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Kalman Filtering for Descriptor Systems: An Alternative Approach and Application to Cell Level Estimation of a Lithium-ion Battery

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***Abstract*— This paper presents an alternative approach for the derivation of the Kalman filter for descriptor systems. The descriptor system is assumed to be regular and of index-one. The proposed filtering algorithm is designed based on the well-known projection theorem approach. The algorithm is simple and convenient for recursive estimation and can be easily extended to complex systems with additive/multiplicative random uncertainty. The developed results are applied to estimate the state of charge and local currents of a parallel connected Lithium-ion battery pack, whose modeling naturally comes out to be a descriptor system. Simulation results demonstrate a significant accuracy of our approach under different levels of initialization error and noise.**

***Index Terms*— Descriptor system, Kalman filtering, Projection theorem, Li-ion battery pack, SOC estimation.**

I. INTRODUCTION

Descriptor systems, also known as differential-algebraic equations (DAEs) or singular systems, arise naturally in many engineering and scientific applications where there are constraints and interdependent equations that describe a system's behaviour [1]. These systems combine differential equations, which describe the system's behaviour over time, with algebraic equations, which impose constraints on the system's behaviour. The importance of descriptor systems emanates from the fact that they provide a powerful tool for modeling complex systems that are difficult or impossible to describe with ordinary differential equations (ODEs) alone. In comparison to normal state space systems, descriptor systems not only preserve the structure of the physical system but also describe static constraints and impulse behaviours [2]. These systems arise in a variety of applications, including electrical circuits, mechanical systems, chemical and biological processes, and robotics [3]–[5]. In this paper, we consider stochastic descriptor systems which are a natural extension of deterministic descriptor systems, where the system behaviour is subject to random uncertainty [6].

Kalman filter is a widely used algorithm for estimating the state of a system based on noisy measurements. The development of Kalman filters for stochastic descriptor systems is an important area of research, as it enables the estimation and control of complex systems subject to both noise and constraints. The literature for the design of Kalman filter for descriptor systems is not rich, and only a limited number of results have been presented [7]–[11]. In [7], the authors have solved the filtering problem based on the least squares method while in [8], the maximum likelihood criterion is considered. The stochastic shuffle algorithm proposed in [9] retains important statistical properties and therefore gives the solution to the descriptor Kalman filtering problem. In [10], the authors have considered the Kalman filter estimation for the descriptor system as a data fitting problem that does not need Gaussian assumption on system noises. For rectangular descriptor systems subjected to uncertainties, a robust Kalman filter is designed in [11]. For the descriptor systems with correlated noises, the optimal and steady-state filter is designed by transferring the system to an equivalent non-singular system with correlated noises in [12]. The optimal unbiased minimum-variance state estimation of descriptor systems via the unknown input filtering method is in [13]. But all these algorithms designed in [7]–[11] have complicated derivations and can not be easily applied to any system.

The projection theorem has a significant physical significance in state estimation for dynamic systems as it projects the system state onto the subspace of the measured data [14], [15]. Specifically, the projection equation provides a way to decompose the system state into a part that lies in the subspace of the measured data and a part that lies in the orthogonal complement of the subspace [16]. It has the benefits of being convenient for recursion and optimal in the sense of minimum variance. In other words, the projection theorem is a kind of orthogonal decomposition, which makes the recursion of the algorithm convenient for implementation.

Lithium-ion (Li-ion) batteries have emerged as the preferred choice for a wide range of applications, including-consumer electronics, electric vehicles (EVs), and renewableenergy storage systems [17]. The performance of these Li-ion batteries is strongly dependent on the battery's state of charge (SOC). Accurate SOC estimation is crucial for a battery management system to effectively monitor and control a Li-ion battery pack, ensuring safe and efficient operation, prolonging the battery pack's lifetime and providing better performance. But the internal states of individual cells within a Li-ion battery pack are likely to differ due to cell inconsistencies caused by manufacturing tolerances and usage conditions [18]. These inconsistencies in the individual cells of a battery pack can lead to issues such as voltage imbalance, which can cause the weaker cells to overcharge

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or over-discharge, leading to a reduction in overall battery pack performance and capacity and, in some cases, even damage or failure of the battery pack [19]. Therefore, it is essential to monitor and balance the individual cells within a battery pack to ensure optimal performance and prolong the life of the battery pack. Moreover, as the battery pack consists of hundreds of cells, monitoring each and every cell naturally puts a cost and sensing constraint. Contrary to series-connected battery packs, which are governed by ODEs only, the dynamics of parallel-connected battery packs are evidently more challenging and require the solution of descriptor systems [18], [20]. In this paper, as an application to our proposed methodology, we estimate SOC and local currents of a parallel connected battery pack realized by a double-capacitor equivalent circuit model (ECM).

In light of the aforementioned discussion, the main contribution of this paper is that we have provided an alternative way to derive the Kalman filtering algorithm for descriptor systems using the projection theorem approach. The advantage of this approach is that it is convenient for recursion and optimal in the sense of minimum mean square error (MMSE). Moreover, the estimation by the designed algorithm is unbiased and can be easily extended to complex systems having multiplicative/additive random uncertainty. To test the practical feasibility of the developed algorithm, it is applied to estimate SOC and local currents of a parallel connected Li-ion battery pack.

