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Distributed Adaptive Control of Disturbed Interconnected Systems with High-Order Tuners

Moh. Kamalul Wafi and Milad Siami, *Senior Member, IEEE*

Abstract— **This paper addresses the challenge of network synchronization under limited communication, involving heterogeneous agents with different dynamics and various network topologies, to achieve a consensus. We investigate the distributed adaptive control for interconnected unknown linear subsystems with a leader and followers, with the presence of input-output disturbance. We enhance the communication within multi-agent systems to achieve consensus under the leadership's guidance. While the measured variable is similar among the followers, the incoming measurements are weighted and constructed based on their proximity to the leader. We also explore the convergence rates across various balanced topologies (Star-like, Cyclic-like, Path, Random), featuring different numbers of agents, using distributed first and high-order tuners. Moreover, we conduct several numerical simulations across various networks, agents and tuners to evaluate the effects of sparsity in the interaction between subsystems using the L_2-norm and $L_\infty-$norm. Some networks exhibit a trend where an increasing number of agents results in smaller errors, although this is not universally the case. Additionally, patterns observed at initial times may not reliably predict overall performance across different networks. Finally, we demonstrate that the proposed modified high-order tuners outperform its counterpart, and we provide related insights along with our conclusions.**

I. Introduction

Multi-agent systems (MAS), spanning areas from robotics, including unmanned ground [1], aerial [2], and underwater vehicles [3], to large-scale societal dynamics [4], have attracted considerable interest. The scope of challenges these systems face extends from internal issues like achieving consensus among agents for coordinated control and stability, to external threats such as disturbances, environmental uncertainties, or attacks [5]. Furthermore, the interconnected nature of networked systems necessitates insights from graph theory. This is underscored by [6], which examines the limits and trade-offs in networks facing stochastic disturbances, and by [7], which explores how denser networks (with more links) affect the number of agents.

In this paper, we explore distributed adaptive control as a foundational element of MAS, sharing similarities with distributed Model Reference Adaptive Control (MRAC). This topic spans from theoretical frameworks aimed at achieving consensus [8], [9] in complex networks to practical applications in large-scale systems [10], [11]. The distributed adaptive control in this paper is adaptable to various agent dynamics, influencing diverse control laws among agents, as closely discussed in [12], [13]. Moreover, specific studies have proposed solutions for nonlinear MAS and neural network-based challenges [14], [15]. Furthermore, inspired from the distributed optimization [16], our study incorporates high-order tuners as adaptive laws to update the gains.

A newly developed algorithm for high-order tuners has been introduced to optimize convex loss functions with time-varying regressors in identification problems. This algorithm leverages Nesterov's method principles, ensuring that parameter estimations stay within predefined bounds when confronted with time-varying regressors [17]. It also accelerates the convergence speed of the tracking error in scenarios where the regressors are constant [18]. With the growing interest in advancing tuners, we have adapted high-order tuners for graph-related problems [18], [19], achieving marginally better outcomes compared to gradient-based methods. Furthermore, we offer insights on designing network weights and selecting parameters in the tuners.

This paper makes the following four main contributions. First, we integrate the concept of adaptive control into networked problems with multiple agents, extending its applicability to complex interconnected systems such as star-like, cyclic-like, path, and random networks. Second, we address the challenge of coordinating an arbitrary number of agents with disturbances to follow a designated leader, similar to distributed MRAC. Third, we compare the performance of three distinct tuning algorithms: the gradient descent and two accelerating tuners, providing a comprehensive evaluation of their effectiveness in networked control systems (NCS). Finally, we not only evaluate the effects of sparsity in subsystem interactions using performance measures (L_2-norm and $L_\infty-$norm) across various network configurations and tuners but also demonstrate that our proposed modified high-order tuners significantly outperform the gradient-based tuner, offering novel insights for future research in NCS.

Notations. $\mathbb{R}^p$ is the $p-$dimensional Euclidean space and $\mathbb{C}^-$ refers the open left-half of the complex plane. A symbol (s) shows the Laplace variable. I_p denotes the identity matrix of size $\mathbb{R}^p$ and $P = \text{diag}\{p_i\}$ is the diagonal matrix with entries $p_i, \forall i$. $\mathbf{1}_p = [1, \ldots, 1]^\top$ is the vector of all ones in $\mathbb{R}^p$. $\otimes$ denotes the Kronecker product and the operators of $\text{tr}[A]$, $|A|$, $\|A\|_2$, and $\|A\|_F$ define the trace, the absolute value, the Euclidean and the Frobenius norm of matrix A respectively.

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M. K. Wafi and M. Siami are with the Department of Electrical & Computer Engineering, Northeastern University, Boston, MA 02115, USA. (e-mails: {wafi.m, m.siami}@northeastern.edu).

II. Preliminaries and Problem Formulation

A. System Setup

We consider an interconnected network of subsystems shown in Fig. 1, which consists of a leader and m unknown unstable subsystems/agents. Let the unknown subsystems for the followers be defined as follows,

$$W_i(s) \sim \begin{cases} \dot{x}_i(t) = A_i x_i(t) + B_i(u_i(t) + v_i^u), \\ y_i(t) = k_{p_i} C_i x_i(t) + v_i^y, \end{cases} \tag{1}$$

where $x_i \in \mathbb{R}^n$ is the state vector, and $u_i, y_i \in \mathbb{R}$ denote the input and output, respectively, for $i = 1, 2, \ldots, m$. The control input u_i and the output measurement y_i are disturbed by the unknown yet constant $v_i^u, v_i^y \in \Omega \subset \mathbb{R}$. The transfer function from u_i to y_i is denoted by $W_i(s)$. The dynamic of the leader as the reference model is written as,

$$W_\ell(s) \sim \begin{cases} \dot{x}_\ell(t) = A_\ell x_\ell(t) + B_\ell r(t), \\ y_\ell(t) = k_\ell C_\ell x_\ell(t), \end{cases} \tag{2}$$

in which r is the reference signal and that is a piecewise-continuous function while $x_\ell \in \mathbb{R}^{n_\ell}$ and $y_\ell \in \mathbb{R}$ represent the reference state and output. Note that $A_i, B_i, C_i, A_\ell, B_\ell, C_\ell$ are constant real matrices with appropriate dimensions whereas k_{p_i} and k_ℓ are the high frequency gains. The goal is to design local control input u_i so that the outputs $y_i, \forall i$ follow that of the known stable leader y_ℓ.

