

A Compositional Approach to Certifying Almost Global Asymptotic Stability of Cascade Systems

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Abstract—In this work, we give sufficient conditions for the almost global asymptotic stability of a cascade in which the subsystems are only almost globally asymptotically stable. The result is extended to upper triangular systems of arbitrary size. In particular, if the unforced subsystems are almost globally asymptotically stable and their only chain recurrent points are hyperbolic equilibria, then the boundedness of forward trajectories is sufficient for the almost global asymptotic stability of the full upper triangular system. We show that unboundedness of such cascades is prohibited by growth rate conditions on the interconnection term and a Lyapunov function for the unforced outer subsystem, and the required structure for the chain recurrent set is enjoyed by classes of systems common in geometric control e.g. dissipative mechanical systems. Our results stand in contrast to prior works that require either time scale separation, strong disturbance robustness properties, or global asymptotic stability in the subsystems.

I. INTRODUCTION

In this work, we are interested in cascades of the form

$$\dot{x} = f(x, y), \quad (1a)$$

$$\dot{y} = g(y), \quad (1b)$$

depicted graphically in in Fig. 1 as system Σ . Throughout, we let x and y evolve on X and Y , which are smooth, connected, complete Riemannian manifolds without boundary (see Remark 1 for an explanation of the choice of setting).

Cascades appear in many interesting and important physical systems. Many underactuated mechanical systems can be rendered as a cascade after a feedback transformation [1], and cascades arise often in robotic systems, either intrinsically or after control design [2]. A long research tradition has studied the implications of cascade structure to simplify analysis and aid in control design [3]. This compositional approach is motivated by the observation that control design for a subsystem is typically easier, due to e.g. lower dimensionality, lower relative degree, or full actuation. However, one must ensure that the recombined system achieves the desired behavior, since (in contrast to linear systems) only *local* asymptotic stability is preserved under cascades for general nonlinear systems [4].

A. Prior Work on Cascade Stability

Approaches to stability certification for nonlinear cascades have exploited a range of structural features. Singular perturbation techniques [5] assume a time scale separation between

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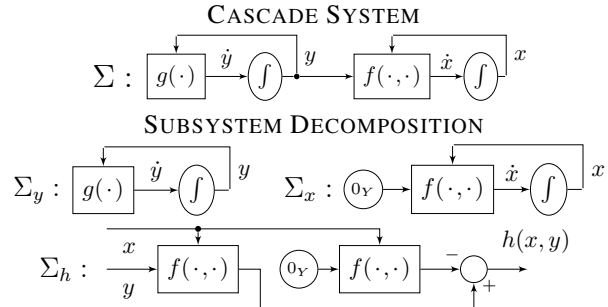


Fig. 1. We give sufficient conditions for the almost global asymptotic stability of a cascade system Σ in terms of qualitative properties of the “inner subsystem” Σ_y and the “unforced outer subsystem” Σ_x , as well as growth rate criteria on the “interconnection term” Σ_h and on a Lyapunov function for Σ_x . Above, 0_Y is the stable equilibrium of Σ_y .

the “fast” inner subsystem and the “slow” outer subsystem, and show that a system’s behavior tends toward that of a “reduced” system as the ratio between convergence rates tends to zero. However, this approach necessitates rapid inner subsystem convergence, which can be challenging to achieve for a control system with realistic input limitations. Another popular approach relies on the robustness of the outer subsystem to disturbances, leveraging the property of *input to state stability*, which roughly requires the asymptotic response of the system under a disturbance input to be bounded by the size of the input (and therefore also implies global asymptotic stability of the system in the absence of disturbances). A classic result then establishes the global asymptotic stability of a cascade for which the outer subsystem is input to state stable and the inner subsystem is globally asymptotically stable [6]. Methods avoiding robustness or time scale assumptions have relied on local exponential stability of the inner subsystem as well as growth restrictions on the “interconnection term” and a suitable Lyapunov function for the unforced outer subsystem to certify asymptotic stability of cascades with globally asymptotically stable subsystems [4].

