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A Set-Theoretic Control Approach to the Trajectory Tracking Problem for Input-Output Linearized Wheeled Mobile Robots

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Abstract— **This paper proposes a set-theoretic receding horizon control scheme to address the trajectory tracking problem for input-constrained differential-drive robots. The proposed solution is derived starting from an input-output linearized description of the robot kinematics and a worst-case characterization of the orientation-dependent input constraint acting on the feedback linearized model. In particular, offline, given a worst-case characterization of the constraint set, we analytically design the smallest robust control invariant region for the tracking error. Moreover, such a region is recursively enlarged by computing a family of robust one-step controllable sets whose union characterizes the controller's domain of attraction. Online, such sets and the knowledge of the current robot's orientation are leveraged to define a non-conservative control law ensuring bounded tracking error. The effectiveness of the proposed strategy is experimentally validated using a Khepera IV robot, and its performance is contrasted with four alternative trajectory tracking algorithms.**

I. Introduction

Mobile robots are becoming increasingly popular in the manufacturing and transportation industry, and search ad rescue operations [1]. To effectively perform various autonomous and cooperative control tasks it is crucial for the robots to accurately follow a desired trajectory [2]. This is a challenging problem due to nonholonomic constraints, model inaccuracies, saturation constraints, and sensor noise. In the last decades, a wide range of trajectory tracking control schemes has been proposed in the literature, ranging from backstepping nonlinear controllers [3], to Lyapunov-based strategies [4] and Model Predictive Control (MPC) [5]. For the MPC solutions, both nonlinear and linear formulations have been investigated. The firsts rely on accurate nonlinear models for predicting the trajectory of the vehicles; consequently, they suffer from high computational complexity and local minima [6]. On the other hand, linear formulations present reduced computational burdens, but they are characterized by suboptimality due to the approximations introduced to describe the system with a linear model [7]. An interesting trade-off between these two MPC formulations is achieved when linear MPC formulations are obtained starting from a feedback-linearized representation of the nonlinear vehicle model. In particular, such an approach allows to define simple linear and convex MPC optimization problems that use exact linearization, see, e.g., [8]. Unfortunately, such an advantage vanishes if the vehicle is subject to input constraints that, under feedback linearization, translate into time-varying and state-dependent constraints causing nonconvex MPC formulations [9]. To the best of the author's knowledge, the first tracking controller capable of dealing with input-constrained feedback-linearized differential-drive robot models has been proposed in [10]. The resulting solution, although effective, does not exploit the orientation and the derivatives of the reference trajectory, which might affect the tracking performance.

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A. Paper's contributions

In this paper, we propose a novel trajectory tracking RHC for input-constrained differential-drive robots. The proposed solution adapts the Set-Theoretic Receding Horizon Control (ST-RHC) algorithm introduced in [11] to deal with the time-varying constraints acting on the input-output feedback linearized model of the robot. In particular, we characterize the linearized error dynamics as a constrained linear system subject to a bounded disturbance depending on the reference trajectory. Worst-case arguments on the disturbance and input constraint sets are leveraged to offline design a stabilizing feedback controller associated with the smallest robust control invariant region. Also, a family of robust one-step controllable sets is computed to enlarge the controller's tracking domain and allow large initial tracking errors. Online, the conservativeness of the offline solution is mitigated by exploiting the knowledge of the current robot's orientation and trajectory-dependent disturbance. The effectiveness of the resulting RHC strategy is validated by means of laboratory experiments. Although the proposed solution borrows from [10] the worst-case characterization of the input constraint set acting on the feedback linearized differential-drive robot (see Lemma 2), there are some key differences between the two strategies. The approach in [10] solves a trajectory tracking problem via a waypoint approach, while here a more challenging and general trajectory tracking problem is considered. The consequence of the above is that unlike [10], the feedback-linearized vehicle's error dynamics are now subject to an additional bounded disturbance term related to the desired reference trajectory. Moreover, differently from [10], the proposed solution is capable of exploiting a larger set of information about the reference trajectory (e.g., reference timing law and its derivatives), which results in improved tracking performance (see Table I). In [10], the waypoint tracking controller is developed by extending the LMI-based receding horizon control framework developed in [12]. On the other hand, the proposed robust controller

(Section III-C) is developed by extending the robust set-theoretic model predictive control paradigm developed in [11]. Finally, unlike [10], the proposed solution ensures that the tracking error trajectory is uniformly ultimately bounded, in a finite number of steps, in the smallest robust control invariant region.

