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Distributed Nash Equilibrium Seeking in N-Cluster Games with Non-Uniform Constant Step-Sizes

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Abstract— **This paper studies a class of non-cooperative games, known as N-cluster game, which subsumes both cooperative and non-cooperative nature among multiple agents in the two problems. Moreover, we consider a *partial-decision information* game setup, *i.e.*, the agents have no direct access to the decisions of other agents in all clusters, and hence need to communicate with each other. We propose a distributed NE seeking algorithm by a synthesis of consensus and gradient tracking. Unlike other existing discrete-time methods for N-cluster games where a common step-size is either publicly known by all agents or only known by agents from the same cluster, the proposed algorithm can work with non-uniform constant step-sizes, which allows the agents (both within and across the clusters) to choose their own preferred step-sizes. We prove that all agents' decisions converge linearly to their corresponding NE so long as the largest step-size and the heterogeneity of the step-sizes are small. We verify the derived results through a numerical example in a Cournot competition game.**

Index Terms— **Nash equilibrium (NE) seeking, distributed methods, non-cooperative games.**

I. Introduction

Simultaneous social cost minimization and Nash equilibrium (NE) seeking among multiple clusters (or coalitions) modeled by N-cluster game have received great attention in recent researches, due to its wide application in many fields, such as business management, transportation systems, political science, sports, to list a few. In such N-cluster games, the agents in the same cluster cooperatively minimize a cluster-level cost function, and collectively act as a virtual player to play an N-player non-cooperative game across clusters.

Related work: Distributed NE seeking under partial-decision information over graphs have been researched, see [1], [2] for unconstrained or locally set constrained games, [3]–[5] for aggregative games, and [6]–[8] for generalized games. Recently, NE seeking agorithms for N-cluster games have started to draw researchers' attention [9]–[12]. Specifically, the work in [9] proposed a NE seeking algorithm based on a dynamic average consensus and the gradient play, assuming all agents' decisions are directly accessible, *i.e.*, full-decision information. Different from the full-decision information setup in [9], the work in [10] modeled the decisions of the agents in the same cluster by a decision vector that needs to be agreed on, and introduced a leader-following hierarchy inter-cluster communication mechanism, where only the cluster leader can exchange information across different clusters. The works in [11], [12] also considered the partial-decision information setup, and supposed that all agents from all clusters are connected. The work in [13] considered the same cluster-level decision vector modeling as in [10], and developed a discrete-time NE seeking algorithm based on gradient play with no explicit communications between clusters. As compared to the primal methods based on gradient play, gradient tracking is preferred for its faster convergence speed without sacrificing the accuracy, and hence has found its great advantages in the fields of N-cluster games recently [14]–[18]. Though being theoretically equivalent, as compared to modeling the collection of all agents as a single decision vector [10], [13], [16], [17], it would be natural to model the decision variable of each agent individually in practical applications [9], [11], [14], [15], [18]. For example, each branch (i.e., agent) of the firm (i.e., cluster) can only decide how much goods it can produce rather than the collection of all branches. In general, it has no control on other branches' production amount, and there is usually no such agreement among branches requiring their production amount to be equal. Moreover, in discrete-time methods, the step-size is usually assumed to be publicly known by all agents, and hence a common step-size agreed by all agents is chosen in the algorithm development [13], [14], [16]–[18]. Concerning a distributed setup, it would be preferred that agents are allowed to select their own preferred step-sizes, as it does not require the central coordination of the step-size in the implementation of the algorithm. However, introducing such non-uniform step-sizes may not always guarantee the convergence, and also brings substantial complexity in the convergence analysis. The work in [15] allows agents from different clusters to use different step-sizes. However, agents from the same cluster still adopt a common step-size, and the full-decision information game setup was considered.

Contributions: The main contributions of this paper are summarized as follows. 1) As compared to modeling the collection of all agents as a single decision vector [10], [13], [16], [17], this paper models the decision variable of each agent in each cluster individually, which is more ameanable to the applications where agents can only control their own actions. Moreover, we consider a partial-decision information game setup, *i.e.*, the agents only have direct access to their own decisions. In contrast to the distributed setups in [16]–[18] where the agent in the same cluster needs

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to reach a common decision through average consensus, this work does not impose a common decision requirement within clusters. Specifically, the agents estimate other agents' decisions via a leader-following consensus protocol across clusters over a high-level network. This is different from the leader-following hierarchy inter-cluster communication mechanism as in [10], [12], [16] where there is a pre-defined leader to exchange information across clusters. Hence, this approach allows more inter-cluster communication channels. Meanwhile, the agents in the same cluster perform the gradient tracking updates over a low-level network. Compared with the two level communication networks in [11], [18], we do not require the intra-cluster communication graph to be a part of the inter-cluster communication graph, and allow the two level networks to be independent of each other, which is more general. 2) Furthermore, in terms of the implementation of the distributed protocols, most works either consider undirected graph or directed graph with doubly stochastic weighting matrix [15], [16], which is relatively straightforward in terms of the analysis. However, for directed graphs, there may not always exist a doubly stochastic weighting matrix [19]. Thus, in our work, we consider row stochastic weighting matrix for the high level network and column stochastic weighting matrix for the low level network, which increases the application range of the method. Though such techniques have also been studied in distributed optimization, the convergence analysis is much more complicated in multi-cluster games. 3) Different from all existing discrete-time methods for N-cluster games where either a common step-size is publicly known by all agents [13], [14], [16], [17] or only known by agents from the same cluster [15], we consider non-uniform step-sizes, *i.e.*, all agents (both among and across) the clusters are allowed to choose their own preferred constant step-sizes. We prove that all agents' decisions converge linearly to their corresponding NE so long as the largest step-size and the heterogeneity of the step-sizes are small enough.

Notations: We use $\mathbf{1}_m$ for an m-dimensional vector with all elements being 1, and I_m for an $m \times m$ identity matrix. For a vector π, we use $\mathrm{diag}(\pi)$ to denote a diagonal matrix formed by the elements of π. For any two vectors u, v, their inner product is denoted by $\langle u, v \rangle$; their weighted inner product due to a positive vector (*i.e.*, vector with all elements being positive) π is denoted by $\langle u, v \rangle_\pi \triangleq u^\top \mathrm{diag}(\pi)^{-1} v$. The transpose of u is denoted by $u^\top$. Moreover, we use $\|u\|_2$ for its standard Euclidean norm, *i.e.*, $\|u\|_2 = \sqrt{\langle u, u \rangle}$, and $\|u\|_\pi$ for its weighted Euclidean norm due to π, *i.e.*, $\|u\|_\pi = \|\mathrm{diag}(\sqrt{\pi})^{-1} u\|_2$. For vector a, we use $[a]_i$ to denote its i-th entry. The transpose and spectral norm of a matrix A are denoted by $A^\top$ and $\|A\|_2$, respectively. The matrix norm $\|A\|_\pi$ induced by $\|\cdot\|_\pi$ is defined as $\|A\|_\pi \triangleq \|\mathrm{diag}(\sqrt{\pi})^{-1} A \ \mathrm{diag}(\sqrt{\pi})\|_2$. We use $\rho(A)$ to represent the spectral radius of a square matrix A, and A_∞ to indicate its infinite power (if it exists) $\lim_{k\to\infty} A^k$.

II. Problem Statement

An N-cluster game, defined by $\Gamma(\mathcal{N}, \{J^i\}, \{\mathbb{R}^{n_i}\})$, is a multi-player non-cooperative game played among N clusters, where each cluster, indexed by $i \in \mathcal{N} \triangleq \{1, 2, \ldots, N\}$, consists of a group of agents, denoted by $\mathcal{V}^i \triangleq \{1, 2, \ldots, n_i\}$, to cooperatively minimize a cluster-level cost function J^i. Denote $n \triangleq \sum_{i=1}^N n_i$. Then, the cluster-level cost function $J^i : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$J^i(\mathbf{x}) \triangleq \tfrac{1}{n_i} \textstyle\sum_{j=1}^{n_i} J_j^i(\mathbf{x}^i, \mathbf{x}^{-i}), \quad \forall i \in \mathcal{N},$$

where $J_j^i(\mathbf{x})$ is a local cost function of agent j in cluster i, $\mathbf{x}^i \triangleq [x_1^{i\top}, \ldots, x_{n_i}^{i\top}]^\top \in \mathbb{R}^{n_i}$ is a collection of all agents' decisions in cluster i with $x_j^i \in \mathbb{R}$ being the action of agent j in cluster i, $\mathbf{x}^{-i} \in \mathbb{R}^{n-n_i}$ denotes a collection of all agents' decisions except cluster i, and $\mathbf{x} \triangleq [\mathbf{x}^{1\top}, \ldots, \mathbf{x}^{N\top}]^\top$. A vector $\mathbf{x}^* \triangleq (\mathbf{x}^{i*}, \mathbf{x}^{-i*}) \in \mathbb{R}^n$ is said to be an NE of the N-cluster non-cooperative game $\Gamma(\mathcal{N}, \{J^i\}, \{\mathbb{R}^{n_i}\})$, if and only if $J^i(\mathbf{x}^{i*}, \mathbf{x}^{-i*}) \le J^i(\mathbf{x}^i, \mathbf{x}^{-i*})$, $\forall i \in \mathcal{N}$.