These global results have important applications, but their utility in geometric control is rather limited. This is because any manifold admitting a smooth globally asymptotically stable vector field is diffeomorphic to \mathbb{R}^n [7], while the state space of many physical systems (e.g. free-flying robots) is not [8]. The most one can achieve in the smooth non-Euclidean setting, for either the subsystems or the cascade, is *almost* global asymptotic stability. This has motivated an almost global notion of input to state stability [9], in which an asymptotic gain holds for all but a measure zero set of initial conditions; a cascade is then guaranteed to be almost globally asymptotically stable if its outer subsystem is almost globally input to state stable and its inner subsystem is almost globally

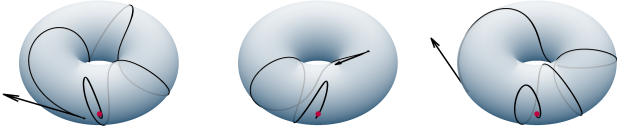


Fig. 2. Trajectories through a sampling of initial conditions for the motivating example (2a)-(2b), projected down to \mathbb{T}^2 from the full state space $T\mathbb{T}^2 = TS^1 \times TS^1$, where the “small” axis of the torus corresponds to θ and the “large” axis corresponds to ϕ . All sampled trajectories converge to $(0, 0, 0, 0) \in T\mathbb{T}^2$, marked in red. Despite the highly non-local and topologically complex behavior of the trajectories, our results certify the almost global asymptotic stability of the system, without the need to guess an explicit Lyapunov function for the full cascade.

asymptotically stable. While verifying almost global input to state stability can be challenging, this can be done under conditions on the exponential instability of other equilibria as well as the “ultimate boundedness” of trajectories of the system even when subjected to arbitrary disturbances [10].

However, not all almost globally asymptotically stable cascades have an outer subsystem enjoying this property; indeed, almost global input to state stability seems to be an inherently stricter property than necessary, since it characterizes the response of the system to arbitrary disturbances, while for our purposes, the outer subsystem is almost always subjected to a converging disturbance. Yet, the lack of a comprehensive understanding of such systems has required bespoke stability certificates for almost globally asymptotically stable cascaded controllers in practice, inhibiting generalization; for example, a single Lyapunov function for the full cascade may be handcrafted via human intuition, even though the cascaded structure inspired the control design in the first place [2].

B. Motivating Example

To motivate our investigation, we present a simple representative example. Consider a cascade of the form (1a)-(1b) evolving on the tangent bundle of \mathbb{T}^2 , given by

$$\ddot{\theta} = -(\sin \theta + \dot{\theta}) \cos 2\phi, \quad (2a)$$

$$\ddot{\phi} = -(\sin \phi + \dot{\phi}), \quad (2b)$$

where $x = (\theta, \dot{\theta}) \in TS^1$, $y = (\phi, \dot{\phi}) \in TS^1$, and we make the identification $TS^1 \cong S^1 \times \mathbb{R}$ for notational convenience. A sampling of system trajectories is shown in Fig. 2. In fact, (2b) describes a damped pendulum with total energy given by $W : (\phi, \dot{\phi}) \mapsto 1 - \cos \phi + \frac{1}{2} \dot{\phi}^2$, and using W as a LaSalle function it can be shown that that $(\phi, \dot{\phi}) = (0, 0)$ is almost globally asymptotically stable for the inner subsystem (2b). By the same reasoning, $(\theta, \dot{\theta}) = (0, 0)$ is also almost globally asymptotically stable for the restriction of the outer subsystem (2a) to the stable equilibrium of the inner subsystem.

It turns out that the entire cascade (2a)-(2b) is almost globally asymptotically stable, but the system does not satisfy the hypotheses of any of the previously discussed results. In particular, the subsystems are not globally asymptotically stable, nor is there a time scale separation between the subsystems. Furthermore, viewing $(\phi, \dot{\phi})$ as a disturbance to (2a), it can be seen that the outer subsystem is *not* almost globally input to state stable [9, Def. 2.1], since the response

to the bounded disturbance $(\phi, \dot{\phi}) = (\frac{\pi}{2}, 0)$ grows unbounded from almost all initial conditions. Nonetheless, the results of this paper will guarantee the almost global asymptotic stability of a class of systems that includes (2a)-(2b).