II. Preliminaries and problem formulation

A. Preliminaries

Definition 1: Given two sets $\mathcal{A}$, $\mathcal{B}$ $\subset$ $\mathbb{R}^n$, their Minkowski/Pontryagin sum ($\oplus$) and difference ($\ominus$) are [13]:

$$\begin{aligned}\mathcal{A} \oplus \mathcal{B} &:= \{a + b : a \in \mathcal{A}, b \in \mathcal{B}\}\\ \mathcal{A} \ominus \mathcal{B} &:= \{a \in \mathbb{R}^n : a + b \in \mathcal{A}, \forall b \in \mathcal{B}\}.\end{aligned}$$

Definition 2: Given the ellipsoidal set

$$\mathcal{E} := \{z \in \mathbb{R}^n : z^T E^{-1} z \leq 1\}, E = E^T > 0, E \in \mathbb{R}^{n \times n}$$

and a matrix $M \in \mathbb{R}^{n \times n}$, then [14]

$$M\mathcal{E} := \{z \in \mathbb{R}^n : z^T (MEM^T)^{-1} z \leq 1\}$$

Property 1: Given two ball sets $\mathcal{C}_1$ and $\mathcal{C}_2$ in the form:

$$\begin{aligned}\mathcal{C}_1 &:= \{z \in \mathbb{R}^2 : z^T Q_1^{-1} z \leq 1\}, \quad Q_1 = r_{\mathcal{C}_1}^2 I\\ \mathcal{C}_2 &:= \{z \in \mathbb{R}^2 : z^T Q_2^{-1} z \leq 1\}, \quad Q_2 = r_{\mathcal{C}_2}^2 I\end{aligned} \tag{1}$$

where $r_{\mathcal{C}_1}$ and $r_{\mathcal{C}_2}$ are the radii of $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively, the Minkowski sum of $\mathcal{C}_1$ and $\mathcal{C}_2$ is defined as follows:

$$\mathcal{C}_1 \oplus \mathcal{C}_2 = \{z \in \mathbb{R}^2 : z^T Q_s^{-1} z\}, Q_s = r_s^2 I, \ r_s = r_{\mathcal{C}_1} + r_{\mathcal{C}_2}$$

while, assuming $r_{\mathcal{C}_1} > r_{\mathcal{C}_2}$ the Minkowski difference can be computed as follows:

$$\mathcal{C}_1 \ominus \mathcal{C}_2 = \{z \in \mathbb{R}^2 : z^T Q_d^{-1} z\}, Q_d = r_d^2 I, r_d = r_{\mathcal{C}_1} - r_{\mathcal{C}_2}$$

and given $M \in \mathbb{R}^{2\times 2} = mI$, $M\mathcal{C}_1 = \{z \in \mathbb{R}^2 : z^T Q_a^{-1} z \leq 1\}$, $Q_a = r_a^2 I$, $r_a = m r_{\mathcal{C}_1}$.

Definition 3: A function $f(t) : \mathbb{R} \to \mathbb{R}^n$ is said uniformly bounded and smooth if $\forall t$, $\exists \Gamma > 0 : \|f(t)\| < \Gamma$ and $f(t) \in \mathcal{C}^2$.

Consider the following discrete-time linear system:

$$z(k+1) = Az(k) + Bu(k) + d(k), \quad u(k) \in \mathcal{U}, \ d(k) \in \mathcal{D} \tag{2}$$

where $k \in \mathbb{Z} := \{0, 1, \ldots\}$, $z \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $d \in \mathbb{R}^n$ and $\mathcal{U} \subset \mathbb{R}^m$, $\mathcal{D} \subset \mathbb{R}^n$ are compact and convex sets containing the origin.

Definition 4: Consider the constrained system (2) *and a target set $\mathcal{T}^i \subset \mathbb{R}^n$. The set of states $\mathcal{T}^{i+1} \subset \mathbb{R}^n$ Robust One-Step Controllable (ROSC) to $\mathcal{T}^i$ for* (2) *is defined as:*

$$\mathcal{T}^{i+1} := \{z \in \mathbb{R}^n : \exists u \in \mathcal{U} \ s.t. \ Az + Bu + d \in \mathcal{T}^i, \forall d \in \mathcal{D}\} \tag{3}$$

Definition 5: The set $\Sigma \subset \mathbb{R}^n$ is said to be Robust Control Invariant (RCI) for (2) *if*

$$\forall z \in \Sigma, \exists u \in \mathcal{U} : Az + Bu + d \in \Sigma, \forall d \in \mathcal{D}$$

B. Set-Theoretic Receding Horizon Control Scheme

The constrained system (2) can be stabilized using the ST-RHC scheme proposed in [11]. Such a dual-mode receding horizon control strategy can be summarized as follows:

- *Offline:* First, by considering model (2) in a disturbance-free scenario (i.e., $d(k) = 0, \forall k$), design a state-feedback controller $u(k) = -Kz(k)$ such that $A - BK$ is asymptotically stable. Then, compute the smallest RCI region, namely $\mathcal{T}^0$, associated to the controlled system. Finally starting from $\mathcal{T}^0$, recursively apply Definition 4 to build a family of ROSC sets $\mathcal{T}^i$ until the set growth saturates (i.e., $\mathcal{T}^{i+1} = \mathcal{T}^i$) or the desired state-space region is covered. Store the computed family $\{\mathcal{T}^i\}_{i=1}^N$, where N is the number of computed ROSC sets.

- *Online ($\forall k$):* Determine the smallest set-membership index $i(k)$ of the ROSC set $\mathcal{T}^{i(k)}$ containing $z(k)$. Then

- **if** $i(k) = 0$, **then** apply $u(k) = -Kz(k)$
- **else** solve the following convex optimization problem:

$$\begin{aligned}u(k) &= \arg\min_u J(z(k), u) \ s.t.\\ Az(k) + Bu &\in (\mathcal{T}^{i(k)-1} \ominus \mathcal{D}), \quad u \in \mathcal{U}\end{aligned} \tag{4}$$

where $J(z(k), u)$ is a convex cost function.