Motivated by the work in [20], we assume the agents are equipped with two level networks: a high-level network for agents' decisions exchange across clusters and N low-level networks for agents' (partial) gradient exchange within the cluster. For the low-level network in each cluster $i \in \mathcal{N}$, it is a directed graph consisting of all agents in the same cluster, denoted by $\mathcal{G}_i(\mathcal{V}^i, \mathcal{E}^i)$ with an adjacency matrix $A^i \triangleq [a_{jk}^i] \in \mathbb{R}^{n_i \times n_i}$, $a_{jk}^i > 0$ if $(k, j) \in \mathcal{E}^i$ and $a_{jk}^i = 0$ otherwise. We assume $(k, k) \in \mathcal{E}^i, \forall k \in \mathcal{V}^i$. For the high-level network, it is a directed graph consisting of all agents in all clusters, denoted by $\bar{\mathcal{G}}(\bar{\mathcal{V}}, \bar{\mathcal{E}})$ with an adjacency matrix $\bar{A} \triangleq [\bar{a}_{pq}] \in \mathbb{R}^{n \times n}$, $\bar{a}_{pq} > 0$ if $(q, p) \in \bar{\mathcal{E}}$ and $\bar{a}_{pq} = 0$ otherwise. We assume $(p, p) \in \bar{\mathcal{E}}, \forall p \in \bar{\mathcal{V}}$. The following two standard assumptions on the two level networks are imposed.

Assumption 1: For $i \in \mathcal{N}$, the digraph $\mathcal{G}_i$ is strongly connected. The associated adjacency matrix A^i is column stochastic, *i.e.*, $\mathbf{1}_{n_i}^\top A^i = \mathbf{1}_{n_i}^\top$.

Assumption 2: The digraph $\bar{\mathcal{G}}$ is strongly connected. Its associated adjacency matrix $\bar{A}$ is row stochastic, *i.e.*, $\bar{A}\mathbf{1}_n = \mathbf{1}_n$.

Under Assumption 1, it is known that A^i is primitive and column stochastic, then we denote its right eigenvector corresponding to the eigenvalue of 1 by $\pi^i \triangleq [\pi_1^i, \ldots, \pi_{n_i}^i]^\top$, such that $\mathbf{1}_{n_i}^\top \pi^i = 1$. Then, π^i corresponds to A^i's non-$\mathbf{1}_{n_i}$ Perron vector with eigenvalue 1, and hence all elements in π^i are positive, and $A_\infty^i = \pi^i \mathbf{1}_{n_i}^\top$. Define $\iota \triangleq \max_{i\in\mathcal{N}} \|\pi^i\|_2$, $\boldsymbol{\pi}^i \triangleq n_i \pi^i$, and $\boldsymbol{\pi} \triangleq [\boldsymbol{\pi}^{1\top}, \ldots, \boldsymbol{\pi}^{N\top}]^\top$. Denote the smallest and largest elements of $\boldsymbol{\pi}$ by $\underline{\boldsymbol{\pi}}$ and $\overline{\boldsymbol{\pi}}$, respectively. With the above notations, we can obtain that $\sqrt{\underline{\boldsymbol{\pi}}}\|\cdot\|_{\boldsymbol{\pi}^i} \le \|\cdot\|_2 \le \sqrt{\overline{\boldsymbol{\pi}}}\|\cdot\|_{\boldsymbol{\pi}^i}$ and $\sqrt{\underline{\boldsymbol{\pi}}}\|\cdot\|_{\boldsymbol{\pi}} \le \|\cdot\|_2 \le \sqrt{\overline{\boldsymbol{\pi}}}\|\cdot\|_{\boldsymbol{\pi}}$ based on the definitions of the weighted Euclidean norm, which will be frequently applied in the subsequent analysis. Moreover, under Assumption 1, it follows from [21, Lemma 1] that the adjacency matrix A^i holds that $\|A^i - A_\infty^i\|_{\boldsymbol{\pi}^i} < 1$, and $\|I_{n_i} - A_\infty^i\|_{\boldsymbol{\pi}^i} = 1$. Define $\sigma_{A^i} \triangleq \|A^i - A_\infty^i\|_{\boldsymbol{\pi}^i}$. Let $\bar{\sigma}_1 \triangleq \max_{i\in\mathcal{N}} \sigma_{A^i}$ and $\varsigma_1 \triangleq \frac{1+\bar{\sigma}_1^2}{1-\bar{\sigma}_1^2}$.

Next, we make the following assumption on the agents' local cost functions.

Assumption 3: For each $j \in \mathcal{V}^i, i \in \mathcal{N}$, the local cost function $J_j^i(\mathbf{x}^i, \mathbf{x}^{-i})$ is convex, continuously differentiable in $\mathbf{x}^i$; the partial gradient with respect to $x_k^i, \forall k \in \mathcal{V}^i$ (denoted by $\nabla_k^i J_j^i(\mathbf{x})$ for simplicity), is $\mathcal{L}$-Lipschitz continuous in $\mathbf{x}$, *i.e.*, for any $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^n$, we have $\|\nabla_k^i J_j^i(\mathbf{x}) - \nabla_k^i J_j^i(\mathbf{x}')\|_2 \leq \mathcal{L}\|\mathbf{x} - \mathbf{x}'\|_2$.

The game mapping of $\Gamma(\mathcal{N}, \{J^i\}, \{\mathbb{R}^{n_i}\})$ is defined as $\boldsymbol{F}(\mathbf{x}) \triangleq [\nabla_{\mathbf{x}^1} J^1(\mathbf{x})^\top, \ldots, \nabla_{\mathbf{x}^N} J^N(\mathbf{x})^\top]^\top$. Then, it follows from Assumption 3 that $\boldsymbol{F}(\mathbf{x})$ is Lipschitz continuous, *i.e.*, for any $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^n$, we have $\|\boldsymbol{F}(\mathbf{x}) - \boldsymbol{F}(\mathbf{x}')\|_2 \leq \sqrt{n}\mathcal{L}\|\mathbf{x} - \mathbf{x}'\|_2$. Next, the following assumption on the game mapping condition is supposed.

Assumption 4: The game mapping $\boldsymbol{F}$ of game Γ is strongly monotone on $\mathbb{R}^n$ with a constant $\chi > 0$, *i.e.*, for any $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^n$, we have $\langle \boldsymbol{F}(\mathbf{x}) - \boldsymbol{F}(\mathbf{x}'), \mathbf{x} - \mathbf{x}' \rangle \geq \chi\|\mathbf{x} - \mathbf{x}'\|_2^2$.

Remark 1: Under Assumptions 3 and 4, game Γ admits a unique NE $\mathbf{x}^*$. Moreover, at NE, $\boldsymbol{F}(\mathbf{x}^*) = \mathbf{0}_n$, and hence $\langle \boldsymbol{F}(\mathbf{x}), \mathbf{x} - \mathbf{x}^* \rangle \geq \chi\|\mathbf{x} - \mathbf{x}^*\|_2^2$.

III. Algorithm

In this section, we present a distributed NE seeking strategy for the N-cluster game under partial-decision information scenario, followed by the detailed convergence analysis.

For the notational convenience, agent $j \in \mathcal{V}^i$ in cluster $i \in \mathcal{N}$ of the low-level network is referred to agent $j^i \triangleq j^i$ with $n_0 = 0$ in the high-level network. Hence, its action variable x_j^i is relabeled by y_{j^i}, *i.e.*, $x_j^i = \mathsf{y}_{j^i}$. Then, each agent $j \in \mathcal{V}^i$, $i \in \mathcal{N}$ needs to maintain the action variable x_j^i, and gradient tracker variables g_{jk}^i for $\forall k \in \mathcal{V}^i$. Moreover, in the high-level network, each agent $p \in \bar{\mathcal{V}}$ also needs to maintain an estimation variable y_q^p for the action y_q of agent $q \in \bar{\mathcal{V}}$. For $i \in \mathcal{N}$, $p \in \bar{\mathcal{V}}$, we denote that $\mathbf{y}_{(i)}^p \triangleq [y_{1^i}^p, \ldots, y_{n_i^i}^p]^\top \in \mathbb{R}^{n_i}, \mathbf{y}^p \triangleq [\mathbf{y}_{(1)}^{p\top}, \ldots, \mathbf{y}_{(N)}^{p\top}]^\top \in \mathbb{R}^n, \mathbf{y}^{(i)} \triangleq [\mathbf{y}^{1^i\top}, \ldots, \mathbf{y}^{n_i^i\top}]^\top \in \mathbb{R}^{n_i n}$. We use subscript t to denote the values of all these variables at time-step t. The update laws are designed as follows.

high-level network: $p, q \in \bar{\mathcal{V}}$

$$y_{q,t+1}^p = \textstyle\sum_{l=1}^n \bar{a}_{pl} y_{q,t}^l + \delta_p \bar{a}_{pq}(\mathsf{y}_{q,t} - y_{q,t}^p) \tag{1a}$$

low-level network: $j, k \in \mathcal{V}^i, i \in \mathcal{N}$

$$x_{j,t+1}^i = x_{j,t}^i - \gamma_j^i g_{jj,t}^i, \tag{1b}$$

$$g_{jk,t+1}^i = \textstyle\sum_{l=1}^{n_i} a_{jl}^i g_{lk,t}^i + \nabla_k^i J_j^i(\mathbf{y}_{t+1}^{j^i}) - \nabla_k^i J_j^i(\mathbf{y}_t^{j^i}), \tag{1c}$$

with arbitrary $x_{j,0}^i \in \mathbb{R}$, $\mathbf{y}_0^{j^i} \in \mathbb{R}^n$ and $g_{jk,0}^i = \nabla_k^i J_j^i(\mathbf{y}_0^{j^i})$, where δ_p is a constant parameter for agent $p \in \bar{\mathcal{V}}$, and $\gamma_j^i > 0$ is a constant step-size sequence adopted by agent $j \in \mathcal{V}^i$, $i \in \mathcal{N}$. Denote the largest step-size by $\gamma_M \triangleq \max_{j \in \mathcal{V}^i, i \in \mathcal{N}} \gamma_j^i$ and the average of all step-sizes by $\bar{\gamma} \triangleq \frac{1}{n}\sum_{j \in \mathcal{V}^i, i \in \mathcal{N}} \gamma_j^i$. Define the heterogeneity of the step-size as the following ratio, $\epsilon_\gamma \triangleq \|\boldsymbol{\gamma} - \bar{\boldsymbol{\gamma}}\|_2 / \|\bar{\boldsymbol{\gamma}}\|_2$, where $\boldsymbol{\gamma} \triangleq [\gamma_1^1, \ldots, \gamma_{n_1}^1, \ldots, \gamma_1^N, \ldots, \gamma_{n_N}^N]^\top$ and $\bar{\boldsymbol{\gamma}} \triangleq \bar{\gamma}\mathbf{1}_n$.