In what follows, we consider cascade systems of the form Σ in Fig. 1 and use properties of the decomposed subsystems shown to certify its almost global asymptotic stability. In Sec. II, we present the main results, which show that when Σ_x and Σ_y are almost globally asymptotically stable and locally exponentially stable and all chain recurrent points of Σ_x are hyperbolic equilibria, then forward boundedness of trajectories is sufficient for almost global asymptotic stability and local exponential stability of the cascade. We then extend the result inductively to upper triangular systems with arbitrarily many subsystems. In Sec. III, we explore the hypotheses of the main results, discussing some important classes of systems enjoying the stated chain recurrence property and also showing that forward boundedness can be verified via growth rate criteria on Σ_h and on a Lyapunov function for Σ_x . In Sec. IV, we apply our results to the motivating example, before discussing the results and concluding the paper in Secs. V and VI.

II. ALMOST GLOBAL ASYMPTOTIC STABILITY

We begin with a brief review of relevant concepts from dynamical systems theory, adopting the definitions of [11], whose perspective on the behavior of asymptotically autonomous semiflows is a central ingredient of our approach.

Definition 1. A *nonautonomous semiflow* on a smooth Riemannian manifold (M, μ) is a continuous map

$$\Phi : \{(t, s) : 0 \leq s \leq t < \infty\} \times M \rightarrow M \quad (3)$$

such that $\Phi(s, s, x) = x$ and $\Phi(t, s, \Phi(s, r, x)) = \Phi(t, r, x)$ for all $t \geq s \geq r > 0$. A semiflow is called *autonomous* when additionally, $\Phi(t + r, s + r, x) = \Phi(t, s, x)$ for all $r > 0$.

In the previous, the parameters s and t can be thought of as respective “start” and “end” times. Hereafter, we will use the shorthands $\Phi^{(t,s)} : x \mapsto \Phi(t, s, x)$ for nonautonomous semiflows and $\Phi^t : x \mapsto \Phi(t, 0, x)$ for autonomous semiflows. Note that nonautonomous semiflows can be generated by “time-varying” vector fields, while autonomous semiflows can be generated by “time-invariant” vector fields.

Definition 2. A nonautonomous semiflow Φ is *asymptotically autonomous with limit semiflow* Θ if, for any sequences $t_j \rightarrow t$, $s_j \rightarrow \infty$, and $x_j \rightarrow x$, it holds that

$$\Phi^{(t_j+s_j, s_j)}(x_j) \rightarrow \Theta^t(x) \text{ as } j \rightarrow \infty. \quad (4)$$

Definition 3. The *equilibrium set* of an autonomous semiflow Φ is given by $\mathcal{E}(\Phi) = \{x \in M : \Phi^t(x) = x \forall t \geq 0\}$. An equilibrium $0_M \in \mathcal{E}(\Phi)$ is *hyperbolic* if its linearization has no purely imaginary eigenvalues [12, p. 149]. Furthermore, 0_M is *almost globally asymptotically stable* if it is (locally) asymptotically stable and its *basin of attraction*, denoted $\mathcal{B}(0_M) \subseteq M$, is full measure and residual, i.e. the complement of $\mathcal{B}(0_M)$ is measure zero [13, p.128] and meager (a countable union of nowhere dense sets) in M .

Definition 4. For an autonomous semiflow Φ on (M, μ) and constants $\varepsilon, T > 0$, an (ε, T) -chain is a pair of finite sequences (x_0, x_1, \dots, x_n) and (t_1, t_2, \dots, t_n) satisfying

$$\text{dist}(\Phi^{t_i}(x_{i-1}), x_i) < \varepsilon \text{ and } t_i > T, \quad i = 1, 2, \dots, n, \quad (5)$$

where $\text{dist} : M \times M \rightarrow \mathbb{R}$ is the distance induced by μ . The chain recurrent set $\mathcal{R}(\Phi)$ consists of all points x for which an (ε, T) -chain with $x = x_0 = x_n$ exists for every $\varepsilon, T > 0$.

The chain recurrent set plays a central role in the topological structure of a dynamical system [12]. Our present interest in chain recurrence will revolve around the following fact.

Fact 1 (Mischaikow, Smith, and Thieme [11]). *Precompact forward trajectories of an asymptotically autonomous semiflow converge to the chain recurrent set of the limit semiflow.*

Remark 1. We define chain recurrence using (ε, T) -chains with respect to a distance function and some $\varepsilon > 0$ because we rely on the results of [11], in which the same choice is made. A Riemannian distance function is appropriate, since then an asymptotically stable equilibrium is locally exponentially stable if and only if it is hyperbolic. In a complete Riemannian manifold (our chosen setting), a set is precompact (i.e. has compact closure) if and only if it is bounded, a more familiar notion in the control community, which will also play a convenient role in our second theorem.