The organization of the remaining paper is as follows. Section II formulates the problem and decomposes the descriptor system by solving the algebraic constraint. Section III presents a new straightforward and convenient approach to designing Kalman filter for descriptor systems based on the well-known projection theorem. Section IV formulates the DAE framework for a parallel connected battery pack under minimal sensing. The effectiveness of the proposed work towards initialization errors and noises is demonstrated in Section V. Finally, Section VI concludes the paper.

II. Problem Formulation

Consider a linear stochastic discrete-time descriptor system of the form:

$$\mathcal{E}x_{k+1} = \mathcal{A}x_k + \mathcal{B}u_k + \mathcal{G}w_k, \tag{1a}$$

$$y_k = \mathcal{C}x_k + v_k, \tag{1b}$$

where $x_k \in \mathbb{R}^n$ is a state vector, $u_k \in \mathbb{R}^m$ is the input vector, $y_k \in \mathbb{R}^p$ is the output vector, $w_k \in \mathbb{R}^h$ is the process noise and $v_k \in \mathbb{R}^p$ is the measurement noise. The system matrices $\mathcal{E}, \mathcal{A} \in \mathbb{R}^{n\times n}$, $\mathcal{B} \in \mathbb{R}^{n\times m}$, $\mathcal{G} \in \mathbb{R}^{n\times h}$ and $\mathcal{C} \in \mathbb{R}^{p\times n}$ are known. Also, the matrix $\mathcal{E}$ is singular *i.e.* rank$(\mathcal{E}) = n_0 < n$.

Before proceeding to the filtering algorithm, we assume the following:

i The matrix pair $(\mathcal{E}, \mathcal{A})$ is regular and of index-one, *i.e.,*

$$\text{rank}\begin{bmatrix} \mathcal{E} & \mathcal{A} \\ 0 & \mathcal{E} \end{bmatrix} = n + \text{rank}\,\mathcal{E}.$$

ii The process noise, w_k and measurement noise, v_k are uncorrelated zero mean white Gaussian with covariance Q_k and R_k respectively, *i.e.,* $\mathbb{E}[w_k v_j^T] = 0$, $\mathbb{E}[w_k w_j^T] = Q_k\delta_{kj}$ and, $\mathbb{E}[v_k v_j^T] = R_k\delta_{kj}$, where δ_{kj} is the Kronecker delta, Q_k and R_k are the symmetric positive definite matrices.

Using singular value decomposition (SVD), we find two non-singular matrices $\mathcal{U}$ and $\mathcal{V}$ such that

$$\mathcal{U}\mathcal{E}\mathcal{V} = \begin{bmatrix} I_{n_0} & 0 \\ 0 & 0 \end{bmatrix}, \mathcal{U}\mathcal{A}\mathcal{V} = \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix},$$

$$\mathcal{U}\mathcal{B} = \begin{bmatrix} \mathcal{B}_1 \\ \mathcal{B}_2 \end{bmatrix}, \mathcal{U}\mathcal{G} = \begin{bmatrix} \mathcal{G}_1 \\ \mathcal{G}_2 \end{bmatrix} \text{ and } \mathcal{C}\mathcal{V} = \begin{bmatrix} \mathcal{C}_1 & \mathcal{C}_2 \end{bmatrix}.$$

Now, the system (1) explicitly can be written in the form

$$x_{d,k+1} = \mathcal{A}_{11}x_{d,k} + \mathcal{A}_{12}x_{a,k} + \mathcal{B}_1u_k + \mathcal{G}_1w_k, \tag{2a}$$

$$0 = \mathcal{A}_{21}x_{d,k} + \mathcal{A}_{22}x_{a,k} + \mathcal{B}_2u_k + \mathcal{G}_2w_k, \tag{2b}$$

$$y_k = \mathcal{C}_1x_{d,k} + \mathcal{C}_2x_{a,k} + v_k, \tag{2c}$$

$$\text{where} \quad x_k = \mathcal{V}\begin{bmatrix} x_{d,k} \\ x_{a,k} \end{bmatrix}. \tag{2d}$$

Since, we assume the system to be regular and of index-one, the matrix $\mathcal{A}_{22}$ is non-singular. Therefore, (2b) implies that

$$x_{a,k} = -\mathcal{A}_{22}^{-1}\left(\mathcal{A}_{21}x_{d,k} + \mathcal{B}_2u_k + \mathcal{G}_2w_k\right). \tag{3}$$

