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Stabilization of a reaction-diffusion equation in H^2-norm with application to saturated Neumann measurement

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***Abstract*— This paper is concerned with the boundary output stabilization of a reaction-diffusion equation in H^2-norm. Stabilizability of reaction-diffusion PDEs is most often studied in L^2-norm and sometimes in H^1-norm. The case of the H^2-norm is much less reported in the literature. In this paper, the study of the system trajectory in H^2-norm is motivated by the fact that such a regularity is required, from a mathematical perspective, to handle a saturated Neumann measurement. More precisely, combining a classical sector condition for saturation functions and spectral methods, we show how the study of the system in H^2-norm allows the local exponential stabilization of the PDE plant while estimating a subset of the domain of attraction.**

I. Introduction

Actuator and sensor saturation mechanisms are known for introducing severe constraints on control design procedures [2], [9] as they introduce harmful nonlinear phenomena with multiple equilibrium points and bounded domains of attraction [3]. Due to its practical importance, this topic has been intensively studied for finite-dimensional systems [1], [20], [23]. One of the most efficient approaches for the local stabilization and estimation of the domain of attraction consists of the combined use of Lyapunov's direct method and a suitable generalized sector condition [20, Lem. 1.6].

The extension of the topic of saturated control to systems described by partial differential equations (PDEs) has attracted much attention in the recent years. This includes the case of the wave and Korteweg-de Vries PDEs [15], [16], [18], as well as the case of reaction-diffusion PDEs [5], [6], [12], [17]. All the above works embrace the case of the saturation of the input. In contrast, the impact of the saturation of the output was studied in [13] in the case of a Dirichlet measurement by combining spectral methods [4] and a generalized sector condition [20, Lem. 1.6]. This allowed to achieve the H^1-norm local exponential boundary feedback stabilization of the plant with estimation of the domain of attraction. However, as described in the conclusion section of [13], the method reported therein does not apply to the case of a saturated Neumann measurement. This is essentially because the H^1 regularity it not sufficient, in the case of the Neumann measurement, to make a connection between the conditions of application of the generalized sector condition and the studied H^1 Lyapunov functional.

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The main contribution of this paper is to solve the problem of local exponential stabilization of a reaction-diffusion equation with saturated Neumann measurement while estimating the domain of attraction. Compared to the case of a saturated Dirichlet measurement [13], this is achieved at the expense of considering a norm with a higher regularity, namely the H^2-norm. It is worth pointing out that most of the stabilization results for reaction-diffusion PDEs reported in the literature hold in L^2-norm, sometimes in H^1-norm. The case of the H^2-norm is far rarer with very few references [21], [22]. In this context, this paper reports one of the very first contributions regarding the output feedback stabilization of a reaction-diffusion PDE in H^2-norm. This is achieved through (i) a detailed analysis of the asymptotic behavior of the eigenstructures of the underlying Sturm-Liouville operator; and (ii) the introduction of a family of "truncated versions" of the H^2 Lyapunov functional to ensure the stability of the closed-loop infinite-dimensional system.

The paper is organized as follows. The problem description is reported in Section II. The adopted control strategy is described in Section III while the related main stability result is presented in Section IV. Some numerical simulations are provided in Section V. Concluding remarks are formulated in Section VI.

Notation. The real spaces $\mathbb{R}^n$ are endowed with the Euclidean norm $\|x\| = \sqrt{x^\top x}$. The associated induced matrix norms are also denoted by $\|\cdot\|$. For any two vectors x and y of arbitrary dimensions, we define $\mathrm{col}(x,y) = [x^\top, y^\top]^\top$. The space of square integrable functions on $(0,1)$ is denoted by $L^2(0,1)$ with the usual inner product $\langle f,g \rangle = \int_0^1 f(x)g(x)\,\mathrm{d}x$ and associated norm denoted by $\|\cdot\|_{L^2}$. For any given integer $m \geq 1$, the Sobolev space of order m is denoted by $H^m(0,1)$ with the usual norm defined by $\|f\|_{H^m}^2 = \sum_{k=0}^m \|f^{(k)}\|_{L^2}^2$. For a symmetric matrix $P \in \mathbb{R}^{n\times n}$, $P \succeq 0$ (resp. $P \succ 0$) means that P is positive semi-definite (resp. positive definite). Considering $(\phi_n)_{n\geq 1}$, an arbitrarily given Hilbert basis of $L^2(0,1)$, we define for any two integers $1 \leq N < M$ and any $f \in L^2(0,1)$ the following operators of projection: $\pi_N f = \left[\langle f,\phi_1\rangle \quad \dots \quad \langle f,\phi_N\rangle\right]^\top$, $\pi_{N,M} f = \left[\langle f,\phi_{N+1}\rangle \quad \dots \quad \langle f,\phi_M\rangle\right]^\top$, and $\mathscr{R}_N f = f - \sum_{n=1}^N \langle f,\phi_n\rangle \phi_n = \sum_{n\geq N+1} \langle f,\phi_n\rangle \phi_n$.

II. Problem description and spectral reduction

A. Problem description

We consider in this work the system described by

$$z_t(t,x) = p z_{xx}(t,x) + q z(t,x) \tag{1a}$$

$$\cos(\theta)z(t,0) - \sin(\theta)z_x(t,0) = 0 \quad (1b)$$
$$z_x(t,1) = u(t) \quad (1c)$$
$$z(0,x) = z_0(x), \quad (1d)$$

for $t > 0$ and $x \in (0,1)$. Here we have $\theta \in (0,\pi/2)$, $p > 0$, and $q \in \mathbb{R}$. The control is $u(t) \in \mathbb{R}$. The state is $z(t,\cdot) \in L^2(0,1)$. The initial condition is $z_0 \in H^2(0,1)$ with $\cos(\theta)z_0(0) - \sin(\theta)z_0'(0) = 0$ and $z_0'(1) = u(0)$. The system output considered in this work is a Neumann trace, defined by

$$y(t) = z_x(t,\xi_p). \quad (2)$$

for some measurement location $\xi_p \in [0,1)$. However, the actually measured output available for feedback control is a saturated version of the Neumann trace, described by

$$y_{\text{sat}_l}(t) = \text{sat}_l(y(t)) = \text{sat}_l(z_x(t,\xi_p)). \quad (3)$$

Here $l > 0$ stands for the saturation level while the saturation function $\text{sat}_l : \mathbb{R} \to \mathbb{R}$ is defined as

$$\text{sat}_l(y) = \begin{cases} y & \text{if } |y| \le l; \\ l\dfrac{y}{|y|} & \text{if } |y| \ge l. \end{cases}$$

The control objective is to design an output feedback controller that achieves the local exponential stabilization in H^2-norm of (1) with saturated Neumann measurement (3) while estimating a subset of the domain of attraction. As we shall see, the capability to study the systems trajectories in H^2-norm will be crucial in the context of the saturated Neumann measurement (3).