IV. Convergence Analysis

The following notations are made throughout the convergence analysis for convenience. For $\forall k \in \mathcal{V}^i, i \in \mathcal{N}$, $\mathbf{g}_{k,t}^i \triangleq [g_{1k,t}^i, \ldots, g_{n_i k,t}^i]^\top \in \mathbb{R}^{n_i}$ denotes the stacked partial gradient tracker with respect to agent k of cluster i, $\bar{\mathbf{g}}_{k,t}^i \triangleq \frac{1}{n_i}\mathbf{1}_{n_i}^\top \mathbf{g}_{k,t}^i \in \mathbb{R}$ denotes the average of the partial gradient tracker with respect to agent k of cluster i, $\mathbf{g}_t \triangleq [g_{11,t}^1, g_{22,t}^1, \ldots, g_{n_N n_N,t}^N]^\top \in \mathbb{R}^n$ denotes the stacked gradient tracker of all clusters, $\nabla_k^i \mathbf{J}^i(\mathbf{y}_t^{(i)}) \triangleq [\nabla_k^i J_1^i(\mathbf{y}_t^{1^i}), \ldots, \nabla_k^i J_{n_i}^i(\mathbf{y}_t^{n_i^i})]^\top \in \mathbb{R}^{n_i}$ denotes the stacked partial gradient with respect to agent k of cluster i evaluated with the the corresponding estimation variable, and $\nabla_k^i \bar{\mathbf{J}}^i(\mathbf{y}_t^{(i)}) \triangleq \frac{1}{n_i}\mathbf{1}_{n_i}^\top \nabla_k^i \mathbf{J}^i(\mathbf{y}_t^{(i)}) \in \mathbb{R}$ denotes the average of the partial gradient with respect to agent k of cluster i evaluated with the the corresponding estimation variable. Then, the concatenated form of (1) is given by

$$\begin{aligned}\mathbf{y}_{q,t+1} &= \bar{A}\mathbf{y}_{q,t} \\ &\quad + \mathrm{diag}([\delta_1 \bar{a}_{1q}, \ldots, \delta_n \bar{a}_{nq}]^\top)(\mathbf{1}_n \mathsf{y}_{q,t} - \mathbf{y}_{q,t}) \quad (2a)\\ \mathbf{x}_{t+1} &= \mathbf{x}_t - \mathrm{diag}(\boldsymbol{\gamma})\mathbf{g}_t, \quad (2b)\\ \mathbf{g}_{k,t+1}^i &= A^i \mathbf{g}_{k,t}^i + \nabla_k^i \mathbf{J}^i(\mathbf{y}_{t+1}^{(i)}) - \nabla_k^i \mathbf{J}^i(\mathbf{y}_t^{(i)}). \quad (2c)\end{aligned}$$

A. Auxiliary Results

We first start with an important property of the adjacency matrix $\bar{A}$ in the following lemma.

Lemma 1: Under Assumption 2, let $\delta_p > 0, p \in \bar{\mathcal{V}}$ be chosen such that $0 \leq \delta_p \bar{a}_{pq} < 2\bar{a}_{pp}$ $\forall q \in \bar{\mathcal{V}}$. Then, the matrix $\tilde{A}_q \triangleq [\tilde{a}_{pm}^q], q \in \bar{\mathcal{V}}$ with its entry given by

$$\tilde{a}_{pm}^q = \begin{cases} \bar{a}_{pm} & \text{if } p \neq m \\ |\bar{a}_{pp} - \delta_p \bar{a}_{pq}| & \text{if } p = m \end{cases}$$

holds that $\rho(\tilde{A}_q) < 1$. Moreover, there exists a matrix norm $\|\cdot\|_E$ such that $\|\tilde{A}_q\|_E < 1$ for $\forall q \in \bar{\mathcal{V}}$.

Proof: The first part of the result (*i.e.*, $\rho(\tilde{A}_q) < 1$) can be readily proved based on [14], [15]. For the second part, we invoke the following lemma to facilitate the proof.

Lemma 2: (see [22, Lemma 5.6.10]) Let $\rho(A)$ be the spectral radius of a (square) matrix A. For any given $\varrho > 0$, there exists a matrix norm $\|\cdot\|_E$ such that $\rho(A) \leq \|A\|_E \leq \rho(A) + \varrho$.

From Lemma 2, we choose $\varrho \in (0, 1 - \max_{q \in \bar{\mathcal{V}}} \rho(\tilde{A}_q))$, then there exists a matrix norm $\|\cdot\|_E$ such that $\|\tilde{A}_q\|_E \leq \rho(\tilde{A}_q)) + \varrho < 1, \forall q \in \bar{\mathcal{V}}$, which completes the proof. $\square$

Denote the vector norm which is compatible with the matrix norm $\|\cdot\|_E$ by $\|\cdot\|_e$, *i.e.*, $\|Av\|_e \leq \|A\|_E\|v\|_e$ for a matrix A and a vector v with compatible size. Due to the equivalence of all norms in a finite-dimensional vector space, there exists $\underline{C} > 0$ and $\overline{C} > 0$ such that $\underline{C}\|v\|_2 \leq \|v\|_e \leq \overline{C}\|v\|_2$. Define $\sigma_{\tilde{A}_q} \triangleq \|\tilde{A}_q\|_E$. Let $\bar{\sigma}_2 \triangleq \max_{q \in \bar{\mathcal{V}}} \sigma_{\tilde{A}_q}$, $\varsigma_2 \triangleq \frac{1+\bar{\sigma}_2^2}{1-\bar{\sigma}_2^2}$.

Next, we provide a bound on the stacked gradient estimator $\mathbf{g}_t$ in the following lemma.

Lemma 3: Under Assumptions 1 and 3, the stacked gradient tracker $\mathbf{g}_t$ holds that

$$\|\mathbf{g}_t\|_2^2 \le 3\overline{\boldsymbol{\pi}}\iota^2\mathcal{L}^2\sum_{i=1}^N n_i^3\|\mathbf{x}_t-\mathbf{x}^*\|_{\boldsymbol{\pi}}^2 + 3\overline{\boldsymbol{\pi}}\sum_{i=1}^N\sum_{k=1}^{n_i} \|\mathbf{g}_{k,t}^i - A_\infty^i\mathbf{g}_{k,t}^i\|_{\boldsymbol{\pi}^i}^2 + \frac{3n^2\iota^2\mathcal{L}^2}{\underline{C}^2}\sum_{q=1}^n\|\mathbf{y}_{q,t}-\mathbf{1}_n\mathrm{y}_{q,t}\|_e^2.$$

Proof: It is noted that for $k\in\mathcal{V}^i, i\in\mathcal{N}$, $\|\mathbf{g}_{k,t}^i\|_2 \le \|\mathbf{g}_{k,t}^i - \mathbf{v}^i\mathbf{1}_{n_i}^\top\mathbf{g}_{k,t}^i\|_2 + \|\mathbf{v}^i\mathbf{1}_{n_i}^\top\mathbf{g}_{k,t}^i\|_2 \le \sqrt{\overline{\boldsymbol{\pi}}}\|\mathbf{g}_{k,t}^i - A_\infty^i\mathbf{g}_{k,t}^i\|_{\boldsymbol{\pi}^i} + n_i\iota\|\bar{\mathbf{g}}_{k,t}^i - \nabla_k^i\bar{\mathbf{J}}^i(\mathbf{y}_t^{(i)})\|_2 + n_i\iota\|\nabla_k^i\bar{\mathbf{J}}^i(\mathbf{y}_t^{(i)}) - \nabla_k^i J^i(\mathbf{x}_t)\|_2 + n_i\iota\|\nabla_k^i J^i(\mathbf{x}_t) - \nabla_k^i J^i(\mathbf{x}^*)\|_2$, where $\nabla_k^i J^i(\mathbf{x}^*)=0$ has been applied. Since A^i is column stochastic, it follows from (2c) that $\bar{\mathbf{g}}_{k,t+1}^i = \bar{\mathbf{g}}_{k,t}^i + \nabla_k^i\bar{\mathbf{J}}^i(\mathbf{y}_{t+1}^{(i)}) - \nabla_k^i\bar{\mathbf{J}}^i(\mathbf{y}_t^{(i)})$. Due to the initial conditions $\mathbf{g}_{k,0}^i = \nabla_k^i\mathbf{J}^i(\mathbf{y}_0^{(i)})$, we obtain

$$\bar{\mathbf{g}}_{k,t}^i = \nabla_k^i\bar{\mathbf{J}}^i(\mathbf{y}_t^{(i)}). \tag{3}$$