Assumption 1. *The dynamics* (1) *are unknown and unstable while* (2) *and the signs of k_{p_i} are known. The numerators of $W_i(s), \forall i$ have roots in $\mathbb{C}^-$ while the denominators of $W_i(s), \forall i$ and $W_\ell(s)$ are monic with relative degree $n_d = 1$.*

B. Communication Network

We describe the m followers and a leader ℓ connected via weighted digraph $\mathcal{G} \coloneqq \{\mathcal{V} = \{1, 2, \ldots, m\} \cup \{\ell\}, \mathcal{E}, w(\cdot)\}$ where $\mathcal{V}$, $\mathcal{E}$, and $w(\cdot)$ represent the set of nodes, directed edges, and the weight function in turn. For simplicity, we denote $w(i,j) = w_{ij}$ where $(i,j) \in \mathcal{E}$. We call the induced subgraph on m followers as $\mathcal{G}_m$ and the leader itself as $\mathcal{G}_\ell$. Also, we assume that there is a directed path from the leader ℓ to all followers. The layering colors indicate the $q-$th group

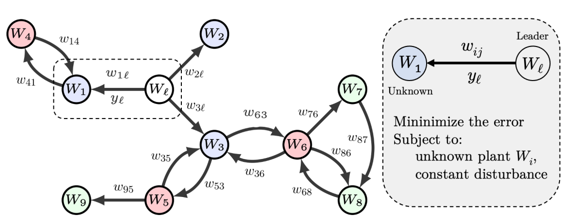


Fig. 1: An example (random graph) of an interconnected network of leader ℓ and $m = 9$ unknown unstable subsystems/followers.

of systems from the leader, where the least is the closest, as example shown in Fig. 1 with $q = \{1, 2, 3\}$. The incoming arrows for $i-$th system represent the measured neighborhood j with respected weight w_{ij}. Note that, the measurement collected to system i from its neighbors j is designed to be 1, where $w_i = \sum_j w_{ij} = 1$, so that the degree matrix for the whole agents is $\mathbb{D} \coloneqq \mathrm{diag}\{D_1, \ldots, D_m\} = I_m$. The measured errors in $\mathcal{G}_m$ are represented as linear operation of its outputs $\bar{y}$ multiplied by the Laplacian-like matrix of $\mathcal{G}_m$, written as $\mathbb{L}_m \coloneqq \mathbb{D} - \mathbb{A}_m$, and subtracted by the leader $\bar{y}_\ell$ using $\mathbb{A}_\ell$, with later definition of $\bar{y}$ and $\bar{y}_\ell$. The matrices of $\mathbb{L}_m$ and $\mathbb{A}_\ell$ are formulated as follows,

$$\underbrace{\begin{bmatrix} w_1 & w_{12} & \cdots & w_{1m} \\ w_{21} & w_2 & \cdots & w_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m1} & w_{m2} & \cdots & w_m \end{bmatrix}}_{\mathbb{L}_m \coloneqq \mathbb{D} - \mathbb{A}_m} \text{ and } \underbrace{\begin{bmatrix} w_{1\ell} & 0 & \cdots & 0 \\ 0 & w_{2\ell} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w_{m\ell} \end{bmatrix}}_{\mathbb{A}_\ell | w_{i\ell} \coloneqq 0, \forall q > 1}$$

in which, $\mathbb{A}_m$ denotes the adjacency matrix of $\mathcal{G}_m$ whereas $\mathbb{A}_\ell \in \mathbb{R}^{m \times m} = \mathrm{diag}\{w_{1\ell}, \ldots, w_{m\ell}\}$ is the diagonal matrix containing the weights from the leader to the subsystems in $q = 1$. Therefore, the error for system i is formulated as,

$$e_i(t) \coloneqq \sum_{j=1}^{m} w_{ij} \left[y_i(t) - y_j(t) \right] - w_{i\ell} y_\ell(t) \tag{3}$$

and the goal is to ensure the boundedness of the errors in $\mathcal{G}_m$, where $\lim_{t \to \infty} \bar{e} \to 0$, $\bar{e} = [e_1, \ldots, e_m]^\top$, leading to the perfect tracking to the leader $\mathcal{G}_\ell$.

Remark 1 (Threshold of network)**.** *The leader weight matrix is positive semi-definite $\mathbb{A}_\ell \succeq 0$, having the eigenvalues ranging from 0 to 1, denoted as $\lambda_i^\ell = [0, 1], \forall i = 1, \ldots, m$, and $\exists \lambda_j^\ell \neq 0$ for $1 \leq j \leq m$. The Laplacian-like matrix of $\mathcal{G}_m$ is positive definite $\mathbb{L}_m \succ 0$, ensuring $\lambda_i^m > 0, \forall i$.*

Remark 2 (Communication network)**.** *The proposed network is balanced $(\mathbb{L}_m - \mathbb{A}_\ell)\mathbf{1}_m = 0$. There is always a directed path from the leader $\mathcal{G}_\ell$ to all followers in $\mathcal{G}_m$, otherwise either $\exists \lambda_i^\ell = 0, \forall i = 1, \ldots, m$ or $\exists \lambda_j^m = 0$ for $1 \leq j \leq m$, violating Remark 1.*

III. Distributed Adaptive Control

We consider the disturbed interconnected systems in (1) be,

$$\mathbf{W}(s) \sim \begin{cases} \dot{\bar{x}}(t) = \mathbf{A}\bar{x}(t) + \mathbf{B}\left(\bar{u}(t) + \nu^u\right), \\ \bar{y}(t) = \mathbf{k}_p \mathbf{C}\bar{x}(t) + \nu^y, \end{cases} \tag{4}$$

where $\bar{x} = [x_1^\top, \ldots, x_m^\top]^\top \in \mathbb{R}^{\bar{n}}$ with $\bar{n} = n \times m$ defines the set of the states, while $\bar{u} = [u_1, \ldots, u_m]^\top \in \mathbb{R}^m$ and $\bar{y} = [y_1, \ldots, y_m]^\top \in \mathbb{R}^m$ represent the set of inputs and outputs respectively. The matrices

$$\begin{aligned} \mathbf{A} &= \mathrm{diag}\{A_1, \ldots, A_m\}, \quad & \mathbf{B} &= \mathrm{diag}\{B_1, \ldots, B_m\}, \\ \mathbf{C} &= \mathrm{diag}\{C_1, \ldots, C_m\}, \quad & \mathbf{k}_p &= \mathrm{diag}\{k_{p_1}, \ldots, k_{p_m}\}, \end{aligned} \tag{5}$$

are diagonal blocks of $\mathcal{G}_m$ with high frequency gains $\mathbf{k}_p$. The transfer function version of $\mathcal{G}_m$ is denoted as $\mathbf{W}(s) = \mathrm{diag}\{W_1(s), \ldots, W_m(s)\}$. We design so that the persistent excitation of $\bar{\nu} = [\nu^{u\top}, \nu^{y\top}]^\top$, $\nu^\alpha = [v_1^\alpha, \ldots, v_m^\alpha]^\top$, with $\alpha = \{u, y\}$ are less than that of the reference $\bar{r}$, where $\sup(\bar{\nu}) < \bar{r}$. We also expand the leader $\mathcal{G}_\ell$ in (2) as follows,