Remark 2.1: The method developed in this paper also applies to the case of an in-domain control described by

$$z_t(t,x) = pz_{xx}(t,x) + qz(t,x) + b(x)u(t) \quad (4a)$$
$$\cos(\theta_1)z(t,0) - \sin(\theta_1)z_x(t,0) = 0 \quad (4b)$$
$$\cos(\theta_2)z(t,1) + \sin(\theta_2)z_x(t,1) = 0 \quad (4c)$$
$$z(0,x) = z_0(x). \quad (4d)$$

where $\theta_1 \in [0,\pi/2)$ and $\theta_2 \in [0,\pi/2]$ provided $b \in H^1(0,1)$ with $b(0) = 0$ if $\theta_1 = 0$ and $b(1) = 0$ if $\theta_2 = 0$. In that case, with the notations introduced in the next subsection, $b \in D(\mathscr{A}^{1/2})$, implying that $b_n = \langle b,\phi_n\rangle$ is such that $\lambda_n b_n^2$ is summable. This allows the application of the approach developed in this paper.

B. Operator representation and H^2-norm

1) Sturm-Liouville operator: Let us introduce the Sturm-Liouville operator $\mathscr{A}f = -pf''$ defined on $D(\mathscr{A}) = \left\{f \in H^2(0,1) : \cos(\theta)f(0) - \sin(\theta)f'(0) = 0, f'(1) = 0\right\}$. Direct computations show that the eigenvalues of $\mathscr{A}$ are $\lambda_n = pr_n^2$ where r_n is the unique solution to $\cos(r_n) = r_n\sin(r_n)\tan(\theta)$ on the interval $((n-1)\pi, n\pi)$ for any integer $n \ge 1$. The associated unit eigenvectors are given by $\phi_n(x) = M_n(r_n\tan(\theta)\cos(r_nx) + \sin(r_nx))$ with $M_n > 0$ expressed by $\frac{1}{M_n^2} = \frac{r_n^2\tan(\theta)^2}{2}\left(1 + \frac{1}{2r_n}\sin(2r_n)\right) + \frac{1}{2}\left(1 - \frac{1}{2r_n}\sin(2r_n)\right) + \frac{\tan(\theta)}{2}(1-\cos(2r_n))$. From the general theory on Sturm-Liouville operators, it is known that $(\phi_n)_{n\ge1}$ forms a Hilbert basis of $L^2(0,1)$. Moreover, from the theory on Riesz-Spectral operators, we have that $\mathscr{A}$ is equivalently represented by $\mathscr{A}f = \sum_{n\ge1}\lambda_n\langle f,\phi_n\rangle\phi_n$ with $D(\mathscr{A}) = \left\{f \in L^2(0,1) : \sum_{n\ge1}\lambda_n^2\langle f,\phi_n\rangle^2 < +\infty\right\}$. Since $\lambda_n > 0$, we introduce for any $\alpha > 0$ the fractional operator $\mathscr{A}^\alpha f = \sum_{n\ge1}\lambda_n^\alpha\langle f,\phi_n\rangle\phi_n$ on the domain $D(\mathscr{A}^\alpha) = \left\{f \in L^2(0,1) : \sum_{n\ge1}\lambda_n^{2\alpha}\langle f,\phi_n\rangle^2 < +\infty\right\}$.

2) Asymptotic behavior: In preparation of later developments, let us note that the following asymptotic behaviors hold when $n \to +\infty$. First, from $r_n \in ((n-1)\pi, n\pi)$, we get that $r_n \sim n\pi$, hence $\lambda_n \sim pn^2\pi^2$. Moreover, introducing $\varepsilon_n = r_n - (n-1)\pi \in (0,\pi)$, it can be observed that $\varepsilon_n \to 0$. Hence, using the identity $r_n = (n-1)\pi + \varepsilon_n$ into the implicit equation $\cos(r_n) = r_n\sin(r_n)\tan(\theta)$ and proceeding with a Taylor expansion, it can be inferred that $\varepsilon_n \sim \frac{\cot(\theta)}{n\pi}$. It can also be observed that $M_n \sim \frac{\sqrt{2}\cot(\theta)}{n\pi}$. Finally, we derive from the analytical expression of the eigenfunctions that $\phi_n(1) = O(1)$ and $\phi_n'(\xi_p) = O(n) = O(\sqrt{\lambda_n})$.

3) Equivalence between the graph norm and the H^2-norm: We say that two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ defined on the same vector space X are equivalent if there exist constants $C_1, C_2 > 0$ such that $C_1\|x\|_1 \le \|x\|_2 \le C_2\|x\|_1$ for all $x \in X$. Since the constants C_1, C_2 are independent of x, we simply write $\|x\|_1 \lesssim \|x\|_2 \lesssim \|x\|_1$.

Noting that $0 \in \rho(\mathscr{A})$, the resolvent set of $\mathscr{A}$, the graph norm $\|f\|_{G(\mathscr{A})} = \|\mathscr{A}f\|_{L^2} + \|f\|_{L^2}$ is equivalent to $\|\mathscr{A}f\|_{L^2}$ for $f \in D(\mathscr{A})$. So, with a slight abuse of vocabulary, we refer to $\|\mathscr{A}\cdot\|_{L^2}$ on $D(\mathscr{A})$ as the graph norm.

Our objective is to study the system trajectories of (1) in H^2-norm. A key element for achieving this relies is the following result whose proof is placed in Appendix.