Thus, system (1) is equivalent to the following system

$$x_{d,k+1} = \mathcal{A}_0x_{d,k} + \mathcal{B}_0u_k + \mathcal{G}_0w_k, \tag{4a}$$

$$y_k = \mathcal{C}_0x_{d,k} + \mathcal{D}_0u_k + \bar{v}_k, \tag{4b}$$

where $\mathcal{A}_0 := \mathcal{A}_{11} - \mathcal{A}_{12}\mathcal{A}_{22}^{-1}\mathcal{A}_{21}$, $\mathcal{B}_0 := \mathcal{B}_1 - \mathcal{A}_{12}\mathcal{A}_{22}^{-1}\mathcal{B}_2$, $\mathcal{G}_0 := \mathcal{G}_1 - \mathcal{A}_{12}\mathcal{A}_{22}^{-1}\mathcal{G}_2$, $\mathcal{C}_0 := \mathcal{C}_1 - \mathcal{C}_2\mathcal{A}_{22}^{-1}\mathcal{A}_{21}$, $\mathcal{D}_0 := -\mathcal{C}_2\mathcal{A}_{22}^{-1}\mathcal{B}_2$, $\mathcal{D}_1 := -\mathcal{C}_2\mathcal{A}_{22}^{-1}\mathcal{G}_2$,
and, $\bar{v}_k := v_k + \mathcal{D}_1w_k$.

The system (4) is non-singular having measurement noise $\bar{v}_k$ of zero mean and correlation coefficient $\mathcal{R}_k = R_k + \mathcal{D}_1Q_k\mathcal{D}_1^T$. Moreover, it is correlated with the process noise w_k with covariance $\mathcal{T}_k = Q_k\mathcal{D}_1^T$.

III. Descriptor Kalman Filtering using Projection Theorem Approach

A. Projection Equation Development

For the system model (4) with correlated noises, we will derive the recursive Kalman filtering algorithm such that it is optimal in the sense of MMSE. The derivation of the algorithm is based on the projection theorem and the projection equation, which leads to the intuitive and simplest form of derivation. By [14], if two random variables A and B are orthogonal, the projection equation can be expressed as

$$\text{Proj}[\text{A}|\text{B}] = \mathbb{E}[\text{A}] + \text{Cov}[\text{A}, \text{B}]\,\text{Var}^{-1}[\text{B}](\text{B} - \mathbb{E}[\text{B}]). \tag{5}$$

In other words, $\text{Proj}[\text{A}|\text{B}]$ is the minimum variance estimation of A based on the measurement data space B.

Let $\hat{x}_{d,k|k}$ be the MMSE estimation of $x_{d,k}$ given the measurement data sequence $y_{0:k} \triangleq [y_0, y_1, \dots, y_k]$ upto time instant k and $\hat{x}_{d,k|k-1}$ be the prior estimation of $x_{d,k}$ given $y_{0:k-1}$, the measurement data sequence given upto time instant $k-1$. More precisely, $\hat{x}_{d,k|k}$ is the projection of $x_{d,k}$ on the subspace generated by $\{y_{0:k}\}$. In the same way, $\hat{x}_{d,k|k-1}$ is the projection of $x_{d,k}$ onto the space $\{y_{0:k-1}\}$. Thus, we define

$$\hat{x}_{d,k|k-1} := \text{Proj}[x_{d,k} \mid y_{0:k-1}], \tag{6a}$$

$$\hat{x}_{d,k|k} := \text{Proj}[x_{d,k} \mid y_{0:k}]. \tag{6b}$$

The corresponding prior and posterior error covariance, respectively, are defined as

$$P_{d,k|k-1} := \mathbb{E}[e_{d,k|k-1}e_{d,k|k-1}^T], \tag{7a}$$

$$P_{d,k|k} := \mathbb{E}[e_{d,k|k}e_{d,k|k}^T], \tag{7b}$$

where $e_{d,k|k-1} = x_{d,k} - \hat{x}_{d,k|k-1}$ is the prior error and $e_{d,k|k} = x_{d,k} - \hat{x}_{d,k|k}$ is the estimation error.
With the help of projection theorem, we can clearly say that the prior error $e_{d,k|k-1}$ and estimation error $e_{d,k|k}$ are perpendicular to the spaces $\{y_{0:k-1}\}$ and $\{y_{0:k}\}$, respectively.

B. Kalman Filtering Algorithm

This subsection will deduce the recursive filtering algorithm for stochastic descriptor systems, which is optimal in the sense of MMSE.

Theorem 1: Under the assumptions (i) and (ii) defined for the system (1), the prior estimate and the prior error covariance are as follows

$$\hat{x}_{d,k|k-1} = \mathcal{A}_0\hat{x}_{d,k-1|k-1} + \mathcal{B}_0u_{k-1}, \tag{8}$$

$$P_{d,k|k-1} = \mathcal{A}_0P_{d,k-1|k-1}\mathcal{A}_0^T + \mathcal{G}_0Q_{k-1}\mathcal{G}_0^T. \tag{9}$$

Proof: For the augmented system (4), the prior estimate from (6a) can be computed as

$$\hat{x}_{d,k|k-1} = \text{Proj}[x_{d,k} \mid y_{0:k-1}] = \mathbb{E}[x_{d,k} \mid y_{0:k-1}].$$