$$\mathbf{W}_\ell(s) \sim \begin{cases} \dot{\bar{x}}_\ell(t) = \mathbf{A}_\ell \bar{x}_\ell(t) + \mathbf{B}_\ell \bar{r}(t), \\ \bar{y}_\ell(t) = \mathbf{k}_\ell \mathbf{C}_\ell \bar{x}_\ell(t), \end{cases} \tag{6}$$

where $\bar{x}_\ell = \mathbf{1}_m \otimes x_\ell$, $\bar{y}_\ell = \mathbf{1}_m \otimes y_\ell$ and $\bar{r} = \mathbf{1}_m \otimes r$ denote the set of states, outputs and references of $\mathcal{G}_\ell$ while $\mathbf{A}_\ell$, $\mathbf{B}_\ell$, and $\mathbf{C}_\ell$ are the diagonal matrices in the forms of,

$$\mathbf{A}_\ell \coloneqq I_m \otimes A_\ell, \quad \mathbf{B}_\ell \coloneqq I_m \otimes B_\ell, \quad \mathbf{C}_\ell \coloneqq I_m \otimes C_\ell, \tag{7}$$

with the similar dimensions to $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ respectively. Likewise, the transfer function of $\mathcal{G}_\ell$ is defined as $\mathbf{W}_\ell(s) = I_m \otimes W_\ell(s)$ with high frequency gains $\mathbf{k}_\ell = I_m \otimes k_\ell$.

It is obvious that if $\mathcal{G}_m$ is known, then the control input $\bar{u}$ satisfying $\lim_{t\to\infty} \bar{e} \coloneqq [e_1, \ldots, e_m]^\top = 0$ is to choose $\bar{u} = \mathbf{M}(s)(\mathbf{1}_m \otimes r)$, where $\mathbf{M}(s) = \mathbf{W}_\ell(s)\mathbf{W}^{-1}(s)$. However, for the unknown $\mathcal{G}_m$, the engineering for unknown constant $\mathbf{k}_p$, the zeros and the poles of $\mathbf{W}(s)$ is required to be solved. Here, we divide the problem into two parts:

1) the unknown $\mathbf{k}_p$ of $\mathbf{W}(s)$
2) the unknown $\mathbf{k}_p$, zeros and poles of $\mathbf{W}(s)$

Regarding the first part, we assume $\mathbf{W}(s) = \mathbf{k}_p\mathbf{W}_\alpha(s)$ and $\mathbf{W}_\ell(s) = \mathbf{k}_\ell\mathbf{W}_\alpha(s)$, where $\mathbf{W}_\alpha(s)$ is the transfer function of $\mathbf{A}_\ell, \mathbf{B}_\ell, \mathbf{C}_\ell$. The optimal estimate for the unknown $\mathbf{k}_p$ is $\mathbf{k}^* = \mathbf{k}_\ell\mathbf{k}_p^{-1}$ and with control input $\bar{u} = (\mathbf{k}^* + \tilde{\mathbf{k}})(\mathbf{1}_m \otimes r)$, then the tracking errors $\bar{e} \coloneqq \mathbb{L}_m\bar{y} - \mathbb{A}_\ell\bar{y}_\ell$ for the unknown $\mathbf{k}_p$ only of $\mathcal{G}_m$ are formulated as follows,

$$\begin{aligned}\bar{e}(t) &= \left[\mathbb{L}_m\mathbf{k}_p\mathbf{W}_\alpha(s)(\mathbf{k}^* + \tilde{\mathbf{k}}) - \mathbb{A}_\ell\mathbf{k}_\ell\mathbf{W}_\alpha(s)\right](\mathbf{1}_m \otimes r(t)) \\ &= \mathbb{L}_m\mathbf{k}_p\mathbf{W}_\alpha(s)\tilde{\mathbf{k}}(\mathbf{1}_m \otimes r(t)),\end{aligned} \tag{8}$$

and due to $\mathbb{D} = I_m =: w_i, \forall i$, then $(\mathbb{L}_m - \mathbb{A}_\ell)\mathbf{1}_m = 0$. As for the second case, we need the Meyer-Kalman-Yakubovic lemma to opt the adaptive laws and guarantee the stability.

Lemma 1. *Consider the networked system in* (4) *where the pairs* $(\mathbf{A}, \mathbf{B})$ *and* $(\mathbf{A}, \mathbf{C})$ *are stabilizable and detectable, assuming the strictly positive realness of the transfer function* $\mathbf{W}_\beta(s) \triangleq \mathbf{C}(sI_{\bar{n}} - \mathbf{A})^{-1}\mathbf{B}$. *Moreover, let the controller be* $\bar{u} \coloneqq \Theta^\top(t)\bar{\eta}(t)$ *where* $\eta_i : \mathbb{R}^+ \to \mathbb{R}^p$, $\bar{\eta} = [\eta_1^\top, \ldots, \eta_m^\top]^\top$, *and* $\bar{y} : \mathbb{R}^+ \to \mathbb{R}^m$ *be the measured time-varying functions while* $\Theta \in \mathbb{R}^{\bar{p}\times m}$ *with* $\bar{p} = p \times m$ *be the adaptive term of,*

$$\dot{\Theta}^\top(t) = -\operatorname{sign}(\mathbf{k}_p)\bar{y}(t)\bar{\eta}^\top(t), \tag{9}$$

then the equilibrium $(\bar{x}, \Theta) \coloneqq 0$ *is uniformly stable in large.*

Proof. Since $\mathbf{W}_\beta(s)$ is strictly positive real (SPR), then $\exists Q = Q^\top \succ 0, P = P^\top \succ 0$ such that,

$$\mathbf{A}^\top P + P\mathbf{A} = -Q, \quad P\mathbf{B} = \mathbf{C}^\top, \tag{10}$$

and choosing the positive definite Lyapunov function of $V(\bar{x}, \Theta) > 0$ leads to the negative semi-definite function of its time derivative $\dot{V}(\bar{x}, \Theta) \leq 0$, as written in the following,

$$\begin{aligned}V &= \bar{x}^\top(t)P\bar{x}(t) + \operatorname{tr}\left[\Theta(t)|\mathbf{k}_p^{-1}|\Theta^\top(t)\right] \\ \dot{V} &= \bar{x}^\top(t)\left[\mathbf{A}^\top P + P\mathbf{A}\right]\bar{x}(t) + 2\bar{x}^\top(t)P\mathbf{B}\Theta^\top(t)\bar{\eta}(t) \\ &\quad - 2\operatorname{tr}\left[\bar{\eta}(t)\bar{y}^\top(t)|\mathbf{k}_p^{-1}|\Theta^\top(t)\right] =: -\bar{x}^\top(t)Q\bar{x}(t) \leq 0,\end{aligned}$$

considering (10) and if we choose $\dot{\Theta}^\top$ as in (9). Note that, it is required to show that $\mathbf{W}_\beta(s)$ be SPR. $\square$