Lemma 2.2: Graph norm and H^2-norm are equivalent on $D(\mathscr{A})$, that is:

$$\|f\|_{H^2}^2 \lesssim \|\mathscr{A}f\|_{L^2}^2 = \sum_{n\ge1}\lambda_n^2\langle f,\phi_n\rangle^2 \lesssim \|f\|_{H^2}^2 \quad (5)$$

for all $f \in D(\mathscr{A})$.

C. Spectral reduction

In order to carry on the spectral reduction, we define first the change of variable formula:

$$w(t,x) = z(t,x) - \frac{1}{2}x^2u(t). \quad (6)$$

In view of the system dynamics (1) and introducing $v = \dot{u}$, we can write

$$\dot{u}(t) = v(t) \quad (7a)$$
$$w_t(t,x) = pw_{xx} + qw(t,x) + a(x)u(t) + b(x)v(t) \quad (7b)$$
$$\cos(\theta)w(t,0) - \sin(\theta)w_x(t,0) = 0 \quad (7c)$$
$$w_x(t,1) = 0 \quad (7d)$$
$$w(0,x) = w_0(x) \quad (7e)$$

where $a(x) = p + \frac{q}{2}x^2$, $b(x) = -\frac{1}{2}x^2$, and $w_0(x) = z_0(x) - \frac{x^2}{2}u(0)$. Under this homogeneous representation, PDE plant (7) reduces to

$$\dot{u}(t) = v(t) \quad (8a)$$

$$w_t(t,\cdot) = \{-\mathscr{A} + q\mathrm{Id}_{L^2}\} w(t,\cdot) + au(t) + bv(t) \tag{8b}$$
$$w(0,\cdot) = w_0 \tag{8c}$$

and is under suitable form for spectral reduction. More precisely, we introduce the following coefficients of projection $z_n(t) = \langle z(t,\cdot), \phi_n \rangle$, $w_n(t) = \langle w(t,\cdot), \phi_n \rangle$, $a_n = \langle a, \phi_n \rangle$, and $b_n = \langle b, \phi_n \rangle$. In particular, one has for classical solutions that $z(t,\cdot) = \sum_{n\geq 1} z_n(t)\phi_n$ with convergence of the series in L^2-norm while $w(t,\cdot) = \sum_{n\geq 1} w_n(t)\phi_n$ with convergence of the series in H^2-norm. In view of (6) we have

$$w_n(t) = z_n(t) + b_n u(t), \quad n \geq 1 \tag{9}$$

while the projection of (8) into $(\phi_n)_{n\geq 1}$ gives

$$\dot{u}(t) = v(t) \tag{10a}$$
$$\dot{w}_n(t) = (-\lambda_n + q) w_n(t) + a_n u(t) + b_n v(t). \tag{10b}$$

Combining the two latter equations we obtain the following representation in original z coordinates:

$$\dot{z}_n(t) = (-\lambda_n + q) z_n(t) + \beta_n u(t) \tag{11}$$

where $\beta_n = a_n + (-\lambda_n + q) b_n$. For classical solutions, the Neumann measurement (2) is given as the series expansion:

$$y(t) = z_x(t,\xi_p) = w_x(t,\xi_p) + \xi_p u(t) = \sum_{n\geq 1} \phi_n'(\xi_p) w_n(t) + \xi_p u(t). \tag{12}$$

In preparation for future developments, we need to establish an asymptotic behavior for a_n and b_n when $n \to +\infty$. First, using the definitions of the coefficients a_n, b_n and the functions a, b, recalling that $\mathscr{A}\phi_n = -p\phi_n'' = \lambda_n \phi_n$ and $\phi_n'(1) = 0$, two successive integration by parts show that $\beta_n = a_n + (-\lambda_n + q) b_n = p\phi_n(1)$. In view of the asymptotic behaviors reported in Subsection II-B.2, we have $\phi_n(1) = O(1)$, so $\beta_n = O(1)$. Since $a \in L^2(0,1)$ we have $a_n = o(1)$ while $\lambda_n \sim pn^2\pi^2$. This implies that $b_n = \frac{\beta_n - a_n}{-\lambda_n + q} = O(1/\lambda_n) = O(1/n^2)$. Now, from the definitions of a, b we have $a(x) = p - qb(x)$ so $a_n = p\int_0^1 \phi_n(x)dx - qb_n$. A direct computation shows that $\int_0^1 \phi_n(x)dx = M_n \left(\tan(\theta)\sin(r_n) + \frac{1-\cos(r_n)}{r_n} \right) = O(1/n^2) = O(1/\lambda_n)$, where we have used that $r_n = (n-1)\pi + \varepsilon_n$ with $\varepsilon_n \sim \frac{\cot(\theta)}{n\pi}$. This implies that $a_n = O(1/n^2) = O(1/\lambda_n)$. This shows in particular that $a, b \in D(\mathscr{A}^{1/2})$.

III. CONTROL STRATEGY AND CLOSED-LOOP SYSTEM REPRESENTATION

A. Control strategy

Let $\delta > 0$ and define an integer $N_0 \geq 1$ such that $-\lambda_{N_0+1} + q < -\delta < 0$. Let $N \geq N_0 + 1$ be arbitrarily fixed for the moment. The adopted control strategy, whose architecture is inspired by the pioneer work [19] and latter reused in [7], [8], [10], [11], [14], is as follows:

$$\hat{w}_n(t) = \hat{z}_n(t) + b_n u(t) \tag{13a}$$
$$\begin{aligned} \dot{\hat{z}}_n(t) = {} & (-\lambda_n + q)\hat{z}_n(t) + \beta_n u(t) \\ & - l_n \left\{ \sum_{k=1}^{N} \hat{w}_k(t)\phi_k'(\xi_p) + \xi_p u(t) - \mathrm{sat}_l(y(t)) \right\} \\ & , \quad 1 \leq n \leq N_0 \end{aligned} \tag{13b}$$
$$\dot{\hat{z}}_n(t) = (-\lambda_n + q)\hat{z}_n(t) + \beta_n u(t), \quad N_0 + 1 \leq n \leq N \tag{13c}$$
$$u(t) = \sum_{n=1}^{N_0} k_n \hat{z}_n(t). \tag{13d}$$

Here $k_n, l_n \in \mathbb{R}$ are the feedback and observer gains, respectively. As detailed in the conclusion section of [13], due to the saturated Neumann measurement, the approach reported therein fails to assess the local stabilization of the plant with controller (13) when evaluating the trajectories in H^1-norm. We show here that this pitfall can be avoided at the expense of a higher regularity of the norm, namely the H^2-norm.