On putting the value of $x_{d,k}$ in the above equation, we obtain (8). Based on this, the prior error is given as

$$e_{d,k|k-1} = \mathcal{A}_0e_{d,k-1|k-1} + \mathcal{G}_0w_{k-1}. \tag{10}$$

On substituting (10) in (7a), we obtain (9). This completes the proof. ∎

Theorem 2: The expression of the posterior estimate and the posterior error covariance are

$$\hat{x}_{d,k|k} = \hat{x}_{d,k|k-1} + \mathcal{K}_k\tilde{y}_{k|k-1}, \tag{11}$$

$$\hat{x}_{a,k|k} = -\mathcal{A}_{22}^{-1}\left[\mathcal{A}_{21}\hat{x}_{d,k|k} + \mathcal{B}_2u_k + \mathcal{G}_2\hat{w}_{k|k}\right], \tag{12}$$

$$P_{d,k|k} = [I - \mathcal{K}_k\mathcal{C}_0]P_{d,k|k-1}[I - \mathcal{K}_k\mathcal{C}_0]^T + \mathcal{K}_k\mathcal{R}_k\mathcal{K}_k^T, \tag{13}$$

$$P_{a,k|k} = \mathcal{A}_{22}^{-1}\mathcal{A}_{21}P_{d,k|k}\left(\mathcal{A}_{22}^{-1}\mathcal{A}_{21}\right)^T + \mathcal{A}_{22}^{-1}\mathcal{G}_2P_\epsilon\left(\mathcal{A}_{22}^{-1}\mathcal{G}_2\right)^T, \tag{14}$$

where

$$\mathcal{K}_k = P_{d,k|k-1}\mathcal{C}_0^T[\mathcal{C}_0P_{d,k|k-1}\mathcal{C}_0^T + \mathcal{R}_k]^{-1}, \tag{15}$$

$\hat{w}_{k|k} = \mathcal{T}_k[\mathcal{C}_0P_{d,k|k-1}\mathcal{C}_0^T + \mathcal{R}_k]^{-1}\tilde{y}_{k|k-1}$, $P_\epsilon = Q_k - \mathcal{T}_k[\mathcal{C}_0P_{d,k|k-1}\mathcal{C}_0^T + \mathcal{R}_k]^{-1}\mathcal{T}_k^T$ and $\tilde{y}_{k|k-1} = y_k - \hat{y}_{k|k-1}$, represents the estimation of process noise, error covariance of process noise and innovation, respectively.

Proof: From (6b), we have

$$\begin{aligned}\hat{x}_{d,k|k} &= \text{Proj}[x_{d,k} \mid y_{0:k}] = \text{Proj}[x_{d,k} \mid y_{0:k-1},\ y_k]\\ &= \text{Proj}[x_{d,k} \mid y_{0:k-1}] + \text{Proj}[e_{d,k|k-1} \mid \tilde{y}_{k|k-1}]\\ &= \hat{x}_{d,k|k-1} + \text{Cov}[e_{d,k|k-1}, \tilde{y}_{k|k-1}]\\ &\times \text{Var}^{-1}(\tilde{y}_{k|k-1})\tilde{y}_{k|k-1}.\end{aligned} \tag{16}$$

Now, for the computation of innovation, we need to find the prior estimate of y_k, which is obtained as

$$\hat{y}_{k|k-1} = \mathcal{C}_0\hat{x}_{k|k-1} + \mathcal{D}_0u_k. \tag{17}$$

Thus, innovation $\tilde{y}_{k|k-1}$ is computed as

$$\tilde{y}_{k|k-1} = y_k - \hat{y}_{k|k-1} = \mathcal{C}_0e_{d,k|k-1} + \bar{v}_k. \tag{18}$$

Based on the above equation, we have

$$\begin{aligned}\text{Cov}[e_{d,k|k-1}, \tilde{y}_{k|k-1}] &= \mathbb{E}[e_{d,k|k-1}(\mathcal{C}_0e_{d,k|k-1} + \bar{v}_k)^T],\\ &= P_{d,k|k-1}\mathcal{C}_0^T.\end{aligned} \tag{19}$$

Thus, the innovation covariance matrix is defined as

$$\begin{aligned}\text{Var}(\tilde{y}_{k|k-1}) &= \mathbb{E}[\tilde{y}_{k|k-1}\tilde{y}_{k|k-1}^T],\\ &= \mathbb{E}[(\mathcal{C}_0e_{d,k|k-1} + \bar{v}_k)(\mathcal{C}_0e_{d,k|k-1} + \bar{v}_k)^T],\\ &= \mathcal{C}_0P_{d,k|k-1}\mathcal{C}_0^T + \mathcal{R}_k.\end{aligned} \tag{20}$$

On substituting (19) and (20) in (16), we get

$$\begin{aligned}\hat{x}_{d,k|k} &= \hat{x}_{d,k|k-1} + P_{d,k|k-1}\mathcal{C}_0^T \times\\ &\quad (\mathcal{C}_0P_{d,k|k-1}\mathcal{C}_0^T + \mathcal{R}_k)^{-1}\tilde{y}_{k|k-1}\\ &= \hat{x}_{d,k|k-1} + \mathcal{K}_k\tilde{y}_{k|k-1}\end{aligned} \tag{21}$$

where $\mathcal{K}_k = P_{d,k|k-1}\mathcal{C}_0^T[\mathcal{C}_0P_{d,k|k-1}\mathcal{C}_0^T + \mathcal{R}_k]^{-1}$ is the Kalman gain. The estimation error $e_{d,k|k}$ is

$$e_{d,k|k} = x_{d,k} - \hat{x}_{d,k|k} = e_{d,k|k-1} - \mathcal{K}_k\tilde{y}_{k|k-1}. \tag{22}$$