Property 2: The ST-RHC controller enjoys the following properties [11]: *(i)* The optimization (4) enjoys recursive feasibility; *(ii)* The state trajectory is uniformly ultimately bounded in $\mathcal{T}^0$ in at most N steps.

C. Robot's Modeling

A slippage-free differential-drive robot can be described through the following discrete-time nonlinear model [10]:

$$\begin{aligned}x(k+1) &= x(k) + T_s \tfrac{R}{2} (\omega_R(k) + \omega_L(k)) \cos(\theta(k))\\ y(k+1) &= y(k) + T_s \tfrac{R}{2} (\omega_R(k) + \omega_L(k)) \sin(\theta(k))\\ \theta(k+1) &= \theta(k) + T_s \tfrac{R}{D} (\omega_R(k) - \omega_L(k))\end{aligned} \tag{5}$$

where, $T_s > 0$ is the sampling time of the system, $q = [x, y, \theta]^T$ the robot's pose (i.e., Cartesian position of the robot's center of mass and orientation), R is the radius of the wheels and D is the wheel's axis length, while $\omega_R, \omega_L \in \mathbb{R}$ are the control inputs, i.e., the left and right wheels' angular velocities subject to the following box-like constraints:

$$\mathcal{U}_d = \{[\omega_R, \omega_L]^T \in \mathbb{R}^2 : H_d [\omega_R, \omega_L]^T \leq 1\}, \tag{6}$$

$$H_d = \begin{bmatrix} \frac{-1}{\overline{\Omega}} & 0 & \frac{1}{\overline{\Omega}} & 0 \\ 0 & \frac{-1}{\overline{\Omega}} & 0 & \frac{1}{\overline{\Omega}} \end{bmatrix}^T \tag{7}$$

where $\overline{\Omega}$ is the maximum angular velocity of the wheels.

The differential-drive kinematics (5) can be recast into a unicycle model (see Fig. 1) by the means of the following transformation:

$$\begin{bmatrix} v(k) \\ \omega(k) \end{bmatrix} = T \begin{bmatrix} \omega_R(k) \\ \omega_L(k) \end{bmatrix}, \quad T = \begin{bmatrix} \frac{r}{2} & \frac{r}{2} \\ \frac{r}{d} & -\frac{r}{d} \end{bmatrix} \tag{8}$$

obtaining:

$$\begin{aligned}x(k+1) &= x(k) + T_s v(k) \cos(\theta(k))\\ y(k+1) &= y(k) + T_s v(k) \sin(\theta(k))\\ \theta(k+1) &= \theta(k) + T_s \omega(k)\end{aligned} \tag{9}$$

where $v, \omega \in \mathbb{R}$ are the longitudinal and angular velocities of the robot. Consequently, the input constraint set (6), mapped into the unicycle input space, transforms into a rhombus-like set, $\mathcal{U}_u \subset \mathbb{R}^2,\ 0_2 = [0,0]^T \in \mathcal{U}_u$, which defines the admissible longitudinal and angular velocities, i.e.,

$$\mathcal{U}_u = \{[v,\omega]^T \in \mathbb{R}^2 \colon H_u\,[v,\omega]^T \leq 1\},\ H_u = H_d T^{-1} \quad (10)$$

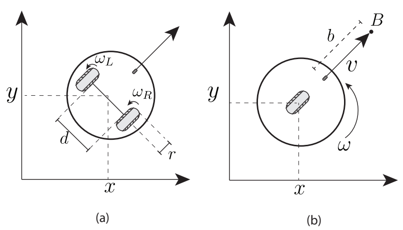

Fig. 1. *(a)* differential-drive, *(b)* unicycle.

D. Problem formulation

Consider a bounded and smooth 2D-trajectory described in terms of Cartesian position $(x_r(t), y_r(t))$, velocity $(\dot{x}_r(t), \dot{y}_r(t)$, and acceleration $(\ddot{x}_r(t), \ddot{y}_r(t))$, where $t \in \mathbb{R}^+$. Then, the robot's reference orientation $\theta_r(t)$, longitudinal velocity $v_r(t)$ and angular velocity $\omega_r(t)$ are [2]:

$$\begin{array}{rcl} \left[\begin{array}{c} v_r(t) \\ \omega_r(t) \end{array}\right] & = & \left[\begin{array}{c} \sqrt{\dot{x}_r(t)^2 + \dot{y}_r(t)^2} \\ \frac{\ddot{y}_r(t)\dot{x}_r(t) - \ddot{x}_r(t)\dot{y}_r(t)}{\dot{x}_r(t)^2 + \dot{y}_r(t)^2} \end{array}\right] \\ \theta_r(t) & = & \mathrm{ATAN}_2\left(\dot{y}_r(t), \dot{x}_r(t)\right) \end{array} \quad (11)$$

Remark 1: The forward Euler discretization of (11) *represents a solution for the discrete-time unicycle model* (9). *Moreover, a procedure to compute $\theta_r(t)$, and $\omega_r(t)$ when $\dot{x}_r(t), \dot{y}_r(t) = 0$ can be found in [2].*

