remark 4 (Comparison with [2]): Another macroscopic traffic model exhibiting oscillating trajectories was proposed in [2]. Their model does not account for information delays and oscillations are due to different reasons: in fact, the model [2] does not enjoy a key monotonicity property [11], which prevents oscillations and is satisfied by our model (2) when $\theta = 0$.

V. Conclusion

In this paper, we have mathematically described how delays, which affect routing recommendations by navigation Apps, influence the traffic flow between an origin-destination pair that is connected by two alternative non-intersecting paths with the same length and free-flow speed. The proposed model shows that an excessive delay can destabilise the equilibrium of the system and give rise to oscillating trajectories, provided the penetration rate and the user demand are large enough. We emphasized how the system becomes more susceptible to delay when the user demand, the penetration rate of the App and the drivers' compliance increase. Depending on their magnitude, oscillations can generate unsatisfied demand and thus seriously deteriorate the functioning of the traffic network.

Future work is mainly headed towards two kinds of extensions of the model. On the one hand, we want to study more complex network topologies, e.g., featuring more than two alternative paths, paths with different lengths and free-flow speeds, and richer traffic dynamics to describe non free-flow conditions. On the other hand, we want to extend the results to more general classes of routing ratios. Such extensions would underline the generality of delay as a key cause of instability and oscillations in traffic flows.

Appendix I
Proof of Lemma 1

First, observe that (14) ensures that d_{CF} and d_{FC} are well-defined. Then, the first condition of Assumption 2 is equivalent to $d_{FC} < 0 < d_{CF}$. For ϕ fixed, one can define a family of functions $\{\rho^{\alpha_j,\eta_k}\}, \alpha_j > \max\{\underline{\alpha}_1, \underline{\alpha}_2\},\ \eta_k > 0$, observing that all of them attain the same value γ at $d = 0$:

$$\gamma := \frac{\phi}{L}\left(\frac{a_2}{B_2} - \left(\frac{a_1}{B_1} + \frac{a_2}{B_2}\right) r_1^0\right).$$

It holds that $\gamma > 0$ when the first inequality in (18) holds, whereas $\gamma < 0$ when the second does. As pointed out in the proof of Proposition 2, the equilibrium point of (7) satisfies to $d^* = L\rho(d^*)/v$, i.e., it is the value of d at which $L\rho(d^*)/v$ and the identity line intersect. Define $d^*_{\alpha_j,\eta_k}$ as the equilibrium point associated to the function ρ^{α_j,η_k}. Suppose

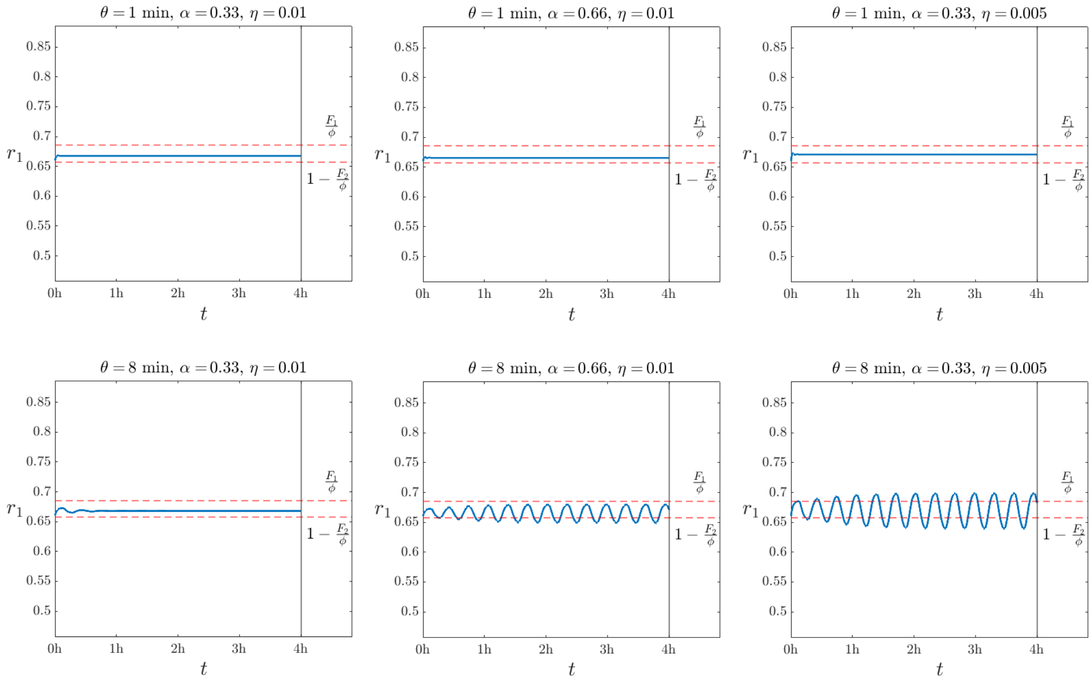

Fig. 2. Each column of plots is associated with different values of the pair of parameters α, $1/\eta$, each line with different delay θ. The red dashed lines delimit the states for which unsatisfied demand is absent (in-between) or present (outside).

that (18) holds, so that $\gamma > 0$ and $\beta_{CF} := \rho(d_{CF}) < 0$. Since neither β_{CF} nor γ depend on α and η, then $0 < d^*_{\alpha_j,\eta_k} < d_{CF}$, for all $\alpha_j \in (\max\{\underline{\alpha}_i, i=1,2\},1]$, $\eta_k > 0$. The proof is complete after observing that we can apply the same process when (18) holds, so that $\gamma < 0$ and $\beta_{FC} := \rho(d_{CF}) > 0$.

Appendix II
Proof of Lemma 2

The second inequality is trivial, since K is the Lipschitz constant of $\rho(d)$. For the first inequality, if we define $U_1(d) := R_1(d) - (1-\alpha)r_1^0$, then

$$\rho'(d) = \frac{\alpha\phi}{\eta L}\left(\frac{a_1}{B_1} + \frac{a_2}{B_2}\right)U_1(U_1 - 1),\ \rho''(d) = \frac{\rho'(1-2U_1)}{\eta}.$$

We see that $|\rho'(d)|$ increases for $U_1(d) \in (0,1/2)$ and decreases for $U_1(d) \in (1/2,1)$, i.e., $|\rho'(d)|$ increases for $d < U_1^{-1}(1/2) = \eta \log(r_2^0/r_1^0)$ and decreases for $d > U_1^{-1}(1/2)$, where U_1^{-1} is the inverse of $U_1(d)$. From (17), $|\rho'(d^*)|$ is greater than at least one between $|\rho'(d_{CF})|$ and $|\rho'(d_{FC})|$.

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