On putting (22) in (7b) and simplifying it, we obtain

$$\begin{aligned}P_{d,k|k} &= \mathbb{E}[e_{d,k|k}e_{d,k|k}^T]\\ &= P_{d,k|k-1} - \mathbb{E}[e_{d,k|k-1}\tilde{y}_{k|k-1}^T]\mathcal{K}_k^T -\\ &\quad \mathcal{K}_k\mathbb{E}[\tilde{y}_{k|k-1}e_{d,k|k-1}{}^T] + \mathcal{K}_k\mathbb{E}[\tilde{y}_{k|k-1}\tilde{y}_{k|k-1}^T]\mathcal{K}_k^T\\ &= P_{d,k|k-1} - P_{d,k|k-1}\mathcal{C}_0^T\mathcal{K}_k^T - \mathcal{K}_k\mathcal{C}_0P_{d,k|k-1}\\ &\quad + \mathcal{K}_k\mathcal{C}_0P_{d,k|k-1}\mathcal{C}_0^T\mathcal{K}_k^T + \mathcal{K}_k\mathcal{R}_k\mathcal{K}_k^T\end{aligned} \tag{23}$$

Rearranging (23), we obtain (13). Now, the estimation of process noise w_k is defined as

$$\begin{aligned}\hat{w}_{k|k} &= \text{Proj}(w_k \mid y_{0:k}),\\ &= \mathbb{E}[w_k] + \text{Cov}[w_k, \tilde{y}_{k|k-1}]\,\text{Var}^{-1}(\tilde{y}_{k|k-1})\tilde{y}_{k|k-1}\\ &= \mathcal{T}_k[\mathcal{C}_0P_{d,k|k-1}\mathcal{C}_0^T + \mathcal{R}_k]^{-1}\tilde{y}_{k|k-1}.\end{aligned} \tag{24}$$

The estimation error w_ϵ and error covariance P_ϵ are

$$\begin{aligned}w_\epsilon &= w_k - \hat{w}_{k|k} = w_k - \mathcal{T}_k[\mathcal{C}_0P_{d,k|k-1}\mathcal{C}_0^T + \mathcal{R}_k]^{-1}\tilde{y}_{k|k-1},\\ P_\epsilon &= \mathbb{E}[w_\epsilon w_\epsilon^T] = Q_k - \mathcal{T}_k[\mathcal{C}_0P_{d,k|k-1}\mathcal{C}_0^T + \mathcal{R}_k]^{-1}\mathcal{T}_k^T.\end{aligned} \tag{25}$$

From (3), we have

$$\hat{x}_{a,k|k} = -\mathcal{A}_{22}^{-1}\left[\mathcal{A}_{21}\hat{x}_{d,k|k} + \mathcal{B}_2 u_k + \mathcal{G}_2\hat{w}_{k|k}\right]. \tag{26}$$

Thus, $\hat{x}_{a,k|k}$ is obtained by substituting (11) and (22) in (26). And, its estimation error $e_{a,k|k}$ is

$$e_{a,k|k} = x_{a,k} - \hat{x}_{a,k|k} = -\mathcal{A}_{22}^{-1}\mathcal{A}_{21}e_{d,k|k} - \mathcal{A}_{22}^{-1}\mathcal{G}_2 w_\epsilon. \tag{27}$$

Thus, posterior error covariance is computed by

$$\begin{aligned} P_{a,k|k} &= \mathbb{E}[e_{a,k|k}e_{a,k|k}^T] \\ &= \mathcal{A}_{22}^{-1}\mathcal{A}_{21}P_{d,k|k}\left(\mathcal{A}_{22}^{-1}\mathcal{A}_{21}\right)^T + \mathcal{A}_{22}^{-1}\mathcal{G}_2 P_\epsilon\left(\mathcal{A}_{22}^{-1}\mathcal{G}_2\right)^T \end{aligned} \tag{28}$$

This completes the proof. ∎

Finally, the updated estimate $\hat{x}_{k|k}$ is given by

$$\hat{x}_{k|k} = \mathcal{V}\begin{bmatrix} \hat{x}_{d,k|k} \\ \hat{x}_{a,k|k} \end{bmatrix}, \tag{29}$$

and filtering estimation error covariance matrix is

$$P_{x,k|k} = \mathcal{V}\begin{bmatrix} P_{d,k|k} & P_{da,k|k} \\ P_{ad,k|k} & P_{a,k|k} \end{bmatrix}\mathcal{V}^T, \tag{30}$$

where

$$\begin{aligned} P_{ad,k|k}^T =& P_{da,k|k} = \mathbb{E}[e_{d,k|k}e_{a,k|k}^T], \\ =& -P_{d,k|k}\left(\mathcal{A}_{22}^{-1}\mathcal{A}_{21}\right)^T. \end{aligned}$$

Thus, the recursive Kalman filtering algorithm for stochastic descriptor system is derived such that it is optimal in the sense of MMSE. The unbiasedness of the estimation is derived in [14].