Now, for the unknown $\mathbf{k}_p$, zeros and poles of $\mathcal{G}_m$, let us define $(\mathbf{N}(s), \mathbf{D}(s)), (\mathbf{N}_\ell(s), \mathbf{D}_\ell(s))$ be the diagonal matrices containing the set of numerators and denominators of $\mathcal{G}_m$ and $\mathcal{G}_\ell$ in turn,

$$\begin{aligned}\mathbf{N}(s) &= \operatorname{diag}\{n_1(s), \ldots, n_m(s)\}, \quad \mathbf{N}_\ell(s) = I_m \otimes n_\ell(s), \\ \mathbf{D}(s) &= \operatorname{diag}\{d_1(s), \ldots, d_m(s)\}, \quad \mathbf{D}_\ell(s) = I_m \otimes d_\ell(s),\end{aligned}$$

where the transfer functions are $\mathbf{W}(s) = \mathbf{k}_p\mathbf{N}(s)\mathbf{D}^{-1}(s)$ and $\mathbf{W}_\ell(s) = \mathbf{k}_\ell\mathbf{N}_\ell(s)\mathbf{D}_\ell^{-1}(s)$. The feed-forward and feedback mechanism to adjust the unknown $\mathbf{N}(s)$ and the unknown $\mathbf{D}(s)$ are written as follows,

$$\Psi_d^{*\top}\mathbf{H}\left[\bar{u}(t) + \nu^u\right] = \Psi_d^{*\top}\bar{z}(t) \tag{11}$$

$$\left[\Phi_d^{*\top}\mathbf{H} + T_d^*\right]\bar{y}(t) = \Phi_d^{*\top}\bar{\omega}(t) + T_d^*\bar{y}(t) \tag{12}$$

where the optimal matrices of $\Psi_d^{*\top} = \operatorname{diag}\{\psi_1^{*\top}, \ldots, \psi_m^{*\top}\}$, $\Phi_d^{*\top} = \operatorname{diag}\{\phi_1^{*\top}, \ldots, \phi_m^{*\top}\}$, and $T_d^* = \operatorname{diag}\{\tau_1^*, \ldots, \tau_m^*\}$ show the adaptive terms with $\psi_i^*, \phi_i^* \in \mathbb{R}^{n-1}$, $\tau_i^* \in \mathbb{R}, \forall i$. The known systems $\mathbf{H}(s)$ are defined as,

$$\mathbf{H}(s) = (sI_{m(n-1)} - \Lambda_{m(n-1)})^{-1}\vartheta_{m(n-1)} \tag{13}$$

in which the pair matrices of $\Lambda_{m(n-1)} = I_m \otimes \Lambda_{n-1}$ and $\vartheta_{m(n-1)} = I_m \otimes \vartheta_{n-1}$ are the stable systems where the pair $(\Lambda_{n-1}, \vartheta_{n-1})$ is of order $n-1, \forall i$. Note that, the vectors $\bar{z} \in \mathbb{R}^{\bar{q}} = [z_1^\top, \ldots, z_m^\top]^\top$ and $\bar{\omega} \in \mathbb{R}^{\bar{q}} = [\omega_1^\top, \ldots, \omega_m^\top]^\top$ with $\bar{q} = (n-1) \times m$ can be also represented as,

$$\dot{\bar{z}}(t) = \Lambda_{m(n-1)}\bar{z}(t) + \vartheta_{m(n-1)}\left[\bar{u}(t) + \nu^u\right], \tag{14}$$

$$\dot{\bar{\omega}}(t) = \Lambda_{m(n-1)}\bar{\omega}(t) + \vartheta_{m(n-1)}\bar{y}(t), \tag{15}$$

therefore, the control is then defined as,

$$\bar{u}(t) = \Theta^\top(t)\bar{\eta}(t) \tag{16}$$

where $\bar{\eta} = [\eta_1^\top, \ldots, \eta_m^\top]^\top$, $\eta_i = [r_i, z_i^\top, \omega_i^\top, y_i]^\top, \forall i = 1, \ldots, m$. Now, we need to show that $\bar{e}, \bar{z}, \bar{w}$ are bounded such that by considering (4), (11)-(16) and the parameter error $\Theta = \Theta^* + \tilde{\Theta}$, the outputs of $\mathcal{G}_m$ are denoted as,

$$\begin{aligned}\dot{\bar{x}}_a(t) &= \mathbf{A}_a\bar{x}_a(t) + \mathbf{B}_a\left[\tilde{\Theta}^\top(t)\bar{\eta}(t) + \mathbf{k}^*(t)\bar{r}(t)\right], \\ \bar{y}(t) &= \mathbf{C}_a\bar{x}_a(t),\end{aligned} \tag{17}$$

where $\bar{x}_a = [\bar{x}^\top \ \bar{z}^\top \ \bar{\omega}^\top]^\top$ and,

$$\begin{aligned}\mathbf{A}_a &= \begin{bmatrix}\mathbf{A} + \mathbf{B}T_d^*\mathbf{k}^*\mathbf{C} & \mathbf{B}\Psi_d^{*\top} & \mathbf{B}\Phi_d^{*\top} \\ \vartheta T_d^*\mathbf{k}^*\mathbf{C} & \Lambda + \vartheta\Psi_d^{*\top} & \vartheta\Phi_d^{*\top} \\ \vartheta\mathbf{k}^*\mathbf{C} & 0 & \Lambda\end{bmatrix}, \\ \mathbf{B}_a &= \begin{bmatrix}\mathbf{B}^\top & \vartheta^\top & 0\end{bmatrix}^\top, \quad \mathbf{C}_a = \begin{bmatrix}\mathbf{k}^*\mathbf{C} & 0 & 0\end{bmatrix}.\end{aligned} \tag{18}$$