Thus, $\|\mathbf{g}_{k,t}^i\|_2^2 \le 3\overline{\boldsymbol{\pi}}\|\mathbf{g}_{k,t}^i - A_\infty^i\mathbf{g}_{k,t}^i\|_{\boldsymbol{\pi}^i}^2 + 3n_i^2\iota^2\|\nabla_k^i\bar{\mathbf{J}}^i(\mathbf{y}_t^{(i)}) - \nabla_k^i J^i(\mathbf{x}_t)\|_2^2 + 3\overline{\boldsymbol{\pi}}n_i^2\iota^2\mathcal{L}^2\|\mathbf{x}_t - \mathbf{x}^*\|_{\boldsymbol{\pi}}^2$. Since $\|\mathbf{g}_t\|_2^2 = \sum_{i=1}^N\sum_{k=1}^{n_i}\|g_{kk,t}^i\|_2^2 \le \sum_{i=1}^N\sum_{k=1}^{n_i}\|\mathbf{g}_{k,t}^i\|_2^2$, substituting the above relation, and noting that

$$\begin{aligned}
&\textstyle\sum_{i=1}^N\sum_{k=1}^{n_i} n_i^2\|\nabla_k^i\bar{\mathbf{J}}^i(\mathbf{y}_t^{(i)}) - \nabla_k^i J^i(\mathbf{x}_t)\|_2^2\\
&\quad\le \mathcal{L}^2\textstyle\sum_{i=1}^N\sum_{k=1}^{n_i}\sum_{j=1}^{n_i} n_i\|\mathbf{y}_t^{j^i} - \mathbf{x}_t\|_2^2\\
&\quad= \mathcal{L}^2\textstyle\sum_{i=1}^N\sum_{k=1}^{n_i}\sum_{j=1}^{n_i}\sum_{q=1}^n n_i\|y_{q,t}^{j^i} - \mathrm{y}_{q,t}\|_2^2\\
&\quad\le \frac{n^2\mathcal{L}^2}{\underline{C}^2}\textstyle\sum_{q=1}^n\|\mathbf{y}_{q,t} - \mathbf{1}_n\mathrm{y}_{q,t}\|_e^2,
\end{aligned}\tag{4}$$

we obtain the desired result. □

The convergence analysis of the proposed algorithm is conducted by establishing a linear system, which is composed of three major expressions: i) the total decision estimation error, ii) the total gradient tracking error, and iii) the gap between all agents' decisions and the NE. Next, we establish the inequality iterations of the three major terms in Lemmas 4, 5 and 6, respectively.

We first derive a bound on the total action estimation error characterized by $\sum_{q=1}^n\|\mathbf{1}_n\mathrm{y}_{q,t} - \mathbf{y}_{q,t}\|_e^2$.

Lemma 4: Under Assumptions 1, 2 and 3, the total action estimation error satisfies:

$$\begin{aligned}
&\textstyle\sum_{q=1}^n\|\mathbf{1}_n\mathrm{y}_{q,t+1} - \mathbf{y}_{q,t+1}\|_e^2\\
&\quad\le [\frac{1+\bar{\sigma}_2^2}{2} + \frac{3n^3\varsigma_2\overline{C}^2\iota^2\mathcal{L}^2\gamma_M^2}{\underline{C}^2}]\textstyle\sum_{q=1}^n\|\mathbf{1}_n\mathrm{y}_{q,t} - \mathbf{y}_{q,t}\|_e^2\\
&\qquad+ 3\overline{\boldsymbol{\pi}}n\varsigma_2\overline{C}^2\gamma_M^2\textstyle\sum_{i=1}^N\sum_{k=1}^{n_i}\|\mathbf{g}_{k,t}^i - A_\infty^i\mathbf{g}_{k,t}^i\|_{\boldsymbol{\pi}^i}^2\\
&\qquad+ 3\overline{\boldsymbol{\pi}}n\varsigma_2\overline{C}^2\iota^2\mathcal{L}^2\textstyle\sum_{i=1}^N n_i^3\gamma_{\underline{M}}^2\|\mathbf{x}_t - \mathbf{x}^*\|_{\boldsymbol{\pi}}^2.
\end{aligned}$$

Proof: From (1b), we have for $q\in\mathcal{V}$, $\mathbf{1}_n\mathrm{y}_{q,t+1} = \mathbf{1}_n\mathrm{y}_{q,t} - \gamma_q\mathbf{1}_n\mathbf{g}_{q,t}$, where γ_q and $\mathbf{g}_{q,t}$ denote the step-size and gradient tracker of agent q, respectively. That is, γ_j^i and $g_{jj,t}^i$ correspond to γ_q and $\mathbf{g}_{q,t}$ with $q=j^i$, if agent $j\in\mathcal{V}^i$ in cluster $i\in\mathcal{N}$ is considered. Subtracting it by (2a) and taking the vector norm $\|\cdot\|_e$ on both sides, we have

$$\begin{aligned}
\|\mathbf{1}_n\mathrm{y}_{q,t+1} - \mathbf{y}_{q,t+1}\|_e &\le \|(\bar{A} - \mathrm{diag}([\delta_1\bar{a}_{1q},\ldots,\delta_n\bar{a}_{nq}]^\top))\\
&\qquad(\mathbf{1}_n\mathrm{y}_{q,t} - \mathbf{y}_{q,t}) - \gamma_q\mathbf{1}_n\mathbf{g}_{q,t}\|_e\\
&\le \|\tilde{A}_q(\mathbf{1}_n\mathrm{y}_{q,t} - \mathbf{y}_{q,t})\|_e + \gamma_q\|\mathbf{1}_n\mathbf{g}_{q,t}\|_e\\
&\le \|\tilde{A}_q\|_E\|\mathbf{1}_n\mathrm{y}_{q,t} - \mathbf{y}_{q,t}\|_e + \sqrt{n}\overline{C}\gamma_M\|\mathbf{g}_{q,t}\|_2.
\end{aligned}$$

Square both sides, we obtain

$$\begin{aligned}
\|\mathbf{1}_n\mathrm{y}_{q,t+1} - \mathbf{y}_{q,t+1}\|_e^2 &\le \bar{\sigma}_2^2\|\mathbf{1}_n\mathrm{y}_{q,t} - \mathbf{y}_{q,t}\|_e^2 + n\gamma_M^2\|\mathbf{g}_{q,t}\|_e^2\\
&\quad+ \frac{1-\bar{\sigma}_2^2}{2}\|\mathbf{1}_n\mathrm{y}_{q,t} - \mathbf{y}_{q,t}\|_e^2 + \frac{2\bar{\sigma}_2^2 n\overline{C}^2\gamma_M^2}{1-\bar{\sigma}_2^2}\|\mathbf{g}_{q,t}\|_2^2\\
&= \frac{1+\bar{\sigma}_2^2}{2}\|\mathbf{1}_n\mathrm{y}_{q,t} - \mathbf{y}_{q,t}\|_e^2 + n\varsigma_2\overline{C}^2\gamma_M^2\|\mathbf{g}_{q,t}\|_2^2,
\end{aligned}\tag{5}$$

Summing over $q=1$ to n gives $\sum_{q=1}^n\|\mathbf{1}_n\mathrm{y}_{q,t+1} - \mathbf{y}_{q,t+1}\|_e^2 \le \frac{1+\bar{\sigma}_2^2}{2}\sum_{q=1}^n\|\mathbf{1}_n\mathrm{y}_{q,t} - \mathbf{y}_{q,t}\|_e^2 + n\varsigma_2\overline{C}^2\gamma_M^2\|\mathbf{g}_t\|_2^2$. Substituting the result in Lemma 3 completes the proof. □

Next, we bound the gap between all agents' decisions and the NE, characterized by $\|\mathbf{x}_t - \mathbf{x}^*\|_{\boldsymbol{\pi}}^2$.