We now summarize the whole filtering procedure in Algorithm 1.

Algorithm 1 Algorithm for the designed Kalman filter

- Set number of measurements N and initialize the filter with $\hat{x}_{0|0}$ and $P_{x,0|0}$.
- Initialize $x_{d,k}$ with $\hat{x}_{d,0|0}$ and $P_{d,0|0} = (P_{x,0|0})_{11}$.
- Compute the matrices $\mathcal{A}_0$, $\mathcal{B}_0$, $\mathcal{G}_0$, $\mathcal{C}_0$, $\mathcal{D}_0$ and $\mathcal{D}_1$.
- Update the measurement noise $\bar{v}_k$.
- for $k = 1 : N$
  - Find $\hat{x}_{d,k|k-1}$ and $P_{d,k|k-1}$ from (8) and (9).
  - Compute the Kalman gain from (15).
  - Find the estimate $\hat{x}_{k|k}$ and covariance $P_{d,k|k}$ from (11) and (13).
  - Compute $\hat{w}_{k|k}$ and P_ϵ by using (22) and (25).
  - Find the estimate $\hat{x}_{a,k|k}$ and $P_{a,k|k}$ by using (13) and (14).
  - Calculate $\hat{x}_{k|k}$ and $P_{x,k|k}$ on using (29) and (30).
- end for

IV. Lithium-Ion Battery Pack Formulation

In this paper, we consider a double capacitor ECM of Li-ion battery, as shown in Fig 1, because of its emerging importance [21]. The electrolytic resistance within the battery is represented by R_t. The $R_s - C_s$ unit accounts for the electrode surface region exposed to the electrolyte, while as $R_f - C_f$ unit represents the bulk inner part of the electrode. The majority of the charge stored in chemical form is accounted by C_f while C_s in comparison to C_f, is very small and consequently experiences quick voltage changes during charging and discharging [21], [22]. The cell current is I which is taken positive for charging and negative for discharging, V_t is the terminal voltage, V_s being the voltages across the capacitance, C_s, and $V_{oc}(\mathcal{Z})$ represents the OCV. It is important to mention here that we use a linear relation between SOC and OCV [23].

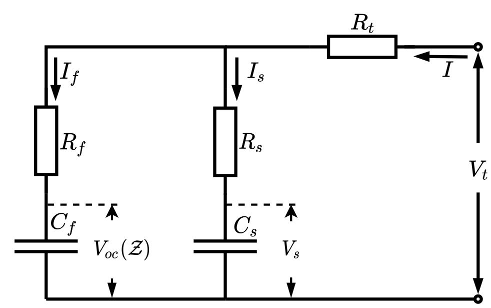

Fig. 1. Double capacitor-based ECM of a Li-ion battery cell

The dynamics of j-th battery cell in a parallel connected battery pack is given by [24]

$$x_{j,k+1} = \tilde{A}_j x_{j,k} + \tilde{B}_j v_{j,k} + \tilde{\Lambda}_j, \tag{31a}$$

$$y_{j,k} = \tilde{C}_j x_{j,k} + \tilde{D}_j v_{j,k} + \tilde{\Psi}_j, \tag{31b}$$

where $x_{j,k} = [\mathcal{Z}_{j,k} \quad V_{sj,k}]^T$ is the state vector, $v_{j,k} = I_{j,k}$ is the input vector and $y_{j,k} = V_{tj,k}$ is the output vector. Here $\mathcal{Z}_{j,k}$ denotes the SOC of the j^{th} cell. The matrices $\tilde{A}_j$, $\tilde{B}_j$, $\tilde{C}_j$, $\tilde{D}_j$, $\tilde{\Lambda}_j$, and $\tilde{\Psi}_j$ are given by $\tilde{\Psi}_j = \frac{R_{sj}\beta_j}{\mathcal{R}_j}$, $\tilde{A}_j = \begin{bmatrix} 1 - \frac{\alpha_j T_s}{C_{fj}\mathcal{R}_j} & \frac{T_s}{C_{fj}\mathcal{R}_j} \\ \frac{\alpha_j T_s}{C_{sj}\mathcal{R}_j} & 1 - \frac{T_s}{C_{sj}\mathcal{R}_j} \end{bmatrix}$, $\tilde{B}_j = \begin{bmatrix} \frac{T_s R_{sj}}{C_{fj}\mathcal{R}_j} \\ \frac{T_s R_{fj}}{C_{sj}\mathcal{R}_j} \end{bmatrix}$, $\tilde{\Lambda}_j = \begin{bmatrix} -\frac{T_s\beta_j}{C_{fj}\mathcal{R}_j} \\ \frac{T_s\beta_j}{C_{sj}\mathcal{R}_j} \end{bmatrix}$, $\tilde{C}_j = \begin{bmatrix} \frac{R_{sj}\alpha_j}{\mathcal{R}_j} & \frac{R_{fj}}{\mathcal{R}_j} \end{bmatrix}$, and, $\tilde{D}_j = \begin{bmatrix} R_{tj} + \frac{R_{sj}R_{fj}}{\mathcal{R}_j} \end{bmatrix}$.