B. Reduced order model representation

The stability analysis reported in the next section relies on a suitable reduced order model for the closed-loop system formed by the plant (1) and the controller (13). As classically done for finite-dimensional systems in the presence of a saturation, this representation makes use of the deadzone nonlinearity $\Phi_l : \mathbb{R} \to \mathbb{R}$ defined, for all $y \in \mathbb{R}$, by

$$\Phi_l(y) = \mathrm{sat}_l(y) - y. \tag{14}$$

The introduction of this deadzone nonlinearity is particularly relevant for stability analysis and estimation of the domain of attraction due to the following generalized sector condition borrowed from [20, Lem. 1.6].

Lemma 3.1: Let $l > 0$ be arbitrarily fixed. For any $y, \omega \in \mathbb{R}$ such that $|y - \omega| \leq l$ we have $\Phi_l(y)(\Phi_l(y) + \omega) \leq 0$.

Let the error of observation be defined by $e_n = z_n - \hat{z}_n$ for $1 \leq n \leq N$. For $N_0 + 1 \leq n \leq N$, let us introduce the scaled error of observation $\tilde{e}_n = \lambda_n e_n$. Combining (12), (13a)-(13b), and (14), we infer that

$$\begin{aligned} \dot{\hat{z}}_n(t) = {} & (-\lambda_n + q)\hat{z}_n(t) + \beta_n u(t) + l_n \sum_{k=1}^{N_0} \phi_k'(\xi_p) e_k(t) \\ & + l_n \sum_{k=N_0+1}^{N} \frac{\phi_k'(\xi_p)}{\lambda_k} \tilde{e}_k(t) + l_n \zeta(t) + l_n \Phi_l(y(t)) \end{aligned} \tag{15}$$

for all $1 \leq n \leq N_0$ where $\zeta = \sum_{n\geq N+1} \phi_n'(\xi_p) w_n$. We now introduce $\tilde{z}_n = \hat{z}_n/\lambda_n$ and the vectors $\hat{Z}^{N_0} = \begin{bmatrix} \hat{z}_1 & \ldots & \hat{z}_{N_0} \end{bmatrix}^\top$, $E^{N_0} = \begin{bmatrix} e_1 & \ldots & e_{N_0} \end{bmatrix}^\top$, $\tilde{Z}^{N-N_0} = \begin{bmatrix} \tilde{z}_{N_0+1} & \ldots & \tilde{z}_N \end{bmatrix}^\top$, and $\tilde{E}^{N-N_0} = \begin{bmatrix} \tilde{e}_{N_0+1} & \ldots & \tilde{e}_N \end{bmatrix}^\top$. Owing to the plant dynamics (11) and the controller dynamics (13c)-(13d) and (15), we infer that

$$u = K\hat{Z}^{N_0} \tag{16a}$$
$$\begin{aligned} \dot{\hat{Z}}^{N_0} = {} & (A_0 + \mathfrak{B}_0 K)\hat{Z}^{N_0} + LC_0 E^{N_0} + L\tilde{C}_1 \tilde{E}^{N-N_0} \\ & + L\zeta + L\Phi_l(y) \end{aligned} \tag{16b}$$
$$\begin{aligned} \dot{E}^{N_0} = {} & (A_0 - LC_0)E^{N_0} - L\tilde{C}_1 \tilde{E}^{N-N_0} \\ & - L\zeta - L\Phi_l(y) \end{aligned} \tag{16c}$$
$$\dot{\tilde{Z}}^{N-N_0} = A_1 \tilde{Z}^{N-N_0} + \tilde{\mathfrak{B}}_1 K \hat{Z}^{N_0} \tag{16d}$$
$$\dot{\tilde{E}}^{N-N_0} = A_1 \tilde{E}^{N-N_0}. \tag{16e}$$

where the different matrices are defined as follows: $A_0 = \mathrm{diag}(-\lambda_1 + q, \ldots, -\lambda_{N_0} + q)$, $A_1 = \mathrm{diag}(-\lambda_{N_0+1} + q, \ldots, -\lambda_N + q)$, $\mathfrak{B}_0 = \begin{bmatrix} \beta_1 & \ldots & \beta_{N_0} \end{bmatrix}^\top$,

$\tilde{\mathfrak{B}}_1 = \begin{bmatrix} \frac{\beta_{N_0+1}}{\lambda_{N_0+1}} & \dots & \frac{\beta_N}{\lambda_N} \end{bmatrix}^\top$, $C_0 = \begin{bmatrix} \phi_1'(\xi_p) & \dots & \phi_{N_0}'(\xi_p) \end{bmatrix}$, $\tilde{C}_1 = \begin{bmatrix} \frac{\phi_{N_0+1}'(\xi_p)}{\lambda_{N_0+1}} & \dots & \frac{\phi_N'(\xi_p)}{\lambda_N} \end{bmatrix}$, $K = \begin{bmatrix} k_1 & \dots & k_{N_0} \end{bmatrix}$, and $L = \begin{bmatrix} l_1 & \dots & l_{N_0} \end{bmatrix}^\top$. Then, we have

$$\dot{X} = FX + \mathscr{L}\zeta + \mathscr{L}\Phi_l(y) \tag{17}$$

where

$$X = \mathrm{col}\left(\hat{Z}^{N_0}, E^{N_0}, \tilde{Z}^{N-N_0}, \tilde{E}^{N-N_0}\right) \tag{18}$$

and

$$F = \begin{bmatrix} A_0 + \mathfrak{B}_0 K & LC_0 & 0 & L\tilde{C}_1 \\ 0 & A_0 - LC_0 & 0 & -L\tilde{C}_1 \\ \tilde{\mathfrak{B}}_1 K & 0 & A_1 & 0 \\ 0 & 0 & 0 & A_1 \end{bmatrix}, \mathscr{L} = \begin{bmatrix} L \\ -L \\ 0 \\ 0 \end{bmatrix}. \tag{19}$$