It also follows that the leader of $\mathcal{G}_\ell$ can be constructed using the optimal gains of $\mathbf{k}^*$, Ψ_d^*, Φ_d^*, and T_d^*, such that it is equal to $\bar{y}_\ell \coloneqq \mathbf{k}_\ell\mathbf{C}_\ell(sI_{\bar{n}} - \mathbf{A}_\ell)^{-1}\mathbf{B}_\ell$, therefore

$$\dot{\bar{x}}_a^*(t) = \mathbf{A}_a\bar{x}_a^*(t) + \mathbf{B}_a\mathbf{k}^*(t)\bar{r}(t), \ \bar{y}_\ell(t) = \mathbf{C}_a\bar{x}_a^*(t) \tag{19}$$

in which by considering the state error $\bar{e}_a = \hat{\mathbb{L}}_m\bar{x}_a - \hat{\mathbb{A}}_\ell\bar{x}_a^*$ and the output error $\bar{e} \coloneqq \mathbb{L}_m\bar{y} - \mathbb{A}_\ell(y_\ell \otimes \mathbf{1}_m)$, where $\hat{\mathbb{L}}_m = \operatorname{diag}\{(\mathbb{L}_m \otimes I_n), I_{\bar{q}}, I_{\bar{q}}\}$ and $\hat{\mathbb{A}}_\ell = \operatorname{diag}\{(\mathbb{A}_\ell \otimes I_n), I_{\bar{q}}, I_{\bar{q}}\}$,

$$\dot{\bar{e}}_a = \mathbf{A}_a\bar{e}_a(t) + \hat{\mathbb{L}}_m\mathbf{B}_a\tilde{\Theta}^\top(t)\bar{\eta}(t) \tag{20a}$$

$$\bar{e} = \mathbb{L}_m\mathbf{C}_a(sI_a - \mathbf{A}_a)^{-1}\mathbf{B}_a\left[\tilde{\Theta}^\top(t)\bar{\eta}(t) + \mathbf{k}^*(t)\bar{r}(t)\right]$$

$$- \mathbb{A}_\ell \mathbf{C}_a (sI_a - \mathbf{A}_a)^{-1} \mathbf{B}_a \mathbf{k}^*(t) \bar{r}(t) \tag{20b}$$

$$= \mathbb{L}_m \mathbf{C}_a (sI_a - \mathbf{A}_a)^{-1} \mathbf{B}_a \tilde{\Theta}^\top(t) \bar{\eta}(t) = \mathbf{C}_a \bar{e}_a(t). \tag{20c}$$

We can generate the adaptive laws using Lemma 1 in which $\exists Q_a = Q_a^\top \succ 0, P_a = P_a^\top \succ 0$ such that, $\mathbf{A}_a^\top P_a + P_a \mathbf{A}_a = -Q_a$, and $P_a \mathbf{B}_a = \mathbf{C}_a^\top$ since $\mathbf{W}_\gamma(s) \triangleq \mathbf{C}_a(sI_a - \mathbf{A}_a)^{-1}\mathbf{B}_a$ is SPR. The stability can be guaranteed from the following Lyapunov function $V(\bar{e}_a, \tilde{\Theta})$,

$$V = \bar{e}_a^\top(t) P \bar{e}_a(t) + \operatorname{tr}\left[\tilde{\Theta}(t)\Gamma_a^{-1}\tilde{\Theta}^\top(t)\right] \tag{21a}$$

$$\begin{aligned} \dot{V} &= -\bar{e}_a^\top(t) Q_a \bar{e}_a(t) + 2\bar{e}_a^\top(t) P_a \hat{\mathbb{L}}_m \mathbf{B}_a \tilde{\Theta}^\top(t) \bar{\eta}(t) \\ &\quad - 2\operatorname{tr}\left[\bar{\eta}(t)\bar{e}^\top(t)\mathbb{L}_m \Gamma_a \Gamma_a^{-1} \tilde{\Theta}^\top(t)\right] \end{aligned} \tag{21b}$$

$$= -\bar{e}_a^\top(t) Q_a \bar{e}_a(t) \leq 0 \tag{21c}$$

if we design the adaptive laws as follows,

$$\dot{\Theta}^\top(t) = -\operatorname{sign}(\mathbf{k}_p)\Gamma_a \mathbb{L}_m^\top \bar{e}(t) \bar{\eta}^\top(t). \tag{22}$$

Remark 3. *The matrix $\mathbf{C}_a$ makes $\mathbb{L}_m$ in (20c) sufficient to capture $\bar{e}_a(t)$ containing $\hat{\mathbb{L}}_m$ in (20a). Also, the cancellation in (21b) occurs due to the fact that $\bar{e}_a^\top(t) P_a \hat{\mathbb{L}}_m \mathbf{B}_a = \bar{e}_a^\top(t) \hat{\mathbb{L}}_m \mathbf{C}_a^\top = \bar{e}^\top(t) \mathbb{L}_m$, considering $P_a \mathbf{B}_a = \mathbf{C}_a^\top$. Given $n_d = 1$ in Assumption 1, and since $\mathbf{H}(s)$ is stable and strictly proper, $\bar{\nu}$ is cancelled out and (17) is valid. This is due to a pole in the feedforward transfer function at $s = 0$, where the disturbance-term decays exponentially to zero.*

Furthermore, we provide one lemma and three theorems with some remarks so that the outputs (4) and the errors $\bar{e}$ of the balanced connected networks are bounded while (17) has a solution and guarantees the tracking of the leader $\mathcal{G}_\ell$.

Lemma 2. *Let the stable systems (13) be shown as $\mathbf{H}(s) := \mathbf{N}_\Lambda(s)\mathbf{D}_\Lambda^{-1}(s)$, where $\mathbf{N}_\Lambda^\top(s) = \operatorname{diag}\{n_{\lambda_1}^\top(s), \ldots, n_{\lambda_m}^\top(s)\}$ and $\mathbf{D}_\Lambda(s) = \operatorname{diag}\{d_{\lambda_1}(s), \ldots, d_{\lambda_m}(s)\}$. There exist the optimal gains $\mathbf{k}^*$, Ψ_d^*, Φ_d^*, and T_d^*, so that the following matching condition is achieved, where*

$$\begin{aligned} &\Psi_d^{*\top}\mathbf{N}_\Lambda \mathbf{D} + \left(\Phi_d^{*\top}\mathbf{N}_\Lambda + T_d^* \mathbf{D}_\Lambda\right)\mathbf{k}_p \mathbf{N} \\ &\qquad = \mathbf{D}_\Lambda \left[\mathbf{D} - \mathbf{k}^* \left(\mathbf{k}_\ell \mathbf{N}_\ell \mathbf{D}_\ell^{-1}\right)^{-1} \mathbf{k}_p \mathbf{N}\right]. \end{aligned} \tag{23}$$

Proof. The complete proof is given in the arXiv [20]. □

Theorem 1. *Given $\mathbf{k}^*$, Ψ_d^*, Φ_d^*, and T_d^* satisfying (23) and $\Theta \equiv \Theta^*$ in (22), then the controller $\bar{u} = \Theta^{*\top}\bar{\eta}$ ensures the boundedness of all the signals in the closed-loop form and*

$$\bar{e} := \mathbb{L}_m \bar{y}(t) - \mathbb{A}_\ell \bar{y}_\ell(t) = \mathbf{1}_m \otimes \epsilon_0 \tag{24}$$

where ϵ_0 denotes the exponentially decaying initial condition.