Lemma 5: Under Assumptions 1, 2, 3 and 4, the agents' decisions $\mathbf{x}_t$ satisfies that

$$\begin{aligned}
\|\mathbf{x}_{t+1} - \mathbf{x}^*\|_{\boldsymbol{\pi}}^2 &\le [\frac{3\overline{\boldsymbol{\pi}}\gamma_M^2}{\underline{\boldsymbol{\pi}}} + \frac{2\gamma_M^2}{\underline{\boldsymbol{\pi}}\chi\bar{\gamma}}]\textstyle\sum_{i=1}^N\sum_{k=1}^{n_i}\|\mathbf{g}_{k,t}^i\\
&\quad- A_\infty^i\mathbf{g}_{k,t}^i\|_{\boldsymbol{\pi}^i}^2 + [1 - \underline{\boldsymbol{\pi}}\chi\bar{\gamma} + \frac{3\overline{\boldsymbol{\pi}}\iota^2\mathcal{L}^2\gamma_M^2}{\underline{\boldsymbol{\pi}}}\textstyle\sum_{i=1}^N n_i^3\\
&\quad+ 2\overline{\boldsymbol{\pi}}\sqrt{n}\mathcal{L}\epsilon_\gamma\bar{\gamma}]\|\mathbf{x}_t - \mathbf{x}^*\|_{\boldsymbol{\pi}}^2 + [\frac{3n^2\iota^2\mathcal{L}^2\gamma_M^2}{\underline{\boldsymbol{\pi}}\underline{C}^2}\\
&\quad+ \frac{2\overline{\boldsymbol{\pi}}n\mathcal{L}^2\gamma_M^2}{\underline{\boldsymbol{\pi}}\underline{C}^2\chi\bar{\gamma}}]\textstyle\sum_{q=1}^n\|\mathbf{y}_{q,t} - \mathbf{1}_n\mathrm{y}_{q,t}\|_e^2.
\end{aligned}$$

Proof: It follows from (2b) that $\mathbf{x}_{t+1} - \mathbf{x}^* = \mathbf{x}_t - \mathrm{diag}(\boldsymbol{\gamma})\mathbf{g}_t - \mathbf{x}^*$. Taking the norm on both sides gives

$$\begin{aligned}
&\|\mathbf{x}_{t+1} - \mathbf{x}^*\|_{\boldsymbol{\pi}}^2 = \|\mathbf{x}_t - \mathrm{diag}(\boldsymbol{\gamma})\mathbf{g}_t - \mathbf{x}^*\|_{\boldsymbol{\pi}}^2\\
&\quad\le \|\mathbf{x}_t - \mathbf{x}^*\|_{\boldsymbol{\pi}}^2 + \frac{\gamma_M^2}{\underline{\boldsymbol{\pi}}}\|\mathbf{g}_t\|_2^2\\
&\qquad- 2\langle\mathbf{x}_t - \mathbf{x}^*, \mathrm{diag}(\boldsymbol{\gamma})(\mathbf{g}_t - \mathrm{diag}(\boldsymbol{\pi})\boldsymbol{F}(\mathbf{x}_t))\rangle_{\boldsymbol{\pi}} &\text{(6a)}\\
&\qquad- 2\langle\mathbf{x}_t - \mathbf{x}^*, \mathrm{diag}(\boldsymbol{\gamma} - \bar{\gamma})\mathrm{diag}((\boldsymbol{\pi})\boldsymbol{F}(\mathbf{x}_t)\rangle_{\boldsymbol{\pi}} &\text{(6b)}\\
&\qquad- 2\bar{\gamma}\langle\mathbf{x}_t - \mathbf{x}^*, \mathrm{diag}(\boldsymbol{\pi})\boldsymbol{F}(\mathbf{x}_t)\rangle_{\boldsymbol{\pi}}. &\text{(6c)}
\end{aligned}$$

For (6a), it follows that

$$\begin{aligned}
&-2\langle\mathbf{x}_t - \mathbf{x}^*, \mathrm{diag}(\boldsymbol{\gamma})(\mathbf{g}_t - \mathrm{diag}(\boldsymbol{\pi})\boldsymbol{F}(\mathbf{x}_t))\rangle_{\boldsymbol{\pi}}\\
&= -2\textstyle\sum_{i=1}^N\sum_{k=1}^{n_i}\gamma_k^i\langle x_{k,t}^i - x_k^{i*}, g_{kk,t}^i - n_i\pi_k^i\nabla_k^i J^i(\mathbf{x}_t)\rangle_{n_i\pi_k^i}\\
&= -2\textstyle\sum_{i=1}^N\sum_{k=1}^{n_i}(\gamma_k^i\langle x_{k,t}^i - x_k^{i*}, g_{kk,t}^i - n_i\pi_k^i\bar{\mathbf{g}}_{k,t}^i\rangle_{n_i\pi_k^i}\\
&\quad+ \gamma_k^i\langle x_{k,t}^i - x_k^{i*}, n_i\pi_k^i(\bar{\mathbf{g}}_{k,t}^i - \nabla_k^i\bar{\mathbf{J}}^i(\mathbf{y}_t^{(i)}))\rangle_{n_i\pi_k^i}\\
&\quad+ \gamma_k^i\langle x_{k,t}^i - x_k^{i*}, n_i\pi_k^i(\nabla_k^i\bar{\mathbf{J}}^i(\mathbf{y}_t^{(i)}) - \nabla_k^i J^i(\mathbf{x}_t))\rangle_{n_i\pi_k^i}).
\end{aligned}$$

The first part holds that

$$\begin{aligned}
&-2\textstyle\sum_{i=1}^N\sum_{k=1}^{n_i}\gamma_k^i\langle x_{k,t}^i - x_k^{i*}, g_{kk,t}^i - n_i\pi_k^i\bar{\mathbf{g}}_{k,t}^i\rangle_{n_i\pi_k^i}\\
&\le \frac{\boldsymbol{\pi}\chi\bar{\gamma}}{2}\|\mathbf{x}_t - \mathbf{x}^*\|_{\boldsymbol{\pi}}^2 + \frac{2\gamma_M^2}{\underline{\boldsymbol{\pi}}\chi\bar{\gamma}}\textstyle\sum_{i=1}^N\sum_{k=1}^{n_i}\|\mathbf{g}_{k,t}^i - A_\infty^i\mathbf{g}_{k,t}^i\|_{\boldsymbol{\pi}^i}^2.
\end{aligned}$$

For the second part, it follows from (3) that $\langle x_{k,t}^i - x_k^{i*}, n_i\pi_k^i(\bar{\mathbf{g}}_{k,t}^i - \nabla_k^i\bar{\mathbf{J}}^i(\mathbf{y}_t^{(i)}))\rangle_{n_i\pi_k^i} = 0$. For the third part, we have

$$\begin{aligned}
&-2\textstyle\sum_{i=1}^N\sum_{k=1}^{n_i}\gamma_k^i\langle x_{k,t}^i - x_k^{i*}, n_i\pi_k^i(\nabla_k^i\bar{\mathbf{J}}^i(\mathbf{y}_t^{(i)})\\
&\quad- \nabla_k^i J^i(\mathbf{x}_t))\rangle_{n_i\pi_k^i} \le \frac{\boldsymbol{\pi}\chi\bar{\gamma}}{2}\|\mathbf{x}_t - \mathbf{x}^*\|_{\boldsymbol{\pi}}^2\\
&\quad+ \frac{2\overline{\boldsymbol{\pi}}\gamma_M^2}{\underline{\boldsymbol{\pi}}\chi\bar{\gamma}}\textstyle\sum_{i=1}^N\sum_{k=1}^{n_i} n_i\|\nabla_k^i\bar{\mathbf{J}}^i(\mathbf{y}_t^{(i)}) - \nabla_k^i J^i(\mathbf{x}_t)\|_2^2.
\end{aligned}$$

The last term follows the same derivation as in (4) that $\sum_{i=1}^N\sum_{k=1}^{n_i} n_i\|\nabla_k^i\bar{\mathbf{J}}^i(\mathbf{y}_t^{(i)}) - \nabla_k^i J^i(\mathbf{x}_t)\|_2^2 \le$

$\frac{n\mathcal{L}^2}{\underline{C}^2}\sum_{q=1}^{n}\|\mathbf{y}_{q,t}-\mathbf{1}_n\mathrm{y}_{q,t}\|_e^2$. Hence, combining the above three parts, we obtain that

$$\begin{aligned}&-2\bar{\gamma}\langle\mathbf{x}_t-\mathbf{x}^*,\mathbf{g}_t-\mathrm{diag}(\boldsymbol{\pi})\boldsymbol{F}(\mathbf{x}_t)\rangle_{\boldsymbol{\pi}}\\&\quad\le\underline{\boldsymbol{\pi}}\chi\bar{\gamma}\|\mathbf{x}_t-\mathbf{x}^*\|_{\boldsymbol{\pi}}^2+\tfrac{2\gamma_M^2}{\underline{\boldsymbol{\pi}}\chi\bar{\gamma}}\textstyle\sum_{i=1}^{N}\sum_{k=1}^{n_i}\|\mathbf{g}_{k,t}^i-A_\infty^i\mathbf{g}_{k,t}^i\|_{\boldsymbol{\pi}^i}^2\\&\quad+\tfrac{2\overline{\boldsymbol{\pi}}n\mathcal{L}^2\gamma_M^2}{\underline{\boldsymbol{\pi}}\underline{C}^2\chi\bar{\gamma}}\textstyle\sum_{q=1}^{n}\|\mathbf{y}_{q,t}-\mathbf{1}_n\mathrm{y}_{q,t}\|_e^2.\end{aligned}\tag{7}$$

For (6b), it follows that

$$\begin{aligned}&-2\langle\mathbf{x}_t-\mathbf{x}^*,\mathrm{diag}(\boldsymbol{\gamma}-\bar{\gamma})\mathrm{diag}((\boldsymbol{\pi})\boldsymbol{F}(\mathbf{x}_t)\rangle_{\boldsymbol{\pi}}\\&\quad\le2\sqrt{\overline{\boldsymbol{\pi}}}\|\mathbf{x}_t-\mathbf{x}^*\|_{\boldsymbol{\pi}}\|\mathrm{diag}(\boldsymbol{\gamma}-\bar{\gamma})(\boldsymbol{F}(\mathbf{x}_t)-\boldsymbol{F}(\mathbf{x}^*))\|_2\\&\quad\le2\overline{\boldsymbol{\pi}}\sqrt{n}\mathcal{L}\epsilon_\gamma\bar{\gamma}\|\mathbf{x}_t-\mathbf{x}^*\|_{\boldsymbol{\pi}}^2,\end{aligned}\tag{8}$$

since $\|\mathrm{diag}(\boldsymbol{\gamma}-\bar{\gamma})(\boldsymbol{F}(\mathbf{x}_t)-\boldsymbol{F}(\mathbf{x}^*))\|_2\le\|\boldsymbol{\gamma}-\bar{\gamma}\|_2\|\boldsymbol{F}(\mathbf{x}_t)-\boldsymbol{F}(\mathbf{x}^*)\|_2\le\sqrt{n}\mathcal{L}\epsilon_\gamma\bar{\gamma}\|\mathbf{x}_t-\mathbf{x}^*\|_2\le\sqrt{\overline{\boldsymbol{\pi}}}\sqrt{n}\mathcal{L}\epsilon_\gamma\bar{\gamma}\|\mathbf{x}_t-\mathbf{x}^*\|_{\boldsymbol{\pi}}$.