In addition, with the augmented vector $\tilde{X} = \mathrm{col}(X, \zeta, \Phi_l(y))$, we obtain that

$$u = K\hat{Z}^{N_0} = \tilde{K}X, \qquad v = \dot{u} = E\tilde{X} \tag{20}$$

where $\tilde{K} = \begin{bmatrix} K & 0 & 0 & 0 \end{bmatrix}$ and $E = K\begin{bmatrix} A_0 + \mathfrak{B}_0 K & LC_0 & 0 & L\tilde{C}_1 & L & L \end{bmatrix}$

Remark 3.2: By Cauchy uniqueness, we note that $\beta_n = p\phi_n(1) \neq 0$ for all integers $n \geq 1$. Hence, since the eigenvalues λ_n are simple, we deduce that the pair $(A_0, \mathfrak{B}_0)$ is controllable. In a similar fashion, the pair (A_0, C_0) is observable if and only if $\phi_n'(\xi_p) \neq 0$, that is $\cos(r_n\xi_p) \neq r_n \sin(r_n\xi_p)\tan(\theta)$, for all integers $1 \leq n \leq N_0$. This latter condition is, of example, fulfilled in the case $\xi_p = 0$.

The final step before presenting the main result of this paper is to write the system output y expressed by (12) into a suitable form that allows the application of the sector condition reported in Lemma 3.1. Recalling that $e_n = z_n - \hat{z}_n$, $\tilde{e}_n = \lambda_n e_n$, and $\tilde{z}_n = \hat{z}_n/\lambda_n$, we obtain the followings:

$$\begin{aligned} y &\overset{(12)}{=} \sum_{n\geq 1} \phi_n'(\xi_p) w_n + \xi_p u = \sum_{n=1}^{N} \phi_n'(\xi_p) w_n + \zeta + \xi_p u \\ &\overset{(9)}{=} \sum_{n=1}^{N} \phi_n'(\xi_p) z_n + \left(\xi_p + \sum_{n=1}^{N} \phi_n'(\xi_p) b_n\right) u + \zeta \\ &= \sum_{n=1}^{N} \phi_n'(\xi_p)\hat{z}_n + \sum_{n=1}^{N} \phi_n'(\xi_p) e_n + \left(\xi_p + \sum_{n=1}^{N} \phi_n'(\xi_p) b_n\right) u + \zeta \\ &= \sum_{n=1}^{N_0} \phi_n'(\xi_p)\hat{z}_n + \sum_{n=1}^{N_0} \phi_n'(\xi_p) e_n + \sum_{n=N_0+1}^{N} \lambda_n \phi_n'(\xi_p)\tilde{z}_n \\ &\quad + \sum_{n=N_0+1}^{N} \frac{\phi_n'(\xi_p)}{\lambda_n}\tilde{e}_n + \left(\xi_p + \sum_{n=1}^{N} \phi_n'(\xi_p) b_n\right) u + \zeta \\ &= H_1 X + H_2 u + \zeta \overset{(20)}{=} HX + \zeta \end{aligned} \tag{21}$$

where $H_1 = \begin{bmatrix} C_0 & C_0 & \lambda_{N_0+1}\phi_{N_0+1}'(\xi_p) & \dots & \lambda_N \phi_N'(\xi_p) & \tilde{C}_1 \end{bmatrix}$, $H_2 = \xi_p + \sum_{n=1}^{N} \phi_n'(\xi_p) b_n$, and $H = H_1 + H_2\tilde{K}$.

IV. Main stability result

Theorem 4.1: Let $\theta \in (0, \pi/2)$, $p > 0$, $q \in \mathbb{R}$, $\xi_p \in [0,1)$, and $l > 0$. Let $\delta > 0$ and $N_0 \geq 1$ be such that $-\lambda_n + q < -\delta$ for all $n \geq N_0 + 1$. Assume that $\phi_n'(\xi_p) \neq 0$, that is $\cos(r_n\xi_p) \neq r_n \sin(r_n\xi_p)\tan(\theta)$, for all integers $1 \leq n \leq N_0$. Let $K \in \mathbb{R}^{1\times N_0}$ and $L \in \mathbb{R}^{N_0}$ be such that $A_0 + \mathfrak{B}_0 K$ and $A_0 - LC_0$ are Hurwitz with eigenvalues that have a real part strictly less than $-\delta < 0$. For a fixed integer $N \geq N_0 + 1$, assume that there exist a symmetric positive definite $P \in \mathbb{R}^{2N\times 2N}$, positive real numbers $\alpha > 1$ and $\beta, \gamma, \mu, T, \kappa > 0$, a matrix $C \in \mathbb{R}^{1\times 2N}$, and a real number $d \in \mathbb{R}$ such that

$$\Theta_1(\kappa) \preceq 0, \quad \Theta_2 \succeq 0, \quad \Theta_3(\kappa) \leq 0 \tag{22}$$

where $\Theta_{1,1}(\kappa) = F^\top P + PF + 2\kappa P + \alpha\gamma \|\mathscr{R}_N \mathscr{A}^{1/2} a\|_{L^2}^2 \tilde{K}^\top \tilde{K}$,

$$\Theta_1(\kappa) = \begin{bmatrix} \Theta_{1,1}(\kappa) & P\mathscr{L} & -TC^\top + P\mathscr{L} \\ \mathscr{L}^\top P & -\beta & -dT \\ -TC + \mathscr{L}^\top P & -dT & -2T \end{bmatrix} + \alpha\gamma \|\mathscr{R}_N \mathscr{A}^{1/2} b\|_{L^2}^2 E^\top E$$

$$\Theta_2 = \begin{bmatrix} P & 0 & (H-C)^\top \\ 0 & \frac{\gamma}{M_\phi} & 1-d \\ H-C & 1-d & \mu l^2 \end{bmatrix}$$

$$\Theta_3(\kappa) = \gamma\left\{-\left(1 - \frac{1}{\alpha}\right)\lambda_{N+1} + q + \kappa\right\} + \beta M_\phi.$$