A. Main Results

Theorem 1 (Almost Global Asymptotic Stability of a Cascade). *Consider a cascade on $X \times Y$ given by*

$$\dot{x} = f(x, y), \quad (6a)$$

$$\dot{y} = g(y), \quad (6b)$$

where (6b) and the unforced outer subsystem

$$\dot{x} = f(x, 0_Y) \quad (7)$$

are almost globally asymptotically stable with respect to $0_Y \in Y$ and $0_X \in X$ respectively, and moreover 0_Y and all chain recurrent points of (7) are hyperbolic equilibria. Then, (6a)-(6b) is almost globally asymptotically stable and locally exponentially stable with respect to $(0_X, 0_Y)$ as long as any forward trajectory starting in $X \times \mathcal{B}(0_Y)$ is bounded.

Proof. Since $X \times \mathcal{B}(0_Y)$ is invariant for (6a)-(6b) and all forward trajectories beginning in $X \times \mathcal{B}(0_Y)$ are bounded, the cascade induces an autonomous semiflow

$$\Psi^t : X \times \mathcal{B}(0_Y) \rightarrow X \times \mathcal{B}(0_Y). \quad (8)$$

Similarly, (7) and (6b) induce the autonomous semiflows

$$\Theta^t : X \rightarrow X, \quad x_0 \mapsto \text{pr}_1 \circ \Psi^t(x_0, 0_Y), \quad (9a)$$

$$\Upsilon^t : \mathcal{B}(0_Y) \rightarrow \mathcal{B}(0_Y), \quad y_0 \mapsto \text{pr}_2 \circ \Psi^t(0_X, y_0), \quad (9b)$$

where pr_1 and pr_2 are the natural projections onto X and Y , and we have carefully chosen the domains of the semiflows. We observe that for each initial condition $y_0 \in \mathcal{B}(0_Y)$, (6a) may be interpreted as time-varying dynamics on X given by

$$\dot{x} = f(x, \Upsilon^t(y_0)). \quad (10)$$

In this manner, each initial condition $y_0 \in \mathcal{B}(0_Y)$ induces a corresponding nonautonomous semiflow on X given by

$$\Phi_{y_0}^{(t,s)} : X \rightarrow X, \quad x_0 \mapsto \text{pr}_1 \circ \Psi^{t-s}(x_0, \Upsilon^s(y_0)), \quad (11)$$

such that we may also conclude

$$\Psi^t(x_0, y_0) = (\Phi_{y_0}^{(t,0)}(x_0), \Upsilon^t(y_0)). \quad (12)$$

With these constructions, we prove the claim in five steps.

STEP 1. $\mathcal{E}(\Psi) = \mathcal{R}(\Theta) \times \{0_Y\}$, and all points in this set are hyperbolic equilibria, of which only $(0_X, 0_Y)$ is stable.

Proof. All equilibria $(x, y) \in X \times \mathcal{B}(0_Y)$ must have $y = 0_Y$ by the definition of $\mathcal{B}(0_Y)$ as a basin of attraction, and therefore $f(x, 0_Y) = 0$ i.e. x is an equilibrium of (7). The equality then follows from the assumption that $\mathcal{R}(\Theta) \subseteq \mathcal{E}(\Theta)$, since equilibria are always chain recurrent i.e. $\mathcal{E}(\Theta) \subseteq \mathcal{R}(\Theta)$. Denoting the vector field on $X \times Y$ describing the full cascade (6a)-(6b) by $F : (x, y) \mapsto (f(x, y), g(y))$, we may express its linearization at any equilibrium $(x, 0_Y) \in X \times \mathcal{B}(0_Y)$ as

$$dF|_{(x, 0_Y)} = \begin{bmatrix} \partial_x f|_{(x, 0_Y)} & \partial_y f|_{(x, 0_Y)} \\ 0 & \partial_y g|_{0_Y} \end{bmatrix}. \quad (13)$$

Since the eigenvalues of a triangular block matrix are simply the eigenvalues of the blocks on the diagonal, the hyperbolicity claim follows directly from the hyperbolicity of 0_Y (6b) and the hyperbolicity of all equilibria of (7).