Problem 1: Consider the input-constrained differential-drive robot model (5)-(7) *and a bounded and smooth trajectory $q_r(k) = [x_r(k), y_r(k), \theta_r(k)]^T$ obtained by means of a forward Euler discretization of* (11)*, with $k \in \{0, 1, \ldots, k_f\}$. Design a trajectory tracking control law $[\omega_R(k), \omega_L(k)]^T = \phi(k, q(k), q_r(k)) \in \mathcal{U}_d$ such that the tracking error $\tilde{q}(k) = q(k) - q_r(k)$ remains bounded $\forall k \geq 0$.*

III. Proposed Solution

In this section, first, the feedback-linearized tracking-error dynamics are derived, and its time-varying input constraints are discussed. Then, the ST-RHC scheme (see Section II-B) is tailored to solve the considered problem.

A. Linearized Vehicle Model via Feedback Linearization

Consider a scalar $b > 0$ and two new outputs

$$z(k) = \left[\ x(k) + b\cos\theta(k),\ y(k) + b\sin\theta(k)\ \right]^T \quad (12)$$

representing the coordinates of an external point $\mathbf{B}$ displaced at a distance b from the robot's center of mass (see Fig. 1.b). Then, the following state-feedback law

$$\left[\begin{array}{c} v(k) \\ \omega(k) \end{array}\right] = T_{FL}(\theta)\left[\begin{array}{c} u_1(k) \\ u_2(k) \end{array}\right],\ T_{FL}(\theta) = \left[\begin{array}{cc} \cos\theta & \sin\theta \\ \frac{-\sin\theta}{b} & \frac{\cos\theta}{b} \end{array}\right] \quad (13)$$

recasts the unicycle model (9) into the following two-single discrete-time integrator model [10, Property 1]:

$$z(k+1) = Az(k) + Bu(k),\ A = I_{2\times 2},\ B = T_s I_{2\times 2} \quad (14a)$$

$$\theta(k+1) = \theta(k) + T_s \frac{-\sin\theta(k)u_1(k) + \cos\theta(k)u_2(k)}{b} \quad (14b)$$

where $u(k) = [u_1(k),\ u_2(k)]^T \in \mathbb{R}^2$ are the control inputs of the feedback-linearized robot model. Note that (14b) defines a nonlinear internal dynamics decoupled from (14a).

B. Input-Ouput Linearized Error Dynamics

Given $q_r(k)$ and the transformation (12), the reference trajectory for feedback linearized robot's model is

$$z_r = \left[\ x_r + b\cos\theta_r,\ y_r + b\sin\theta_r\ \right]^T \quad (15)$$

By defining the linearized tracking error as $\tilde{z}(k) = z(k) - z_r(k)$, we have that

$$\begin{array}{rcl} \tilde{z}(k+1) & = & Az(k) + Bu(k) - Az_r(k) - Bu_r(k) \\ & = & A\tilde{z}(k) + Bu(k) + d(k) \end{array} \quad (16)$$

where $d(k) = -Bu_r(k)$ and

$$u_r(k) = T_{FL}^{-1}(\theta_r(k))\,[v_r(k), \omega_r(k)]^T \quad (17)$$

is the reference input associated to the trajectory.

Remark 2: If the reference trajectory is bounded, then $d(k)$ is a bounded disturbance with $d(k) \in \mathcal{D} \subset \mathbb{R}^2$. Moreover, if $d(k)$ is a-priori known, then $\mathcal{D}$ can be over-approximated with a ball of radius r_d, i.e,

$$\mathcal{D} = \{d \in \mathbb{R}^2 : d^T Q_d^{-1} d \leq 1\},\ Q_d = r_d^2 I_{2\times 2} \quad (18)$$

Lemma 1: [15] *If $u(\cdot)$ stabilizes* (16)*, the point $\mathbf{B}$ tracks any reference trajectory with a bounded internal dynamics. Consequently, also the tracking error $\tilde{q}(k)$ is bounded.* □

Lemma 2: [10, Section III.B] *The set of admissible inputs for* (16) *is the following time-varying and orientation-dependent polyhedral set*

$$\begin{array}{c} \mathcal{U}(\theta) = \{[u_1, u_2]^T \in \mathbb{R}^2 :\ H(\theta)\,[u_1, u_2]^T \leq 1\}, \\ H(\theta) = H_d T^{-1} T_{FL}(\theta) = \\ = \left[\begin{array}{cc} \frac{D\sin\theta - 2\cos\theta}{2\overline{\Omega}Rb} & \frac{-D\cos\theta - 2\sin\theta}{2\overline{\Omega}Rb} \\ \frac{-D\sin\theta - 2\cos\theta}{2\overline{\Omega}Rb} & \frac{D\cos\theta - 2\sin\theta}{2\overline{\Omega}Rb} \\ \frac{-D\sin\theta + 2\cos\theta}{2\overline{\Omega}Rb} & \frac{D\cos\theta + 2\sin\theta}{2\overline{\Omega}Rb} \\ \frac{D\sin\theta + 2\cos\theta}{2\overline{\Omega}Rb} & \frac{-D\cos\theta + 2\sin\theta}{2\overline{\Omega}Rb} \end{array}\right] \end{array} \quad (19)$$

which admits the following worst-case internal and circular approximation:

$$\hat{\mathcal{U}} = \bigcap_{\forall\theta} \mathcal{U}(\theta) = \{u \in \mathbb{R}^2\,|u^T Q_u^{-1} u \leq 1\},\ Q_u = r_u^2 I_{2\times 2} \quad (20)$$

where $r_u = \frac{2\overline{\Omega}Rb}{\sqrt{4b^2 + D^2}}$.