with $M_\phi = \sum_{n\geq N+1} \frac{\phi_n'(\xi_p)^2}{\lambda_n^2} < +\infty$. Consider the block representation $P = (P_{i,j})_{1\leq i,j\leq 4}$ with dimensions that are compatible with (18) and define

$$\mathscr{E} = \left\{ w \in D(\mathscr{A}) : \begin{bmatrix} \pi_{N_0} w \\ \pi_{N_0,N}\mathscr{A} w \end{bmatrix}^\top \begin{bmatrix} P_{2,2} & P_{2,4} \\ P_{4,2} & P_{4,4} \end{bmatrix} \begin{bmatrix} \pi_{N_0} w \\ \pi_{N_0,N}\mathscr{A} w \end{bmatrix} + \gamma \|\mathscr{R}_N \mathscr{A} w\|_{L^2}^2 < \frac{1}{\mu} \right\}. \tag{23}$$

Then, the closed-loop system composed of the PDE (1), the saturated Neumann measurement (3), and the controller (13), is locally exponentially stable in H^2-norm in the sense that there exists $M \geq 1$ such that for any initial condition $z_0 \in \mathscr{E}$ and with a zero initial condition of the observer (i.e., $\hat{z}_n(0) = 0$ for all $1 \leq n \leq N$), the system trajectory satisfies

$$\|z(t,\cdot)\|_{H^2}^2 + \sum_{n=1}^{N} \hat{z}_n(t)^2 \leq Me^{-2\kappa t}\|z_0\|_{H^2}^2 \tag{24}$$

for all $t \geq 0$. Moreover, for any fixed $\kappa \in (0, \delta]$, the constraints (22) are always feasible for N selected to be large enough.

Proof: Let $N \geq N_0 + 1$, $P \succ 0$, $\alpha > 1$, $\beta, \gamma, \mu, T, \kappa > 0$, $C \in \mathbb{R}^{1\times 2N}$, and $d \in \mathbb{R}$ be fixed as in the statement of the theorem. Let the Lyapunov functional candidate defined by

$$V_\infty(X, w) = X^\top PX + \gamma \sum_{n\geq N+1} \lambda_n^2 \langle w, \phi_n \rangle^2 \tag{25}$$

for all $X \in \mathbb{R}^{2N}$ and all $w \in D(\mathscr{A})$. Here, one would like to compute the time derivative of V along (10b) and (17). This is not possible because the series $\sum \lambda_n^3 \langle w, \phi_n \rangle^2$ is not convergent, in general. To avoid this pitfall, let us introduce the following "truncated versions" of the Lyapunov functional:

$$V_M(X, w) = X^\top PX + \gamma \sum_{n=N+1}^{M} \lambda_n^2 \langle w, \phi_n \rangle^2 \tag{26}$$

for arbitrary integers $M \geq N+1$. The time derivative of V_M along (10b) and (17) reads

$$\dot{V}_M + 2\kappa V_M = X^\top (F^\top P + PF + 2\kappa P)X + 2X^\top P\mathscr{L}(\zeta + \Phi_l(y)) + 2\gamma \sum_{n=N+1}^{M} \lambda_n^2 ((-\lambda_n + q + \kappa)w_n + a_n u + b_n v) w_n.$$

Invoking now Young's inequality, we infer that $2\lambda_n^2 w_n a_n u = 2(\lambda_n^{3/2} w_n)(\lambda_n^{1/2} a_n u) \leq \frac{1}{\alpha}\lambda_n^3 w_n^2 + \alpha\lambda_n a_n^2 u^2$, hence $2\sum_{n=N+1}^{M} \lambda_n^2 w_n a_n u \leq \frac{1}{\alpha}\sum_{n=N+1}^{M} \lambda_n^3 w_n^2 + \alpha\|\mathscr{R}_N \mathscr{A}^{1/2} a\|_{L^2}^2 u^2$. Here, we recall that $\|\mathscr{R}_N \mathscr{A}^{1/2} a\|_{L^2}^2 = \sum_{n\geq N+1} \lambda_n a_n^2 < \infty$ because $a \in D(\mathscr{A}^{1/2})$ as discussed at the end of Subsection II-C. Similarly, one has $2\sum_{n=N+1}^{M} \lambda_n^2 w_n b_n v \leq \frac{1}{\alpha}\sum_{n=N+1}^{M} \lambda_n^3 w_n^2 + \alpha\|\mathscr{R}_N \mathscr{A}^{1/2} b\|_{L^2}^2 v^2$ with $\|\mathscr{R}_N \mathscr{A}^{1/2} b\|_{L^2}^2 = \sum_{n\geq N+1} \lambda_n b_n^2 < \infty$ because $b \in D(\mathscr{A}^{1/2})$. Furthermore, recalling that $\zeta = \sum_{n\geq N+1} \phi_n'(\xi_p) w_n$ and using Cauchy-Schwartz inequality, we infer that

$$\begin{aligned}
\zeta^2 &\leq 2\left(\sum_{n=N+1}^{M} \phi_n'(\xi_p) w_n\right)^2 + 2\left(\sum_{n\geq M+1} \phi_n'(\xi_p) w_n\right)^2 \\
&\leq 2\sum_{n=N+1}^{M} \frac{\phi_n'(\xi_p)^2}{\lambda_n^2} \sum_{n=N+1}^{M} \lambda_n^2 w_n^2 \\
&\quad + 2\sum_{n\geq M+1} \frac{\phi_n'(\xi_p)^2}{\lambda_n^2} \sum_{n\geq M+1} \lambda_n^2 w_n^2 \\
&\leq 2M_\phi \sum_{n=N+1}^{M} \lambda_n^2 w_n^2 + 2R_M \|\mathscr{A} w\|_{L^2}^2
\end{aligned}$$

with $M_\phi = \sum_{n\geq N+1} \frac{\phi_n'(\xi_p)^2}{\lambda_n^2} < +\infty$ and $R_M = \sum_{n\geq M+1} \frac{\phi_n'(\xi_p)^2}{\lambda_n^2} \to 0$ as $M \to +\infty$ due to the asymptotic behaviors described in Subsection II-B.2. Hence, with $\tilde{X} = \mathrm{col}(X, \zeta, \Phi_l(y))$ and invoking (20), we infer