For (6c), by Assumption 4, we have

$$\begin{aligned}&-2\bar{\gamma}\langle\mathbf{x}_t-\mathbf{x}^*,\mathrm{diag}(\boldsymbol{\pi})\boldsymbol{F}(\mathbf{x}_t)\rangle_{\boldsymbol{\pi}}=-2\bar{\gamma}\langle\mathbf{x}_t-\mathbf{x}^*,\boldsymbol{F}(\mathbf{x}_t)\rangle\\&\quad\le-2\chi\bar{\gamma}\|\mathbf{x}_t-\mathbf{x}^*\|_2^2\le-2\underline{\boldsymbol{\pi}}\chi\bar{\gamma}\|\mathbf{x}_t-\mathbf{x}^*\|_{\boldsymbol{\pi}}^2.\end{aligned}\tag{9}$$

Substituting (7), (8), (9) and Lemma 3 into (6) yields the desired result. □

Finally, we quantify the total gradient tracking error, measured by $\sum_{i=1}^{N}\sum_{k=1}^{n_i}\|\mathbf{g}_{k,t}^i-A_\infty^i\mathbf{g}_{k,t}^i\|_{\boldsymbol{\pi}^i}^2$.

Lemma 6: Under Assumptions 1, 2 and 3, the total gradient tracking error $\sum_{i=1}^{N}\sum_{k=1}^{n_i}\|\mathbf{g}_{k,t}^i-A_\infty^i\mathbf{g}_{k,t}^i\|_{\boldsymbol{\pi}^i}^2$ satisfies

$$\begin{aligned}&\textstyle\sum_{i=1}^{N}\sum_{k=1}^{n_i}\|\mathbf{g}_{k,t+1}^i-A_\infty^i\mathbf{g}_{k,t+1}^i\|_{\boldsymbol{\pi}^i}^2\le[\frac{1+\bar{\sigma}_1^2}{2}\\&\quad+\tfrac{9\overline{\boldsymbol{\pi}}n^2\varsigma_1(1+n\varsigma_2\overline{C}^2)\mathcal{L}^2\gamma_M^2}{\underline{\boldsymbol{\pi}}}]\textstyle\sum_{i=1}^{N}\sum_{k=1}^{n_i}\|\mathbf{g}_{k,t}^i-A_\infty^i\mathbf{g}_{k,t}^i\|_{\boldsymbol{\pi}^i}^2\\&\quad+[\tfrac{9n^4\varsigma_1(1+n\varsigma_2\overline{C}^2)\iota^2\mathcal{L}^4\gamma_M^2}{\underline{\boldsymbol{\pi}}\underline{C}^2}+\tfrac{3n\varsigma_1(3+\bar{\sigma}_2^2)\mathcal{L}^2}{2\underline{\boldsymbol{\pi}}\underline{C}^2}]\textstyle\sum_{q=1}^{n}\|\mathbf{y}_{q,t}\\&\quad-\mathbf{1}_n\mathrm{y}_{q,t}\|_e^2+\tfrac{9\overline{\boldsymbol{\pi}}n^2\varsigma_1(1+n\varsigma_2\overline{C}^2)\iota^2\mathcal{L}^4\gamma_M^2}{\underline{\boldsymbol{\pi}}}\textstyle\sum_{i=1}^{N}n_i^3\|\mathbf{x}_t-\mathbf{x}^*\|_{\boldsymbol{\pi}}^2.\end{aligned}$$

Proof: It is obtained from (2c) that

$$\begin{aligned}&\|\mathbf{g}_{k,t+1}^i-A_\infty^i\mathbf{g}_{k,t+1}^i\|_{\boldsymbol{\pi}^i}^2=\|A^i\mathbf{g}_{k,t}^i-A_\infty^i\mathbf{g}_{k,t}^i\|_{\boldsymbol{\pi}^i}^2+\|(I_{n_i}\\&-A_\infty^i)(\nabla_k^i\mathbf{J}^i(\mathbf{y}_{t+1}^{(i)})-\nabla_k^i\mathbf{J}^i(\mathbf{y}_t^{(i)}))\|_{\boldsymbol{\pi}^i}^2+2\langle A^i\mathbf{g}_{k,t}^i\\&-A_\infty^i\mathbf{g}_{k,t}^i,(I_{n_i}-A_\infty^i)(\nabla_k^i\mathbf{J}^i(\mathbf{y}_{t+1}^{(i)})-\nabla_k^i\mathbf{J}^i(\mathbf{y}_t^{(i)}))\rangle_{\boldsymbol{\pi}^i}.\end{aligned}$$

It is noted that $\|I_{n_i}-A_\infty^i\|_{\boldsymbol{\pi}^i}=1$, then

$$\begin{aligned}&\|\mathbf{g}_{k,t+1}^i-A_\infty^i\mathbf{g}_{k,t+1}^i\|_{\boldsymbol{\pi}^i}^2\le\bar{\sigma}_1^2\|\mathbf{g}_{k,t}^i-A_\infty^i\mathbf{g}_{k,t}^i\|_{\boldsymbol{\pi}^i}^2\\&\quad+\|\nabla_k^i\mathbf{J}^i(\mathbf{y}_{t+1}^{(i)})-\nabla_k^i\mathbf{J}^i(\mathbf{y}_t^{(i)})\|_{\boldsymbol{\pi}^i}^2+2\|A^i\mathbf{g}_{k,t}^i\\&\quad-A_\infty^i\mathbf{g}_{k,t}^i\|_{\boldsymbol{\pi}^i}\|\nabla_k^i\mathbf{J}^i(\mathbf{y}_{t+1}^{(i)})-\nabla_k^i\mathbf{J}^i(\mathbf{y}_t^{(i)})\|_{\boldsymbol{\pi}^i}\\&\le\tfrac{1+\bar{\sigma}_1^2}{2}\|\mathbf{g}_{k,t}^i-A_\infty^i\mathbf{g}_{k,t}^i\|_{\boldsymbol{\pi}^i}^2\\&\quad+\varsigma_1\|\nabla_k^i\mathbf{J}^i(\mathbf{y}_{t+1}^{(i)})-\nabla_k^i\mathbf{J}^i(\mathbf{y}_t^{(i)})\|_{\boldsymbol{\pi}^i}^2,\end{aligned}\tag{10}$$

where

$$\begin{aligned}&\|\nabla_k^i\mathbf{J}^i(\mathbf{y}_{t+1}^{(i)})-\nabla_k^i\mathbf{J}^i(\mathbf{y}_t^{(i)})\|_{\boldsymbol{\pi}^i}^2\le\tfrac{\mathcal{L}^2}{\underline{\boldsymbol{\pi}}}\textstyle\sum_{j=1}^{n_i}\|\mathbf{y}_{t+1}^{j^i}-\mathbf{y}_t^{j^i}\|_2^2\\&\quad=\tfrac{\mathcal{L}^2}{\underline{\boldsymbol{\pi}}}\textstyle\sum_{q=1}^{n}\sum_{j=1}^{n_i}\|y_{q,t+1}^{j^i}-y_{q,t}^{j^i}\|_2^2.\end{aligned}$$

Hence,

$$\begin{aligned}&\textstyle\sum_{i=1}^{N}\sum_{k=1}^{n_i}\|\nabla_k^i\mathbf{J}^i(\mathbf{y}_{t+1}^{(i)})-\nabla_k^i\mathbf{J}^i(\mathbf{y}_t^{(i)})\|_{\boldsymbol{\pi}^i}^2\le\frac{n\mathcal{L}^2}{\underline{\boldsymbol{\pi}}}\sum_{q=1}^{n}\\&\quad\|\mathbf{y}_{q,t+1}-\mathbf{y}_{q,t}\|_2^2\le\tfrac{3n\mathcal{L}^2}{\underline{\boldsymbol{\pi}}}\textstyle\sum_{q=1}^{n}(\|\mathbf{y}_{q,t+1}-\mathbf{1}_n\mathrm{y}_{q,t+1}\|_2^2\\&\quad+\|\mathbf{y}_{q,t}-\mathbf{1}_n\mathrm{y}_{q,t}\|_2^2+\|\mathbf{1}_n\mathrm{y}_{q,t+1}-\mathbf{1}_n\mathrm{y}_{q,t}\|_2^2)\\&\le\tfrac{3n\mathcal{L}^2}{\underline{\boldsymbol{\pi}}}\textstyle\sum_{q=1}^{n}(\frac{3+\bar{\sigma}_2^2}{2\underline{C}^2}\|\mathbf{y}_{q,t}-\mathbf{1}_n\mathrm{y}_{q,t}\|_e^2\\&\quad+\|\mathbf{1}_n\mathrm{y}_{q,t+1}-\mathbf{1}_n\mathrm{y}_{q,t}\|_2^2+n^2\varsigma_2\overline{C}^2\gamma_M^2\|\mathbf{g}_{q,t}\|_2^2),\end{aligned}$$

where the last inequality follows from (5). It is noted that $\sum_{q=1}^{n}\|\mathbf{1}_n\mathrm{y}_{q,t+1}-\mathbf{1}_n\mathrm{y}_{q,t}\|_2^2=n\|\mathbf{x}_{t+1}-\mathbf{x}_t\|_2^2=n\|\mathbf{x}_t-\mathrm{diag}(\boldsymbol{\gamma})\mathbf{g}_t-\mathbf{x}_t\|_2^2\le n\gamma_M^2\|\mathbf{g}_t\|_2^2$. Summing over $k=1$ to n_i, $i=1$ to N for (10) and substituting the above result and Lemma 3 complete the proof. □