C. Proposed Receding Horizon Controller

Here, the control scheme presented in Section II-B is tailored to solve constrained trajectory tracking problem starting from the tracking error dynamics (16) and its worst-case input constraint set (20).

Assumption 1: The set $B\hat{\mathcal{U}}$ contains $\mathcal{D}$. □

Remark 3: Assumption 1 ensures that the controller has sufficient authority over the disturbance caused by the reference trajectory. It can be offline verified, and it imposes a feasibility condition for the reference trajectory.

The linearized robot error dynamics (16) are subject to the time-varying input constraint $\mathcal{U}(\theta(k))$ and bounded disturbance $d(k) = -Bu_r(k) \in \mathcal{D}$. Consequently, to perform the offline phase of the ST-MPC scheme, the only possibility is to consider the worst-case input constraint $\hat{\mathcal{U}} \subset \mathcal{U}(\theta(k))$, $\forall \theta(k)$ (see (20)) to compute the RCI region $\mathcal{T}^0$ and a family of ROSC sets $\mathcal{T}^i$ that are valid $\forall \theta(k)$. Nevertheless, online, such a source of conservativeness will be mitigated exploiting the knowledge of $\theta(k)$ to determine the actual input constraint $\mathcal{U}(\theta(k)), \forall k$. The following propositions show that for the linearized robot error dynamics, the sets $\mathcal{T}^0$ and ROSC sets $\mathcal{T}^i$ can be analytically computed.

Proposition 1: Consider the model (16) *under the constraint $u(k) \in \hat{\mathcal{U}}$ and disturbance $d(k) \in \mathcal{D}$. The terminal set $\mathcal{T}^0 = \mathcal{D}$ is the smallest RCI under the control law*

$$u(k) = -B^{-1}\tilde{z}(k) \tag{21}$$

Proof: First, under Assumption 1, $\forall \tilde{z} \in \mathcal{D}$, the control law $u = -B^{-1}\tilde{z} \in \hat{\mathcal{U}}$, or equivalently that $B^{-1}\mathcal{D} \subset \hat{\mathcal{U}}$. Indeed, by noticing that B is invertible, $B^{-1}\mathcal{D} \subset \hat{\mathcal{U}} \iff -B^{-1}Q_d(-B^{-1})^T \leq Q_u \iff \frac{r_d}{T_s} \leq r_u$. By cross multiplying the last inequality by B^{-1} on the left and $(B^{-1})^T$ on the right, we obtain $BB^{-1}Q_d(B^{-1})^T B^T \leq BQ_uB^T \iff Q_d \leq BQ_uB^T \iff r_d \leq T_s r_u \iff \mathcal{D} \subseteq B\hat{\mathcal{U}}$. Now, if $d(k) = 0, \forall k$, and $u(k) = -B^{-1}\tilde{z}(k)$ we have that $\tilde{z}(k+1) = A\tilde{z}(k) + B(-B^{-1})\tilde{z}(k) = \tilde{z}(k) - \tilde{z}(k) = 0_2$. Consequently, for any disturbance realization $d(k) \in \mathcal{D}$, it is also true that the one-step evolution is bounded by $\mathcal{D}$ and that $\mathcal{T}^0 = \mathcal{D}$ is the smallest RCI set. ■

Proposition 2: Consider the model (16) *under the constraint $u(k) \in \hat{\mathcal{U}}$ and disturbance $d(k) \in \mathcal{D}$. Given a target ball set $\mathcal{T}^{i-1} \subset \mathbb{R}^2$ of radius $r_{i-1} > 0$, the set of states ROSC to $\mathcal{T}^{i-1}$ is*

$$\mathcal{T}^i = \{\tilde{z} \in \mathbb{R}^2 : \tilde{z}^T Q_i^{-1} \tilde{z} \leq 1\}, \quad Q_i = r_i^2 I_{2\times 2} \tag{22}$$

$$r_i = r_{i-1} - r_d + T_s r_u \tag{23}$$

Proof: The set $\mathcal{T}^i$ ROSC to $\mathcal{T}^{i-1}$ for (16) can be computed as $\mathcal{T}^i = ((\mathcal{T}^{i-1} \ominus \mathcal{D}) \oplus (-B\hat{\mathcal{U}}))A$, see [13, Sec. 11.3.2]. Since $\mathcal{D}$, $\hat{\mathcal{U}}$, $\mathcal{T}^{i-1}$ are ball sets and $A = I_{2\times 2}$, $B = T_s I_{2\times 2}$, then also $\mathcal{T}^i$ is a ball of radius r_i computed as in (23) (see Property 1), concluding the proof. ■