$$\dot{V}_M + 2\kappa V_M \leq 2\sum_{n=N+1}^{M} \lambda_n^2 \Gamma_n w_n^2 + 2\beta R_M \|\mathscr{A} w\|_{L^2}^2 + \tilde{X}^\top \left(\begin{bmatrix} \Theta_{1,1}(\kappa) & P\mathscr{L} & P\mathscr{L} \\ \mathscr{L}^\top P & -\beta & 0 \\ \mathscr{L}^\top P & 0 & 0 \end{bmatrix} + \alpha\gamma\|\mathscr{R}_N \mathscr{A}^{1/2} b\|_{L^2}^2 E^\top E \right) \tilde{X}$$

where $\Gamma_n = \gamma\left\{-\left(1-\frac{1}{\alpha}\right)\lambda_n + q + \kappa\right\} + \beta M_\phi$ for all $n \geq N+1$.

We now have to invoke the sector condition from Lemma 3.1. More precisely, provided $X \in \mathbb{R}^{2N}$ and $w \in D(\mathscr{A})$ are such that $|y - (CX + d\zeta)| \leq l$, Lemma 3.1 gives $\Phi_l(y)(\Phi_l(y) + CX + d\zeta) \leq 0$. This implies that

$$\dot{V}_M + 2\kappa V_M \leq \tilde{X}^\top \Theta_1(\kappa)\tilde{X} + 2\sum_{n=N+1}^{M} \lambda_n^2 \Gamma_n w_n^2 + 2\beta R_M \|\mathscr{A} w\|_{L^2}^2$$

as soon as, in view of (21), $|(H-C)X + (1-d)\zeta| = |y - (CX + d\zeta)| \leq l$. With $\alpha > 1$ and in view of (22), we deduce that $\Gamma_n \leq \Theta_3(\kappa) \leq 0$ for all $n \geq N+1$ while $\Theta_1(\kappa) \preceq 0$. This implies that

$$\dot{V}_M + 2\kappa V_M \leq 2\beta R_M \|\mathscr{A} w\|_{L^2}^2, \quad \forall M \geq N+1 \tag{27}$$

for all $X \in \mathbb{R}^{2N}$ and $w \in D(\mathscr{A})$ so that $|(H-C)X + (1-d)\zeta| \leq l$.

We now need to establish a connection between the constraint $|(H-C)X + (1-d)\zeta| \leq l$ and the Lyapunov function candidate. To do so, consider $X \in \mathbb{R}^{2N}$ and $w \in D(\mathscr{A})$ so that $V_\infty(X, w) \leq 1/\mu$. From $\Theta_2 \succeq 0$ and the Schur complement, we have

$$\frac{1}{\mu l^2} \begin{bmatrix} H-C & 1-d \end{bmatrix}^\top \begin{bmatrix} H-C & 1-d \end{bmatrix} \preceq \begin{bmatrix} P & 0 \\ 0 & \frac{\gamma}{M_\phi} \end{bmatrix}.$$

Recall that $\zeta = \sum_{n\geq N+1} \phi_n'(\xi_p) w_n$ and $M_\phi = \sum_{n\geq N+1} \frac{\phi_n'(\xi_p)^2}{\lambda_n^2} < +\infty$. Hence, the use of Cauchy-Schwartz inequality gives $\zeta^2 \leq M_\phi \sum_{n\geq N+1} \lambda_n^2 w_n^2$. So, we deduce from the two latter inequalities that

$$\begin{aligned}
\frac{1}{\mu l^2}|(H-C)X + (1-d)\zeta|^2 &\leq \begin{bmatrix} X \\ \zeta \end{bmatrix}^\top \begin{bmatrix} P & 0 \\ 0 & \frac{\gamma}{M_\phi} \end{bmatrix} \begin{bmatrix} X \\ \zeta \end{bmatrix} \\
&= X^\top P X + \frac{\gamma}{M_\phi}\zeta^2 \leq V_\infty(X, w) \leq \frac{1}{\mu},
\end{aligned} \tag{28}$$

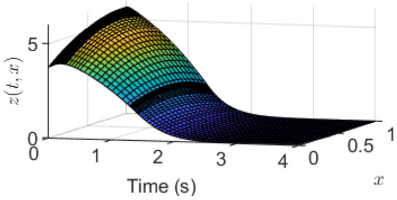
which implies that $|(H-C)X + (1-d)\zeta| \leq l$, and so (27) holds true.

Let now $w_0 \in \mathscr{E}$ and consider zero initial conditions for the observer, that is $\hat{z}_n(0) = 0$ for all $1 \leq n \leq N$. This implies that $z_0 = w_0$ while $X(0) = \mathrm{col}(0, \pi_{N_0} z_0, 0, \pi_{N_0,N} \mathscr{A} z_0)$ and $V_\infty(X(0), w_0) < 1/\mu$. Assume that there exists $t > 0$ such that $V_\infty(X(t), w(t,\cdot)) \geq 1/\mu$. Then, a continuity argument shows the existence of $t_0 > 0$ such that $V_\infty(X(t), w(t,\cdot)) < 1/\mu$ for all $t \in [0, t_0)$ while $V_\infty(X(t_0), w(t_0,\cdot)) = 1/\mu$. From (28) we deduce that (27) holds for all $t \in [0, t_0]$. An integration on that time interval gives $V_M(X(t_0), w(t_0,\cdot)) \leq e^{-2\kappa t_0} V_M(X(0), w_0) + 2\beta R_M \int_0^{t_0} e^{-2\kappa(t_0-\tau)} \|\mathscr{A} w(\tau,\cdot)\|_{L^2}^2 d\tau$. Letting $M \to +\infty$, we have $R_M \to 0$, hence $V_\infty(X(t_0), w(t_0,\cdot)) \leq e^{-2\kappa t_0} V_M(X(0), w_0) < \frac{1}{\mu}$, giving a contradiction. This shows that $V_\infty(X(t), w(t,\cdot)) < 1/\mu$ for all $t \geq 0$. From (28), (27) holds for all $t \geq 0$. Integrating the latter equation and letting $M \to +\infty$ implies that $V_\infty(X(t), w(t,\cdot)) \leq e^{-2\kappa t} V_M(X(0), w_0)$ for all $t \geq 0$. The claimed H^2-stability estimate easily follows from the definition (25) of V_∞ and Lemma 2.2.