Proof. Considering the stable systems of both $\mathbf{D}_\Lambda$ and $\mathbf{N}$, and operating both sides of (23) with $\mathbb{L}_m \bar{y}$, then

$$\begin{aligned} \mathbb{L}_m \bar{u}(t) &= \mathbb{L}_m \Psi_d^{*\top}\mathbf{H}\bar{u}(t) + \mathbb{L}_m \Phi_d^{*\top}\mathbf{H}\bar{y}(t) + \mathbb{L}_m T_d^* \bar{y}(t) \\ &\quad + \mathbb{L}_m \mathbf{k}^* \mathbf{W}_\ell^{-1}\bar{y}(t) + (\mathbf{1}_m \otimes \epsilon_1), \end{aligned} \tag{25}$$

and let $\Theta \equiv \Theta^*$, then $\bar{u} = \Theta^{*\top}\bar{\eta}$, such that by adding the zero term $(\mathbb{L}_m - \mathbb{A}_\ell)\mathbf{k}^*\bar{r} := 0$ into (25), we have

$$\mathbf{k}^* \mathbf{W}_\ell^{-1}(s)\left[\mathbb{L}_m \bar{y}(t) - \mathbb{A}_\ell \bar{y}_\ell(t)\right] + (\mathbf{1}_m \otimes \epsilon_1) = 0. \tag{26}$$

Since $\mathbf{W}_\ell$ is stable and using (4), then $\bar{y} \in \mathcal{L}^\infty$, $\bar{u} \in \mathcal{L}^\infty$, while $\bar{z}$ and $\bar{\omega}$ are bounded. □

Let $\Gamma_a \in \mathbb{R}^{m\times m} \succ 0$, $Q_a = Q_a^\top \succ 0$, and $P_a = P_a^\top \succ 0$ such that $\mathbf{A}_a^\top P_a + P_a \mathbf{A}_a = -Q_a$, and $P_a \mathbf{B}_a = \mathbf{C}_a^\top$ then the following theorem holds.

Theorem 2. *Consider the networked system (4) of $\mathcal{G}_m$ and (6) of $\mathcal{G}_\ell$ with the Laplacian-like $\mathbb{L}_m$ and the leader weight $\mathbb{A}_\ell$ satisfying Remarks 1 and 2 along with the disturbance-term in Remark 3. The pair $(\mathbf{A}, \mathbf{B})$ is stabilizable satisfying Assumption 1 and let the controller be $\bar{u} := \Theta^\top(t)\bar{\eta}(t)$ where $\eta_i : \mathbb{R}^+ \to \mathbb{R}^p$, $\bar{\eta} = [\eta_1^\top, \ldots, \eta_m^\top]^\top$, and $\bar{y} : \mathbb{R}^+ \to \mathbb{R}^m$ be the measured time-varying functions while $\Theta \in \mathbb{R}^{\bar{p}\times m}$ with $\bar{p} = p \times m$ be the adaptive term of the form (22), then the boundedness of $\bar{e}, \bar{z}, \bar{\omega}$ in $\mathcal{G}_m$ is guaranteed, leading to the asymptotic tracking to the leader $\mathcal{G}_\ell$.*

Proof. Using (4), (14), (15), (16), and the parameter error $\Theta = \Theta^* + \tilde{\Theta}$, then (17) is obtained to describe $\mathcal{G}_m$ while $\mathcal{G}_\ell$ is defined in (19), showing the perfect matching of (4)-(5), using $\bar{u}^*$ as in (23). Considering the errors of $\bar{e}_a = \hat{\mathbb{L}}_m \bar{x}_a - \hat{\mathbb{A}}_\ell \bar{x}_a^*$ and $\bar{e} := \mathbb{L}_m \bar{y} - \mathbb{A}_\ell \bar{y}_\ell$, then (20a)-(20c) is bounded, proven by (21a)-(21c) if the adaptive law in (22) is chosen. □

IV. Distributed High-Order Tuners

We discuss two common errors in adaptive system, the tracking error $\bar{e}$ between the leader $\mathcal{G}_\ell$ and the followers $\mathcal{G}_m$ and the parameter estimation error $\tilde{\Theta} = \Theta - \Theta^*$. We propose two high-order tuners, $\Theta_1, \Theta_2 \in \mathbb{R}^{mp_i\times m}$ inspired by [18], against the gradient-based tuner in (22). The two tuners come from the Bregman Lagrangian $\mathscr{L}(\Theta_j^\top, \dot{\Theta}_j^\top, t)$ in the form of,

$$\mathscr{L}(\cdot) = e^{\bar{\alpha}_j - \bar{\gamma}_j}\left[\mathbb{D}_b(f(\Theta_j^\top), \Theta_2^\top) - e^{\bar{\beta}_j} L(\Theta_j^\top)\right] \tag{27}$$

where $f(\Theta_j^\top) = \Theta_j^\top + e^{-\bar{\alpha}_j}\dot{\Theta}_j^\top$ and the Bregman divergence is denoted as $\mathbb{D}_b(y, x) = b(y) - b(x) - \operatorname{tr}\left[(y - x)\nabla b(x)^\top\right]$ with $b(x) = 0.5\|x\|_F^2$ for all $j = 1, 2$. Moreover, $L(\cdot)$ defines the time-varying loss function from (20a), where

$$L(\cdot) = \frac{1}{2}\left(\frac{d}{dt}\bar{e}_a^\top(t) P_a \bar{e}_a(t) + \bar{e}_a^\top(t) Q_a \bar{e}_a(t)\right) \tag{28}$$

resulting the update laws for specific $\dot{\Theta}_j^\top$ as $\Gamma_\gamma \nabla_{\Theta_j} L(\Theta_j)$. Given $\Gamma_\gamma := \gamma I_m, \Gamma_\beta := \beta I_m \in \mathbb{R}^{m\times m} \succ 0$ such that $\operatorname{tr}[\Gamma_\gamma] := \gamma \times m = \gamma_m$, $\operatorname{tr}[\Gamma_\beta] := \beta \times m = \beta_m$ and the normalization $\mathcal{N} = 1 + \mu\bar{\eta}^\top\bar{\eta}$, such that by substituting $\bar{\alpha}_1 = \ln(\Gamma_\beta \mathcal{N}^{-1})$, $\bar{\alpha}_2 = 0$, $\bar{\beta}_1 = \ln(\Gamma_\gamma \Gamma_\beta^{-1} \mathcal{N}^{-1})$, $\bar{\beta}_2 = \ln(\Gamma_\gamma \Gamma_\beta \mathcal{N}^{-1})$, and

$$\bar{\gamma}_1 = \int_{t_0}^{t} \Gamma_\beta \mathcal{N}\, ds, \quad \bar{\gamma}_2 = \Gamma_\beta(t - t_0) \tag{29a}$$

then we have,

$$\mathscr{L}_1(\cdot) = e^{\bar{\gamma}_1}\left(\frac{1}{2}\Gamma_\beta^{-1}\mathcal{N}^{-1}\|\dot{\Theta}_1(t)\|_F^2 - \Gamma_\gamma L(\Theta_1(t))\right) \tag{30a}$$

$$\mathscr{L}_2(\cdot) = e^{\bar{\gamma}_2}\left(\frac{1}{2}\|\dot{\Theta}_2(t)\|_F^2 - \Gamma_\gamma \Gamma_\beta \mathcal{N}^{-1} L(\Theta_2(t))\right). \tag{30b}$$