Remark 4: Given the results of Propositions 1-2, it is possible to solve Problem 1 by implementing the ST-RHC controller detailed in Section II-B, where: (2) *is replaced by* (16)*; $K = B^{-1}$, $\mathcal{T}^0 = \mathcal{D}$ as in Proposition 1; $\mathcal{U} = \hat{\mathcal{U}}$ (i.e., the worst-case input constraint set* (20)*); $\{\mathcal{T}^i\}_{i=1}^N$ are recursively computed as in Proposition 2. the set-membership index $i(k)$ is computed as*

$$i(k) := \min\{i : \tilde{z}(k)^T Q_i^{-1} \tilde{z}(k) \leq 1\} \tag{24}$$

The solution described in Remark 4 is conservative because it uses the worst-case input constraint set $\hat{\mathcal{U}} \subset \mathcal{U}(\theta), \forall \theta$ and it assumes that $d(k) = -Bu_r(k)$ is an unknown disturbance. However, online and for any k, both $d(k)$ and $\mathcal{U}(\theta)$ can be determined starting from the reference trajectory $q_r(k)$ and robot's orientation $\theta(k)$, respectively. By taking advantage of such information, the following proposition describes a non-conservative control strategy solving Problem 1.

Algorithm 1 Tracking Set-Theoretic Receding Horizon Controller (T-ST-RHC)

Offline:

1: Set $\mathcal{U} = \hat{\mathcal{U}}$, $K = B^{-1}$, $\mathcal{T}^0 = \mathcal{D}$; Build $\{\mathcal{T}^i\}_{i=1}^N$ using (22); Store $\{\mathcal{T}^i\}_{i=0}^N$.

Online:

1: Measure $x(k)$, $y(k)$, $\theta(k)$ and compute $\tilde{z}(k) = z(k) - z_r(k)$, with $z(k)$ as in (12), $z_r(k)$ as in (15);
2: Compute $\mathcal{U}(\theta)$ as in (19) and $u_r(k)$ as in (17);
3: Find $i(k)$ as in (24);
4: **if** $i(k) > 0$, **then**

$$u(k) = \arg\min_u J(x,u) \; s.t. \tag{25a}$$

$$A\tilde{z}(k) + Bu - Bu_r(k) \in \mathcal{T}^{i(k)-1}, \; u \in \mathcal{U}(\theta) \tag{25b}$$

5: **else**

$$u(k) = -B^{-1}\tilde{z}(k) + \hat{u}_r(k), \quad \text{where} \tag{26}$$

$$\hat{u}_r(k) = \arg\min_{\hat{u}_r} \|\hat{u}_r - u_r(k)\|_2^2 \quad s.t. \tag{27a}$$

$$-B^{-1}\tilde{z}(k) + \hat{u}_r \in \mathcal{U}(\theta) \tag{27b}$$

6: **end if**
7: Compute

$$[\omega_r(k), \omega_L(k)]^T = T^{-1}T_{FL}u(k) \tag{28}$$

and apply it to the robot; $k \leftarrow k+1$, go to 1;

Theorem 1: For any $\tilde{z}(0) \in \bigcup_{i=0}^N \mathcal{T}^i$, the tracking ST-RHC strategy described in Algorithm 1 provides a solution to Problem 1.

Proof: The proof can be divided in four parts: *(I) - Opt* (25) *always admits a solution.* First, by construction, the optimization (4) is feasible for any $d \in \mathcal{D}$ (see Property 1). Consequently, (25) is admissible because the input constraint set is enlarged (i.e., $\mathcal{U}(\theta) \supset \hat{\mathcal{U}}$, $\forall \theta(k)$) and the conservative Minkowski difference is replaced by $d(k)$; *(II) - Opt* (27) *is always feasible and $\mathcal{T}^0$ is RCI under $u(k) = -B^{-1}\tilde{z}(k) + \hat{u}_r(k)$.* Indeed, $\hat{u}_r(k) = 0$ is always a feasible solution that corresponds to the terminal control law for which $\mathcal{T}^0$ is RCI for any $d(k) \in \mathcal{D}$, see Proposition 1. On the other hand, the opt. (27) selects the optimal $\hat{u}_r(k)$, compatible with the input constraint $\mathcal{U}(\theta)$, that compensates (totally or partially) for the disturbance realization $d(k) = -B\hat{u}_r(k)$. Consequently, if $\tilde{z}(k) \in \mathcal{T}^0$, then $\tilde{z}(k+j) \in \mathcal{T}^0$, $\forall j \geq 1$ and $u(k+j) \in \mathcal{U}(\theta), \forall j \geq 0$; *(III) - Feasibility and Uniformly Ultimately Boundedness (UUB).* Recursive feasibility trivially holds since, by construction, both (25) and (27) always admits a feasible solution compatible with the given constraints and worst-case disturbance realization. Consequently, starting from any admissible initial tracking error $\tilde{z}(0) \in \bigcup_{i=0}^{N} \mathcal{T}^i$, the set-membership index $i(k)$ monotonically decreases, at each k, until $i = 0$ is reached. Consequently, the tracking error of the feedback linearized model reaches the RCI set $\mathcal{T}^0$ in at most N steps where it is UUB under the effect of (26); *(IV) - Bounded tracking error.* First, $u(k)$ computed by Algorithm 1 stabilizes the feedback linearized error dynamics. Given the result of Lemma 1, the input transformation (13), the control law (28) solves the considered reference tracking problem with a bounded tracking error $\tilde{q}(k)$, concluding the proof. ■