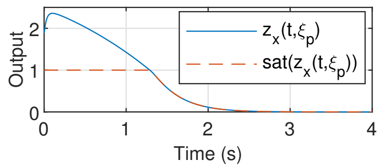
To conclude, let us show that, for any given $\kappa \in (0, \delta]$, the conditions (22) can always be satisfied for N sufficiently large. To do so, we fix $\alpha > 1$ arbitrarily and we set $C = 0$, $d = 0$, $\beta = T = N$, and $\gamma = 1/\sqrt{N}$. Now, because $\|\tilde{C}_1\| = O(1)$ and $\|\tilde{\mathfrak{B}}_1\| = O(1)$ as $N \to +\infty$, the application of the lemma in appendix of [11] to the matrix $F + \kappa I$ implies that the solution $P \succ 0$ to $F^\top P + PF + 2\kappa P = -I$ satisfies $\|P\| = O(1)$ as $N \to +\infty$. Since $\|\mathscr{L}\| = \sqrt{2}\|L\|$ and $\|\tilde{K}\| = \|K\|$ are independent of N while $\|P\| = O(1)$ and $\|E\| = O(1)$ as $N \to +\infty$, we infer from the Schur complement that for $N \geq N_0 + 1$ set large enough we have $\Theta_1(\kappa) \preceq 0$ and $\Theta_3(\kappa) \leq 0$. This fixes the order N of the observer along with the decision variables P, β, γ, T. Applying again the Schur complement, we infer that $\Theta_2 \succeq 0$ for $\mu > 0$ selected to be sufficiently large. This completes the proof. ■

V. Numerical illustration

We consider the PDE plant described by (1) with $p = q = 2$ and $\theta = \pi/4$, giving an unstable open-loop system. The



(a) State $z(t,x)$



(b) Output $z(t,\xi_p)$ and saturated output $\mathrm{sat}_l(z_x(t,\xi_p))$

Fig. 1. Closed-loop system

system output is the saturated Neumann measurement (3) at $\xi_p = 0.5$ with saturation level $l = 1$. We set the controller and observer gain as $K = -2.01$ and $L = 11.18$. With exponential decay rate $\kappa = 0.5$, the conditions of application of Theorem 4.1 are found feasible for $N = 3$.

We consider the initial condition $z_0 \in \mathscr{E}$ expressed by $z_0(x) = 3.76 + 3.76x - 1.88x^2$. The closed-loop system is simulated based on the 100 first modes of the PDE plant. The time domain behavior for the closed-loop system is depicted in Fig. 1. As predicted by Theorem 4.1, we achieve the exponential decay of the PDE state despite the impact of the saturation on the ouput, as depicted in Fig. 1(b).

VI. Conclusion

This paper has studied the local boundary feedback stabilization of a reaction-diffusion equation in the presence of a saturated Neumann measurement. This was made possible by studying the stability of the closed-loop system in H^2-norm.

Appendix

Proof of Lemma 2.2. Since $\mathscr{A}^{1/2}$ is self-adjoint, we have $\|\mathscr{A}^{1/2}f\|_{L^2}^2 = \langle \mathscr{A}^{1/2}f, \mathscr{A}^{1/2}f\rangle = \langle \mathscr{A}f, f\rangle$. So, it can be inferred on one side (using the spectral representation of $\mathscr{A}$) that $\|\mathscr{A}^{1/2}f\|_{L^2}^2 = \sum_{n\geq 1} \lambda_n \langle f, \phi_n\rangle^2$ while, on the other side (using the functional definition of $\mathscr{A}$ and an integration by parts) $\|\mathscr{A}^{1/2}f\|_{L^2}^2 = -p\int_0^1 f''(x)f(x)dx = p\cot(\theta)f(0)^2 + p\int_0^1 f'(x)^2dx$. Invoking the continuous embedding $H^1(0,1) \subset L^\infty(0,1)$, we deduce that $\|\mathscr{A}^{1/2}f\|_{L^2} \lesssim \|f\|_{H^1}$. Moreover, since $\theta \in (0,\pi/2)$ and $p > 0$, we have $\|f'\|_{L^2} \lesssim \|\mathscr{A}^{1/2}f\|_{L^2}$. We also have $\|f\|_{L^2}^2 = \sum_{n\geq 1}\langle f, \phi_n\rangle^2 \lesssim \sum_{n\geq 1}\lambda_n\langle f, \phi_n\rangle^2 = \|\mathscr{A}^{1/2}f\|_{L^2}^2$ because $0 < \lambda_1 \leq \lambda_n$ for all $n \geq 1$. The two latter inequalities show that $\|f\|_{H^1} \lesssim \|\mathscr{A}^{1/2}f\|_{L^2}$. Adopting a similar approach, we have $\|\mathscr{A}f\|_{L^2}^2 = \sum_{n\geq 1}\lambda_n^2\langle f, \phi_n\rangle^2 = p^2\|f''\|_{L^2}^2$. Using again that $0 < \lambda_1 \leq \lambda_n$ for all $n \geq 1$, we get $\|\mathscr{A}^{1/2}f\|_{L^2}^2 = \sum_{n\geq 1}\lambda_n\langle f, \phi_n\rangle^2 \lesssim \sum_{n\geq 1}\lambda_n^2\langle f, \phi_n\rangle^2 = \|\mathscr{A}f\|_{L^2}^2$. Combining all the above results, we infer that $\|f\|_{H^2}^2 = \|f\|_{H^1}^2 + \|f''\|_{L^2}^2 \lesssim \|\mathscr{A}^{1/2}f\|_{L^2}^2 + \|\mathscr{A}f\|_{L^2}^2 \lesssim \|\mathscr{A}f\|_{L^2}^2 = p^2\|f''\|_{L^2}^2 \lesssim \|f\|_{H^2}^2$.

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