The Lagrangian functions in (30a)-(30b) act as the basis of high-order tuners in this letter. Using a cost function $J(\Theta_j)$ as the integral of the functions for some time interval t_θ, the Euler-Lagrangian equation of $\frac{d}{dt}\nabla_{\dot{\Theta}_j}\mathcal{L}(\cdot) = \nabla_{\Theta_j}\mathcal{L}(\cdot)$ and neglecting the time derivative of the normalization $\dot{\mathcal{N}}$, the high-order tuners yield in,

$$\ddot{\Theta}_1^\top(t) + \Gamma_\beta \mathcal{N}\dot{\Theta}_1^\top(t) = -\Gamma_\gamma\Gamma_\beta\mathcal{N}\nabla_{\Theta_1}L(\Theta_1(t)) \tag{31a}$$

$$\ddot{\Theta}_2^\top(t) + \Gamma_\beta \dot{\Theta}_2^\top(t) = -\Gamma_\gamma\Gamma_\beta\mathcal{N}^{-1}\nabla_{\Theta_2}L(\Theta_2(t)) \tag{31b}$$

or similarly for (31a), we can rewrite it in the following fashion using a new variable Ξ_1,

$$\dot{\Theta}_1^\top(t) = -\Gamma_\beta\mathcal{N}\left[\Theta_1^\top(t) - \Xi_1^\top(t)\right]. \tag{32}$$

Remark 4. *The normalization $\mathcal{N}$ in* (31a)-(31b) *is required for stability proof with $\mu \geq 2(\gamma_m/\beta_m)\|\mathbf{B}_a^\top \hat{\mathbb{L}}_m P_a\|_F^2$ while Γ_β and Γ_γ show the damping and the forcing term of the methods, respectively. Also, $\exists Q_a = Q_a^\top \succeq 2I_a$ that solves $\mathbf{A}_a^\top P_a + P_a\mathbf{A}_a = -Q_a$ and $P_a\mathbf{B}_a = \mathbf{C}_a^\top$.*

Theorem 3. *Using the followers $\mathcal{G}_m$ in* (17), *the leader $\mathcal{G}_\ell$ in* (19), *the adaptive laws of $\dot{\Xi}_1 = \dot{\Theta}$ in* (22) *and* (32), *and the controller* (16) *for the given $\mathcal{N}, \mu, \Gamma_\beta, \Gamma_\gamma, Q_a$, in* Remark 4 *results in bounded solutions $\bar{e}_a \in \mathcal{L}^\infty$, $\bar{e} \in \mathcal{L}^\infty$, $\tilde{\Xi}_1 := (\Xi_1 - \Theta_1^*) \in \mathcal{L}^\infty$, $(\Theta_1 - \Xi) \in \mathcal{L}^\infty$ for arbitrary initial conditions with $\lim_{t\to\infty}\bar{e}_a = 0$. Also, if $\eta, \dot{\eta} \in \mathcal{L}^\infty$, then $\lim_{t\to\infty}\dot{\Xi}_1 = 0$, $\lim_{t\to\infty}\dot{\Theta}_1 = 0$, and $\lim_{t\to\infty}\Theta_1 - \Xi_1 = 0$.*

Proof. Let us introduce (22) into different fashion as follows,

$$\dot{\Xi}_1^\top(t) = -\operatorname{sign}(\mathbf{k}_p)\Gamma_\gamma \mathbb{L}_m^\top \bar{e}(t)\bar{\eta}^\top(t). \tag{33}$$

and by defining $\tilde{\Theta}_1 := \Theta_1 - \Theta_1^*$ and $\tilde{\Xi}_1 = \Xi_1 - \Theta_1^*$, then using tuner in (32), the Lyapunov candidate $V(\bar{e}_a, \Theta_1, \Xi_1)$ is chosen as,

$$V = \bar{e}_a^\top(t)P\bar{e}_a(t) + \operatorname{tr}\left[\tilde{\Theta}_1(t)\Gamma_\gamma^{-1}\tilde{\Theta}_1(t)^\top\right] \tag{34}$$

having the time derivative along the trajectory of (20a)-(20c),

$$\dot{V} \leq -\left(\|\bar{e}_a\| - 2\|\mathbf{B}_a^\top\hat{\mathbb{L}}_m P_a\|_F\|(\Theta_1 - \Xi_1)^\top\bar{\eta}\|_F\right)^2 - \|\bar{e}_a\|^2 - \frac{2\beta_m}{\gamma_m}\|(\Theta_1 - \Xi_1)^\top\|_F^2 \leq 0 \tag{35}$$

using Cauchy–Schwarz inequality $\|AB\|_F \leq \|A\|_F\|B\|_2$. It concludes that the boundedness in Theorem 3 holds. The complete proof is given in the arXiv version [20]. □

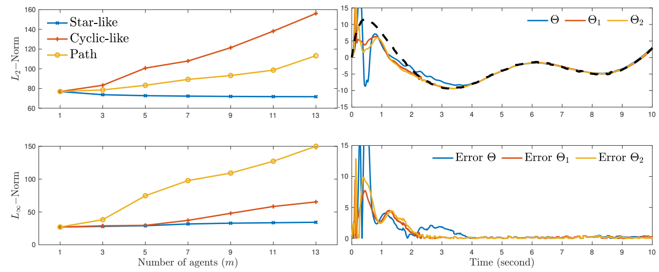
Remark 5. *A complementary proof for* (31b) *using $\dot{\Xi}_2 = -\Gamma_\gamma\mathcal{N}^{-1}\nabla L(\tilde{\Xi}_2)$, $\mathcal{N}^{-1} = N_m$ and modifying* (20a) *to accommodate $\mathcal{N}$, with $V_2(\bar{e}_a, \Theta_2, \Xi_2)$ similar to* (34), *results in bounded solutions in which the time derivative of* (34) *along the trajectory of* (20a)-(20c) *using Ξ_2, yields in $\dot{V}_2 \leq N_m\dot{V} \leq 0$. Also, given the tuners of Θ_1 and Θ_2 in* (31a) *and* (31b), *the values of Γ_β and Γ_γ should be chosen larger as systems becoming far away ($q > 2$) from the leader $\mathcal{G}_\ell$.*