Remark 5: As prescribed by Algorithm 1, the quadratically constrained quadratic program (25) *must be solved, at most, for the first N steps (until $\mathcal{T}^0$ is reached). Afterwards, the linearly constrained quadratic program* (27) *is solved.*

IV. Experimental Results

The proposed trajectory tracking control has been validated by means of hardware-in-the-loop laboratory experiments carried out using a Khepera IV differential-drive robot. A demo of the hereafter presented experiment can be found at the following weblink: `https://youtu.be/A0Tlbgr08tY`. The robot parameters are $R = 0.021\ [m]$, $D = 0.0884\ [m]$, the maximum velocity is set to $\overline{\Omega} = 10\ [rad/sec]$, and $T_s = 0.15\ [sec]$ The robot's pose vector has been estimated using the wheels encoder's measurements and odometry calculations as outlined in [2]. Algorithm 1 has been implemented on a Windows 10 computer equipped with an *Intel i7-8750H* processor and Matlab R2022b. The optimizations (25) and (27) have been solved using the Matlab's functions *fmincon* and *quadprog*, respectively. Moreover, a wireless TCP channel has been used for communicating with the robot, see Fig. 2. The performance of the proposed tracking algorithm has been compared with four alternative strategies: (i) the RHC strategy developed by the same authors in [10], (ii) the Lyapunov-based controller in [4], (iii) the unconstrained linear MPC solution in [5], (iv) the dynamic feedback-linearization controller in [2]. All the competitor schemes have been configured using the same parameters described in [10, Sec. IV.A]. By denoting with $e(t) = \sqrt{(x_r(t) - x(t))^2 + (y_r(t) - y(t))^2}$ the tracking error, the tracking performance has been evaluated using four different indices: (a) integral absolute error (IAE) ($\int_0^{k_f} |e(t)|dt$), (b) integral square error (ISE) ($\int_0^{k_f} e(t)^2 dt$), (c) integral time-weighted absolute error (ITAE) ($\int_0^{k_f} t|e(t)|dt$), (d) integral time squared error (ITSE) ($\int_0^{k_f} te(t)^2 dt$). The performed experiments have considered the following lemniscate trajectory (see Fig. 3)

$$\begin{bmatrix} x_r(t) \\ y_r(t) \end{bmatrix} = \begin{bmatrix} 0.6\sin(\frac{t}{3.5}) \\ 0.6\sin(\frac{t}{7}) \end{bmatrix},\ t \in [0, k_f],\ k_f = 44$$

with a robot's initial pose $q(0) = [x(0),\ y(0),\ \theta(0)]^T = [0.6,\ 0,\ \pi]^T$. It is straightforward to verify that for the given trajectory and robot constraints, the associated disturbance set (18) is a ball of radius $r_d = 0.0287$ and that $\hat{\mathcal{U}}$ is a ball of radius $r_u = 0.1982$ (see (20)). Consequently, since $B\hat{\mathcal{U}}$ is a ball of radius $T_s r_u = 0.0297$, the condition $\mathcal{D} \subset B\hat{\mathcal{U}}$ is satisfied, see Assumption 1. To cover the initial tracking error $\tilde{z}(0) = z_r(0) - z(0) = [0.4106,\ -0.0447]^T$, a family of $N = 396$ ROSC sets has been computed using (22). Moreover, the ST-RHC algorithm has been configured to use a multi-objective cost function $J(\tilde{z}, u) = \|A\tilde{z}(k) + Bu - Bu_r(k)\|_2^2 + 0.5\|u\|_2^2$, where the first term takes into account the tracking error and the second one the control effort.

The obtained experimental results are shown in Figs. 3-4 and Table I. Fig. 3 shows the reference robot's trajectories where it can be noted that the proposed T-ST-MPC strategy, similarly to [5], [10], allows the robot's trajectory to quickly converge to the reference. Indeed, from Fig. 4 it is evident that the tracking error enters the RCI region $\mathcal{T}^0$ at $k = 1.8[\mathrm{sec}]$ where it remains confined thereafter. In the same figure, it is possible to appreciate how the compensated control action (26) allows $\tilde{z}(k)$ to remain bounded in a neighbourhood of the origin that is much smaller than the worst-case region $\mathcal{T}^0$ (obtained for $\hat{u}_r(k) = 0,\ \forall\, k$). Fig. 5 shows that the computed left and right wheel angular velocities fulfill the prescribed constraints and that the robot's orientation error remains bounded. Also it can be appreciated a comparison of the algorithms in terms of the norm of the tracking error $\|e(k)\|_2$. Table I summarizes and contrasts the tracking performance of the proposed T-ST-RCH strategy with the selected alternative schemes. The obtained numerical results confirm that the tracking performance obtained by T-ST-RHC is superior to the ones obtained by the competitors. The latter can be justified as follows. The controller in [10] only used instantaneous information about the (x_r, y_r) coordinates of the reference trajectory, while the proposed solution exploits their first and second derivatives. The solutions [2], [4], [5] are developed without explicitly taking into account the robot's maximum velocity constraints. Consequently, saturation phenomena arising from the large initial tracking error degrade their performance. Finally, the average CPU times required by the proposed solution to solve (27) and (25) are $1.58\ [ms]$ and $13.02\ [ms]$. The CPU times of the competitor's control laws are $2.1\ [ms]$ for [10], $0.78\ [ms]$ for [5], $0.025\ [ms]$ for [2], and $0.032\ [ms]$ for [4].