Now, consider a block of n cells connected in parallel. Under the reduced sensing scenario, we assume only the total current is measurable and local currents are unknown. Since cells are connected in parallel, measuring the terminal voltage of one of the cells is sufficient. Mathematically, Kirchhoff's voltage law puts the following algebraic constraint

$$\frac{R_{sp}\alpha_p}{\mathcal{R}_p}\mathcal{Z}_{p,k}+\frac{R_{fp}}{\mathcal{R}_p}V_{sp,k}+(R_{tp}+\frac{R_{sp}R_{fp}}{\mathcal{R}_p})I_{p,k}+\frac{R_{sp}\beta_p}{\mathcal{R}_p}= \frac{R_{sq}\alpha_q}{\mathcal{R}_q}\mathcal{Z}_{q,k}+\frac{R_{fq}}{\mathcal{R}_q}V_{sq,k}+(R_{tq}+\frac{R_{sq}R_{fq}}{\mathcal{R}_q})I_{q,k}+\frac{R_{sq}\beta_q}{\mathcal{R}_q}, \quad \forall p,q\in\{1,2,...,n\}, p\neq q. \tag{32}$$

Similarly, Kirchhoff's current law imposes the following algebraic constraint $\sum_{j=1}^{n} I_{j,k}=I_{t,k}$, where $I_{t,k}$ is the total current applied to battery system. Therefore, under reduced sensing scenario, the parallel connected battery pack of n cells, naturally gives rise to descriptor system [1], which takes the form given by (2), where $x_k=[x_{d,k}\;\; x_{a,k}]^T$ and $u_k=[I_{t,k}\;\; 1]$. $x_{d,k}=\begin{bmatrix} x_{1,k} & x_{2,k} & \cdots & x_{n,k}\end{bmatrix}^T$, $x_{a,k}=\begin{bmatrix} I_{1,k} & I_{2,k} & \cdots & I_{n,k}\end{bmatrix}^T$, $\bar{y}_k=\begin{bmatrix} y_{1,k} & y_{2,k} & \cdots & y_{n,k}\end{bmatrix}^T$, $y_k=\bar{y}_k-\Psi$ and $x_{d,k}\in\mathbb{R}^{2n}$, and, $x_{a,k}$, $\bar{y}_k\in\mathbb{R}^n$. Furthermore, $\Psi=\begin{bmatrix}\tilde{\Psi}_1 & \tilde{\Psi}_2 & \cdots & \tilde{\Psi}_n\end{bmatrix}^T$, $\mathcal{A}_{11}=diag(\tilde{A}_1,\;\tilde{A}_2,\;\cdots,\tilde{A}_n)$, $\mathcal{A}_{12}=diag(\tilde{B}_1,\;\tilde{B}_2,\;\cdots,\tilde{B}_n)$, $\mathcal{C}_1=diag(\tilde{C}_1,\;\tilde{C}_2,\;\cdots,\tilde{C}_n)$, $\mathcal{C}_2=diag(\tilde{D}_1,\;\tilde{D}_2,\;\cdots,\tilde{D}_n)$,

$$\mathcal{A}_{21}=\begin{bmatrix}\tilde{C}_1 & -\tilde{C}_2 & 0 & \cdots & 0\\ \tilde{C}_1 & 0 & -\tilde{C}_3 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \tilde{C}_1 & 0 & 0 & \cdots & -\tilde{C}_n\\ 0 & 0 & 0 & \cdots & 0\end{bmatrix},$$

$$\mathcal{A}_{22}=\begin{bmatrix}\tilde{D}_1 & -\tilde{D}_2 & 0 & \cdots & 0\\ \tilde{D}_1 & 0 & -\tilde{D}_3 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \tilde{D}_1 & 0 & 0 & \cdots & -\tilde{D}_n\\ 1 & 1 & 1 & \cdots & 1\end{bmatrix},$$

$$\mathcal{B}_1=\begin{bmatrix}0 & \tilde{\Lambda}_1\\ 0 & \tilde{\Lambda}_2\\ \vdots & \vdots\\ 0 & \tilde{\Lambda}_n\end{bmatrix},\;\mathcal{B}_2=\begin{bmatrix}0 & \tilde{\Psi}_1-\tilde{\Psi}_2\\ 0 & \tilde{\Psi}_1-\tilde{\Psi}_3\\ \vdots & \vdots\\ 0 & \tilde{\Psi}_1-\tilde{\Psi}_n\\ -1 & 0\end{bmatrix}.$$

V. Results and Discussion

To evaluate the effectiveness of our proposed filter design approach, we have considered a battery pack with two cells in parallel. The capacity and nominal voltage of the two cylindrical Li-ion battery cells considered in our study are 2.6 Ah, 3.7 V and 2.3 Ah, 3.6 V, respectively. The input current to the battery pack is shown in Fig. 2. which is generated from an urban dynamometer driving schedule (UDDS) and scaled to a discharging range of 0–6 A [21]. The values of the parameters of DC-ECM for the two cells, as shown in Table I, are evaluated using circuit analysis methods [25]. To incorporate process and measurement noise we take, $Q_k=10^{-4}$ and $R_k=10^{-3}I_2$, where I_2 represents the identity matrix of dimension two. To evaluate the robustness of our approach, we randomize the initial conditions of both SOC and current and run the simulation for 1000 Monte Carlo runs. The sampling time T_s is taken as $1s$.