V. Numerical Simulations and Findings

In this section, we simulate four different networks as shown in Fig. 1 and Fig. 3, namely: Random, Star-like, Cyclic-like, and Path, with m numbers of agents where $m \in \{1, 3, 5, 7, 9, 11, 13\}$. The leader ℓ has a transfer function $W_\ell(s)$ and meanwhile, the m followers are characterized by individual unstable transfer functions, each $W_i(s)$ defined as:

$$W_\ell(s) := \frac{3s+3}{s^2+5s+6},\ W_i(s) := \frac{s+k+4}{(s-1-k)(s-2-k)}$$

where $i = 1, 2, \ldots, m$, when $m = 1$, $k = 9i$ is used; when $m = 3$, $k = 4i - 3$ is used; when $m = 5$, $k = 2i - 1$ is used; when $m = 7$, $k = (4i-1)/3$ is used; when $m = 9$, $k = i$ is used; when $m = 11$, $k = (4i+1)/5$ is used; and when $m = 13$, $k = (2i+1)/3$ is used. We design the weights w_{ij} for the incoming measurements of node i from its neighbors j based on the level of q where agents in $q := 1$ gain more weights than those of $q = \{2, 3, \ldots\}$, with the disturbances $\nu^\alpha = [5, 0.5]^\top \otimes \mathbf{1}_m$, $\alpha = \{u, y\}$.



(a) L_2-norm and $L_\infty-$norm **(b)** $m = 9$

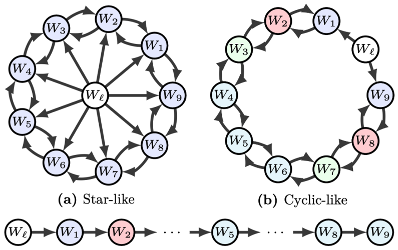
Fig. 2: (a) Performance measures of various m in Fig. 3; and (b) Random graph given by Fig. 1 with various tuners.

TABLE I: L_2-norm and $L_\infty-$norm of the random graph in Fig. 1.

	L_2-Norm			L_∞-Norm		
	Θ	Θ_1	Θ_2	Θ	Θ_1	Θ_2
Random Graph	104.76	97.72	98.26	58.45	37.61	39.79

For $m = 1$, this is the classic adaptive problem with weights of $\mathbb{L}_m = \mathbb{A}_\ell = I_m$ while for other m, we simulate three networks (Star-like, Cyclic-like, and Path) because they are comparable and the topologies are unchanged for various m. Furthermore, we compete three methods of tuners; first-order tuner Θ in (22), high-order tuners Θ_1 in (31a) and Θ_2 in (31b). One should note that, due to Remark 5, we adjust the constants of Γ_β and Γ_γ to be higher as W_i becomes worse for the three tuners. Finally, we discuss the results using L_2-norm and $L_\infty-$norm of the forms, $\|f\|_2^2 = \int_{\mathbb{T}}|f(t)|^2\,dt$ and $\|f\|_\infty = \sup_{t\in\mathbb{T}}|f(t)|$, in three different parts; **(a)** Fig. 4 for $m = 9$; **(b)** Fig. 2a for various m; and **(c)** Fig. 2b and Table. I for three tuners.

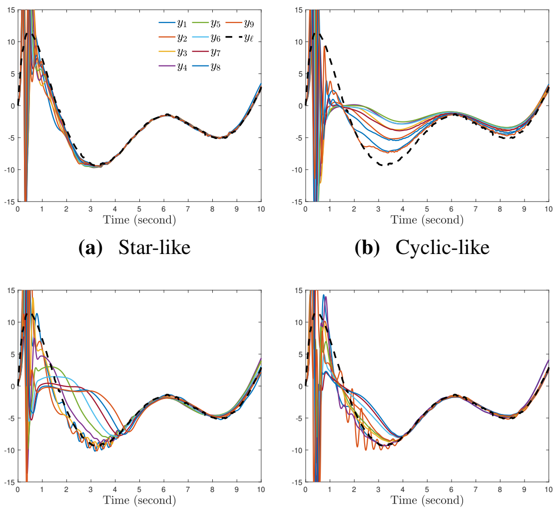
Part (**a**). Among the networks, Star is the best since the whole agents connect directly to the leader with $w_{i\ell} = 0.5, \forall i$ and $w_{ij} = 0.25, \forall j \neq \ell$ weights. Regarding random graph, the performance lies in the closeness to the leader (level of q) and the weights, with the most outer agents as the worst. It is confirmed for cyclic-graph that W_4, W_5, and W_6 are the worst and the latest to reach the consensus with the largest q. Furthermore, Path-graph has the highest $L_\infty-$norm since the consensus should wait for the preceding agents to be the



(a) Star-like (b) Cyclic-like

(c) Path

Fig. 3: Three connected (not strongly connected) balanced networks ($\mathbb{D} = I_m$) used in the simulations (in addition to the graph in Fig. 1).



(a) Star-like **(b)** Cyclic-like

(c) Path **(d)** Random

Fig. 4: Simulation results from four graphs with $m = 9$.

same as leader W_ℓ. However, the highest $L_\infty-$norm happens for some initial time, and does not guarantee the the overall performance, ensured by the better L_2-norm for various m.

Part (**b**). For $m = 1$, any networks define the classic MRAC problem, resulting the same errors. For the Star, it is interesting as m is increasing, it yields the smaller L_2-norm because the systems in between (W_1, W_9) are becoming less apart which do not happen to the other networks. Regarding Path, even though for some initial time the $L_\infty-$norm for Path is more than that of the Cyclic, the L_2-norm for various m of Path outperforms its counterpart. This is due to the fact that the measurement gained for W_i is $w_{ij} = 1$ from the starting W_ℓ while for the Cyclic, the consensus is slightly slower due to many communications from non-leader agents.

Part (**c**). The advantages of high-order tuners (HT) lies in 1) the stability with time-varying regressors and 2) an accelerated method with $\mathcal{O}(1/\sqrt{\epsilon})$ for a convex loss function, as opposed to the classic gradient descent $\mathcal{O}(1/\epsilon)$ [19]. The result shows that the modified high-order tuners, Θ_1 and Θ_2, perform better than the standard gradient-based Θ. Note that, for Parts (**a**) and (**b**), we use Θ in (22).

VI. Conclusion and Future Work

We study the distributed adaptive control with network perspective using different agents and tuners. We provide the mathematical foundation, the designs, and the comparative illustrations. The results conclude some interesting trends based on the topologies and the increasing agents. There exists a stable network in Star-like while the highest $L_\infty-$norm of Path does not reflect the overall performance, outperforming Cyclic-like with lower L_2-norm. Moreover, we also show that the modified high-order tuners outperform the gradient-based method. Finally, the future research focuses on adding the delays using the control-oriented learning [21].

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