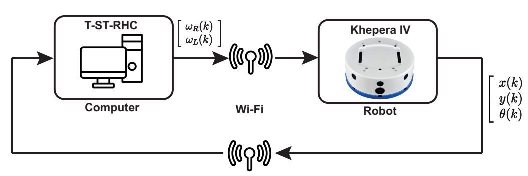

Fig. 2. Hardware-in-the-loop setup with Khepera IV.

V. Conclusions

In this paper, a novel set-theoretic receding horizon control strategy has been proposed to solve the trajectory tracking problem for input-constrained differential-drive robots. By

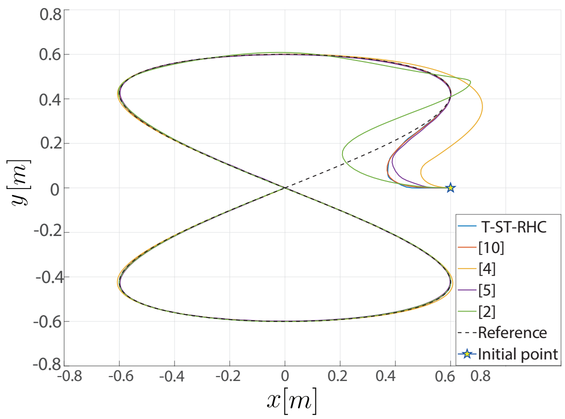

Fig. 3. Trajectories performed by the robot.

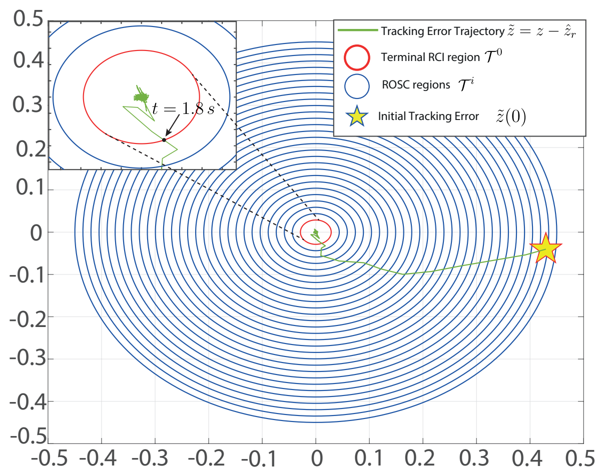

Fig. 4. Computed $\{\mathcal{T}^i\}$ and trajectory tracking error for the feedback linearized robot. For the sake of clarity, only some of the computed ROSC sets are shown.

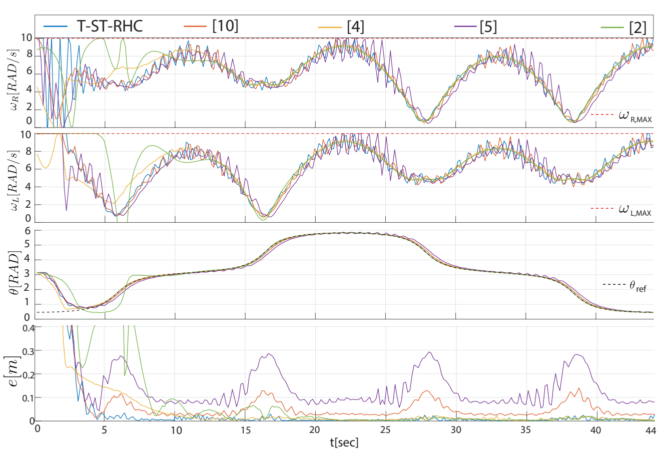

Fig. 5. Wheels angular velocities and robot's orientation.

considering an input-output linearized description of the vehicle kinematics, the strategy has been designed to take into account the associated time-varying and orientation-dependent constraints. To this end, a worst-case approximation of the constraint set has been exploited to offline design the smallest control invariant region for the tracking error and a family of robust one-step controllable sets whose union characterizes the worst-case domain of attraction of the proposed controller. Then, online, non-conservative, and constraint-admissible control inputs have been computed resorting to a receding horizon strategy exploiting the knowledge of the robot's orientation and reference trajectory. Experimental results obtained using a Khepera IV differential-drive robot and comparison with four alternative schemes have shown the superior tracking performance of the proposed solution.

TABLE I
AVERAGE TRACKING PERFORMANCE INDICES.

	IAE	ISE	ITAE	ITSE
T-ST-RHC	0.690	0.225	1.942	0.148
[10]	1.433	0.258	19.320	0.560
[4]	1.767	0.400	10.653	0.735
[5]	2.853	0.378	53.130	3.341
[2]	1.421	0.325	6.503	0.554

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