B. Main Results

Now, we are ready for the analysis on the convergence of the proposed algorithm. With the inequality iterations derived in Lemmas 4, 5 and 6, we can establish the following linear dynamical system

$$\mathbf{u}_{t+1}\le\mathbf{T}\mathbf{u}_t,\tag{11}$$

with

$$\mathbf{u}_t\triangleq\begin{bmatrix}\|\mathbf{x}_t-\mathbf{x}^*\|_{\boldsymbol{\pi}}^2\\\sum_{i=1}^{N}\sum_{k=1}^{n_i}\|\mathbf{g}_{k,t}^i-A_\infty^i\mathbf{g}_{k,t}^i\|_{\boldsymbol{\pi}^i}^2\\\sum_{q=1}^{n}\|\mathbf{y}_{q,t}-\mathbf{1}_n\mathrm{y}_{q,t}\|_e^2\end{bmatrix},$$

$$\mathbf{T}\triangleq\begin{bmatrix}\begin{matrix}1-k_1\bar{\gamma}+k_2\gamma_M^2\\+k_3\epsilon_\gamma\bar{\gamma}\end{matrix}&k_4\gamma_M^2+k_5\gamma_M^2/\bar{\gamma}&k_6\gamma_M^2+k_7\gamma_M^2/\bar{\gamma}\\k_8\gamma_M^2&1-k_{10}+k_9\gamma_M^2&k_{11}+k_{12}\gamma_M^2\\k_{13}\gamma_M^2&k_{14}\gamma_M^2&1-k_{16}+k_{15}\gamma_M^2\end{bmatrix},$$

where $k_1\triangleq\underline{\boldsymbol{\pi}}\chi$, $k_2\triangleq\frac{3\overline{\boldsymbol{\pi}}\iota^2\mathcal{L}^2}{\underline{\boldsymbol{\pi}}}\sum_{i=1}^{N}n_i^3$, $k_3\triangleq2\overline{\boldsymbol{\pi}}\sqrt{n}\mathcal{L}$, $k_4\triangleq\frac{3\overline{\boldsymbol{\pi}}}{\underline{\boldsymbol{\pi}}}$, $k_5\triangleq\frac{2}{\underline{\boldsymbol{\pi}}\chi}$, $k_6\triangleq\frac{3n^2\iota^2\mathcal{L}^2}{\underline{\boldsymbol{\pi}}\underline{C}^2}$, $k_7\triangleq\frac{2\overline{\boldsymbol{\pi}}n\mathcal{L}^2}{\underline{\boldsymbol{\pi}}\underline{C}^2\chi}$, $k_8\triangleq\frac{9\overline{\boldsymbol{\pi}}n^2\varsigma_1(1+n\varsigma_2\overline{C}^2)\iota^2\mathcal{L}^4}{\underline{\boldsymbol{\pi}}}\sum_{i=1}^{N}n_i^3$, $k_9\triangleq\frac{9\overline{\boldsymbol{\pi}}n^2\varsigma_1(1+n\varsigma_2\overline{C}^2)\mathcal{L}^2}{\underline{\boldsymbol{\pi}}}\sum_{i=1}^{N}n_i^3$, $k_{10}\triangleq\frac{1-\bar{\sigma}_1^2}{2}$, $k_{11}\triangleq\frac{3n\varsigma_1(3+\bar{\sigma}_2^2)\mathcal{L}^2}{2\underline{\boldsymbol{\pi}}\underline{C}^2}$, $k_{12}\triangleq\frac{9n^4\varsigma_1(1+n\varsigma_2\overline{C}^2)\iota^2\mathcal{L}^4}{\underline{\boldsymbol{\pi}}\underline{C}^2}$, $k_{13}\triangleq3\overline{\boldsymbol{\pi}}n\varsigma_2\overline{C}^2\iota^2\mathcal{L}^2\sum_{i=1}^{N}n_i^3$, $k_{14}\triangleq3\overline{\boldsymbol{\pi}}n\varsigma_2\overline{C}^2$, $k_{15}\triangleq\frac{3n^3\varsigma_2\overline{C}^2\iota^2\mathcal{L}^2}{\underline{C}^2}$, $k_{16}\triangleq\frac{1-\bar{\sigma}_2^2}{2}$.

Then, the convergence results of all agents' decisions to the NE $\mathbf{x}^*$ can be established based on the convergence of the dynamical system (11), which are summarized in the following theorem.

Theorem 1: Suppose Assumptions 1, 2, 3 and 4 hold. Generate the agent's action $\{x_{j,t}^i\}_{t\ge0}$, gradient tracker $\{g_{jk,t}^i\}_{t\ge0}$ and estimation variable $\{y_{q,t}^p\}_{t\ge0}$ by (1) with the non-uniform constant step-size γ_j^i satisfying

$$0\le\epsilon_\gamma<\tfrac{k_1}{k_3},0<\gamma_M<\min\{\tfrac{1}{k_1},\gamma_1^*,\gamma_2^*,\gamma_3^*\},$$

where γ_1^*,γ_2^* and γ_3^* are some constants related to the heterogeneity ϵ_γ. Then, all players' decisions $\{\mathbf{x}_t\}_{t\ge0}$ converge to their corresponding NE $\mathbf{x}^*$ at a rate of $\rho(\mathbf{T})$.

Proof: For system (11), if $\rho(\mathbf{T})<1$, then $\mathbf{T}^t$ converges to $\mathbf{0}$ at a geometric rate with exponent $\rho(\mathbf{T})$ [22], which implies that the supremum of each individual component converges to 0 with the same rate.

Lemma 7: (see [22, Cor. 8.1.29]) Let $A \in \mathbb{R}^{m\times m}$ be a matrix with non-negative entries and $\boldsymbol{\theta} \in \mathbb{R}^m$ be a vector with positive entries. If there exists a constant $\lambda \geq 0$ such that $A\boldsymbol{\theta} < \lambda\boldsymbol{\theta}$, then $\rho(A) < \lambda$.

To invoke Lemma 7, the matrix $\mathbf{T}$ has to be non-negative. Thus, it suffices to have $0 < \bar{\gamma} \leq \frac{1}{k_1}$. According to Lemma 7, to ensure $\rho(\mathbf{T}) < 1$, one needs to seek for some positive vector $\boldsymbol{\theta} \triangleq [\theta_1, \theta_2, \theta_3]^\top$, where $\theta_1 > 0$, $\theta_2 > 0$ and $\theta_3 > 0$, such that $\mathbf{T}\boldsymbol{\theta} < \boldsymbol{\theta}$. Without the loss of generality, we can set $\theta_3 = 1$. Then we obtain

$$\begin{cases} (k_2\theta_1 + k_4\theta_2 + k_6)\bar{\gamma} \\ \qquad < (k_1 - k_3\epsilon_\gamma)\theta_1\bar{\gamma}^2/\gamma_M^2 - (k_5\theta_2 + k_7), \\ (k_8\theta_1 + k_9\theta_2 + k_{12})\gamma_M^2 < k_{10}\theta_2 - k_{11}, \\ (k_{13}\theta_1 + k_{14}\theta_2 + k_{15})\gamma_M^2 < k_{16}. \end{cases} \tag{12}$$

Therefore, we would like to find the range of the step-size such that (12) hold simultanenously for some $\theta_1 > 0$, $\theta_2 > 0$. To ensure the existence of solution γ_M and $\bar{\gamma}$, the right-hand-side of (12) needs to be positive, *i.e.*, $\epsilon_\gamma < \frac{k_1}{k_3}, \theta_1 > \frac{(k_5\theta_2+k_7)\gamma_M^2/\bar{\gamma}^2}{k_1-k_3\epsilon_\gamma}, \theta_2 > \frac{k_{11}}{k_{10}}$. Thus, we can set $\theta_1 = \frac{2(k_5k_{11}+k_7k_{10})\gamma_M^2/\bar{\gamma}^2}{k_{10}(k_1-k_3\epsilon_\gamma)}, \theta_2 = \frac{2k_{11}}{k_{10}}$.