TABLE I
DC-ECM Parameters used in Simulation Study

Parameter	Cell 1	Cell 2	Units
R_t	0.015	0.010	[Ω]
R_s	0.045	0.030	[Ω]
R_f	0.055	0.040	[Ω]
C_s	110	200	[F]
C_f	9100	5630	[F]

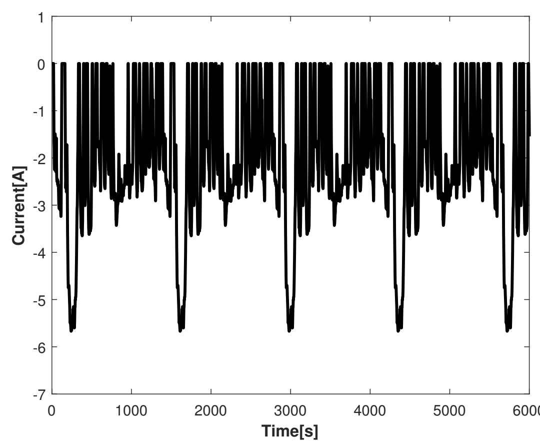

Fig. 2. Applied total input current I_t from UDDS drive cycle.

Fig. 3 denotes the true and estimated SOC of cell 1 and cell 2, while Fig. 4 represents the truth and estimation of local currents of cell 1 and cell 2. Despite a large initial estimation error, the SOC estimates and the local cell current estimates quickly converge the truth. Root mean square error (RMSE) of both SOC and local cell current is plotted in Fig. 5, which clearly demonstrates that SOC and current for the cells are rapidly converging to zero.

In practice, the battery management system might not always be aware of the initial SOC of the concerned cell. Moreover, the presence of high noise levels, particularly in measurement, can not be ignored. Therefore, for practical applicability, our method must be robust to SOC initialization error and should be able to appropriately track the SOC in the presence of high levels of process and measurement noise. In order to evaluate the effectiveness of our proposed filtering methodology towards different initialization errors and noise, we developed and carried out several simulations based on combinations of three factors: the initialization error, process noise level, and measurement noise level, and accordingly calculated the error (%) in SOC. The six different cases considered in the simulation study are outlined in Table II. As evident from Table II, the proposed method has significant accuracy and is effective towards different levels of initialization error and noises.

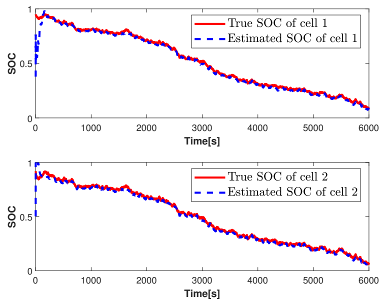

Fig. 3. True and estimated SOC's of cell 1 and cell 2.

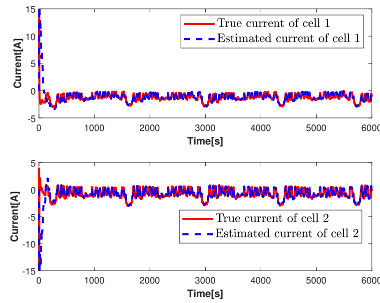

Fig. 4. True and estimated current of cell 1 and cell 2.

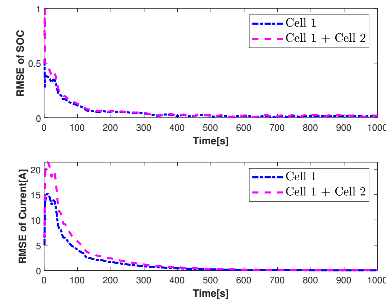

Fig. 5. RMSE plot of SOC and current.

TABLE II

SIMULATION TEST DESIGN

Test #	Initialization Error	Process Noise Level	Measurement Noise Level	Error(%) SOC
1	5%	0.0001	$0.001I_2$	1.12
2	5%	0.001	$0.01I_2$	2.16
3	5%	0.001	$0.1I_2$	3.54
4	20%	0.0001	$0.001I_2$	1.34
5	20%	0.001	$0.01I_2$	3.07
6	20%	0.001	$0.1I_2$	4.48

VI. CONCLUSION

In this paper, we have provided an alternative approach to derive the Kalman filter for descriptor systems using the well-known projection theorem approach. The descriptor system is assumed to be regular and of index-one. The proposed approach is simple and convenient and can be easily extended to complex systems with additive and multiplicative random uncertainties. To test the applicability of our proposed methodology, we estimated the state of charge and local currents of a parallel connected Li-ion battery pack, whose modeling naturally comes out to be a descriptor system. As evident from the simulation results, despite large initial estimation errors and noise, the filter is able to track the truth quickly, which demonstrates the effectiveness of our approach.

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