Then, we can solve the three inequalities in (12) respectively by noting that $\bar{\gamma} \leq \gamma_M$ and $\gamma_M/\bar{\gamma} < n$. We obtain $\gamma_M < \gamma_1^*$, $\gamma_M < \gamma_2^*$, $\gamma_M < \gamma_3^*$, where $\gamma_1^* \triangleq \frac{k_7k_{10}(k_1-k_3\epsilon_\gamma)}{q_1^*}$, $\gamma_2^* \triangleq \sqrt{\frac{k_{10}k_{11}(k_1-k_3\epsilon_\gamma)}{q_2^*}}$, $\gamma_3^* \triangleq \sqrt{\frac{k_{10}k_{16}(k_1-k_3\epsilon_\gamma)}{q_3^*}}$, $q_1^* \triangleq 2n^2k_2(k_5k_{11}+k_7k_{10}) + (2k_4k_{11}+k_6k_{10})(k_1-k_3\epsilon_\gamma)$, $q_2^* \triangleq 2n^2k_8(k_5k_{11}+k_7k_{10}) + (2k_9k_{11}+k_{10}k_{12})(k_1-k_3\epsilon_\gamma)$, $q_3^* \triangleq 2n^2k_{13}(k_5k_{11}+k_7k_{10}) + (2k_{11}k_{14}+k_{10}k_{15})(k_1-k_3\epsilon_\gamma)$. $\square$

Remark 2: Theorem 1 shows that the linear convergence of all agents' decisions to the NE is guaranteed when both the largest step-size and its heterogeneity are sufficiently small. Besides, it should be remarked that the bounds on both the largest step-size and heterogeneity are only sufficient but not necessary conditions for the convergence results. For the upper bound on the step-size, checking the conditions may still require some global information that is difficult to compute in a distributed manner. We remark the derived upper bound is just a sufficient condition (not necessary) to ensure the convergence to the NE. Hence, the bound is not tight, and better bounds may be obtained with better choices in the analysis.

Next, we analyze the convergence of the algorithm when all agents adopt uniform constant step-size, summarized in the following corollary.

Corollary 1: Suppose Assumptions 1, 2, 3 and 4 hold. Generate the agent's action $\{x_{j,t}^i\}_{t\geq 0}$, gradient tracker $\{g_{jk,t}^i\}_{t\geq 0}$ and estimation variable $\{y_{q,t}^p\}_{t\geq 0}$ by (1) with uniform constant step-size γ satisfying

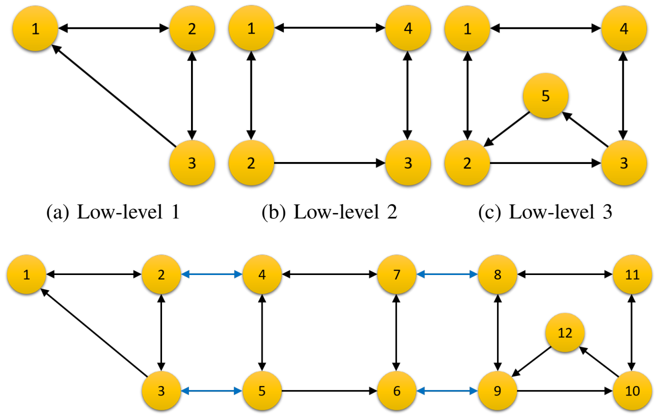
$$0 < \gamma < \min\{\tfrac{1}{k_1}, \gamma_{1,c}^*, \gamma_{2,c}^*, \gamma_{3,c}^*\},$$

where $\gamma_{1,c}^*, \gamma_{2,c}^*$ and $\gamma_{3,c}^*$ are some constants. Then, all players' decisions $\{\mathbf{x}_t\}_{t\geq 0}$ converge to their corresponding NE $\mathbf{x}^*$ at a rate of $\rho(\mathbf{T})$.

Proof: It directly follows from Theorem 1 by noting that $\epsilon_\gamma = 0$ and $\gamma_M/\bar{\gamma} = 1$, which gives

$$\gamma_{1,c}^* \triangleq \frac{k_1k_7k_{10}}{2k_2k_5k_{11}+2k_2k_7k_{10}+2k_1k_4k_{11}+k_1k_6k_{10}},$$
$$\gamma_{2,c}^* \triangleq \sqrt{\frac{k_1k_{10}k_{11}}{2k_5k_8k_{11}+2k_7k_8k_{10}+2k_1k_9k_{11}+k_1k_{10}k_{12}}},$$
$$\gamma_{3,c}^* \triangleq \sqrt{\frac{k_1k_{10}k_{16}}{2k_5k_{11}k_{13}+2k_7k_{10}k_{13}+2k_1k_{11}k_{14}+k_1k_{10}k_{15}}}.$$

Then, following the same arguments in Theorem 1, when $0 < \gamma < \min\{\frac{1}{k_1}, \gamma_{1,c}^*, \gamma_{2,c}^*, \gamma_{3,c}^*\}$, the convergence of the algorithm is guaranteed. $\square$



(a) Low-level 1 (b) Low-level 2 (c) Low-level 3

(d) High-level network

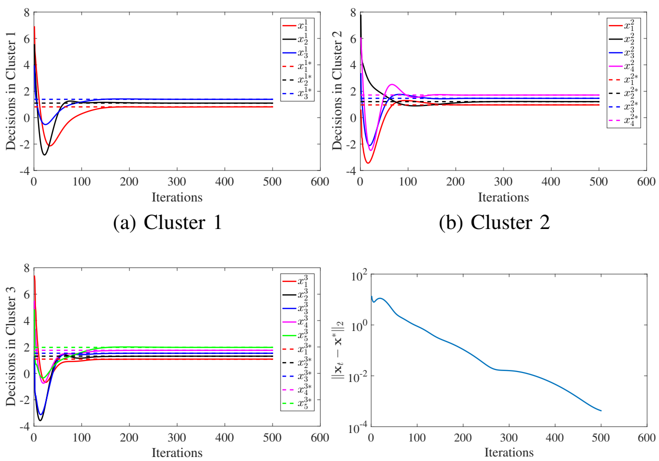
Fig. 1. Graph topology of two level networks.

V. Numerical Simulations

In this section, we validate the performance of the proposed algorithm by a Cournot competition game. In particular, we consider N firms, and each firm $i \in \mathcal{N}$ consists of n_i branches to help produce goods. For $j \in \mathcal{V}^i, i \in \mathcal{N}$, let x_j^i be the quantity of goods produced by branch j of firm i, then its local cost function $J_j^i(\mathbf{x})$ is modeled by the following function $J_j^i(\mathbf{x}) = c_j^i(x_j^i) - p_j^i(\mathbf{x})x_j^i$, where $c_j^i(x_j^i) = a_j^i(x_j^i)^2 + b_j^i(x_j^i)$ models the cost incurred by generating x_j^i quantity of goods, $p_j^i(\mathbf{x}) = d_j^i - w_j^{i\top}\mathbf{x}$ models the selling price of such goods, $a_j^i, b_j^i, d_j^i \in \mathbb{R}$ and $w_j^i \in \mathbb{R}^n$ are constant parameters. As a numerical setting, we set $N = 3$, $n_i = 3, 4$ and 5, respectively. For constant parameters, we let $a_j^i = 1$, $d_j^i = 10 + i + j$, b_j^i and each element of w_j^i be uniformly drawn from $[0, 1]$, respectively. The two level networks are given in Fig. 1, which are strongly connected. The inital conditions of $\mathbf{x}$ and $\mathbf{y}_q$ are set to some arbitrary values, and $\delta_q = 0.5$, $\forall q \in \bar{\mathcal{V}}$.

A. Algorithm Convergence

In this part, we focus on the verification of the convergence result derived in Theorem 1. The step-size γ_j^i is evenly selected from $[0.045, 0.1]$, giving a heterogeneity of 0.2381. Then, the trajectories of the decisions of all firms (and branches) and the NE gap $\|\mathbf{x}_t - \mathbf{x}^*\|_2$ are plotted in Fig. 2. As can be seen, the convergence to the NE is obtained and the rate is linear.



(a) Cluster 1 (b) Cluster 2

(c) Cluster 3 (d) NE gap

Fig. 2. Trajectories of agents' decisions in different clusters.

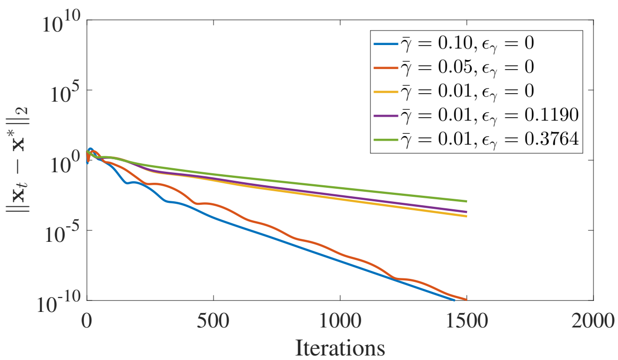


Fig. 3. Influence of the step-size and heterogeneity on the rate of convergence.

B. Influence of step-size on the convergence

In this part, we investigate the influence of the step-size including the heterogeneity on the convergence. Specifically, we let the agents' step-sizes be selected within $(0, 0.1]$. The initial conditions of $\mathbf{x}$ and $\mathbf{y}_q$ are set to zero, while the rest of the parameters are kept the same as in Sec. V-A. Fig. 3 plots the NE gaps under various step-size cases with different averaged step-size and different heterogeneity. Three cases for uniform step-size $\epsilon_\gamma = 0$ are also included for comparison. As can be seen, smaller heterogeneity of the step-size and larger averaged step-size lead to a faster rate of convergence.

VI. Conclusions

This paper has considered the N-cluster game under partial-decision information settings, where a distributed Nash equilibrium (NE) seeking algorithm has been proposed with non-uniform constant step-sizes among all agents. It has been shown that all agents' decisions linearly converge to their corresponding NE when the largest step-size and the heterogeneity of the step-size are small.

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