An almost globally asymptotically stable system has only one stable equilibrium (since all other equilibria lie on the boundary of its basin of attraction). Therefore, at $x = 0_X$ all eigenvalues of the top left block have negative real part, but one or more eigenvalues at all other equilibria of (7) have positive real part. Hence $(0_X, 0_Y)$ is locally exponentially stable and all other equilibria in $X \times \mathcal{B}(0_Y)$ are unstable. \blacktriangledown

To complete the proof, it will therefore suffice to show that the stable equilibrium $(0_X, 0_Y)$ is almost globally attractive.

STEP 2. For any $y_0 \in \mathcal{B}(0_Y)$, the nonautonomous semiflow Φ_{y_0} is asymptotically autonomous with limit semiflow Θ .

Proof. For any sequences $t_j \rightarrow t$, $s_j \rightarrow \infty$, and $x_j \rightarrow x$,

$$\begin{aligned} \lim_{j \rightarrow \infty} \Phi_{y_0}^{(t_j + s_j, s_j)}(x_j) &= \lim_{j \rightarrow \infty} \text{pr}_1 \circ \Psi^{t_j}(x_j, \Upsilon^{s_j}(y_0)) \end{aligned} \quad (14)$$

$$= \text{pr}_1 \circ \Psi^{\lim_{j \rightarrow \infty} t_j} \left(\lim_{j \rightarrow \infty} x_j, \lim_{j \rightarrow \infty} \Upsilon^{s_j}(y_0) \right) \quad (15)$$

$$= \text{pr}_1 \circ \Psi^t(x, 0_Y) = \Theta^t(x), \quad (16)$$

where (14) follows immediately from (11), (15) is obtained by the continuity of pr_1 and Ψ , and (16) relies on the attractivity of 0_Y . Thus for any $y_0 \in \mathcal{B}(0_Y)$, by definition Φ_{y_0} is asymptotically autonomous with limit semiflow Θ . \blacktriangledown

STEP 3. Every trajectory of Ψ converges to a hyperbolic equilibrium.

Proof. Together, Step 2, Fact 1, and the boundedness of forward trajectories of Ψ imply that for each $y_0 \in \mathcal{B}(0_Y)$,

every trajectory of Φ_{y_0} converges to $\mathcal{R}(\Theta)$, and asymptotic stability of (6b) ensures that every trajectory of Υ converges to 0_Y . Thus (12) shows that every trajectory of Ψ converges to $\mathcal{R}(\Theta) \times \{0_Y\}$ which consists of hyperbolic equilibria by Step 1. Since hyperbolic equilibria are isolated, by continuity every trajectory converges to a hyperbolic equilibrium. \blacktriangledown

STEP 4. *Almost no trajectories of (6a)-(6b) converge to an unstable equilibrium.*

Proof. All points converging to a hyperbolic equilibrium lie on its global stable manifold, which for an unstable equilibrium is the union of countably many embedded submanifolds of positive codimension (see [14, p. 73] or [15, Sec. 2.1]). Hence, this is a meager set of measure zero. Also, all unstable equilibria in $X \times \mathcal{B}(0_Y)$ are hyperbolic by Step 1, and there are countably many of these equilibria due to the isolation of hyperbolic equilibria and the second countability of $X \times \mathcal{B}(0_Y)$ [16, Thm 2.50 and Prop. 3.11]. Thus, the set of points in $X \times \mathcal{B}(0_Y)$ converging to an unstable equilibrium is a countable union of meager sets of measure zero and is thus meager (essentially by definition) and measure zero [13, p. 128] in $X \times Y$. \blacktriangledown

STEP 5. *Almost every trajectory of (6a)-(6b) converges to the stable equilibrium $(0_X, 0_Y)$.*

Proof. Since $\mathcal{B}(0_Y)$ is full measure and residual in Y by assumption, $X \times \mathcal{B}(0_Y)$ is full measure and residual in $X \times Y$. By Step 3, every initial condition in this set converges to a hyperbolic equilibrium, and by Step 4, the subset converging to an unstable equilibrium is meager and measure zero in $X \times Y$. Since the difference of a residual set of full measure by a meager set of measure zero is residual and full measure, the complement constitutes a residual, full measure set of points converging to the unique stable equilibrium $(0_X, 0_Y)$. \blacksquare

Remark 2. The main potential pitfall of the unforced outer subsystem being only *almost* globally asymptotically stable is the possibility of “funneling” a non-negligible (i.e. non-meager or positive measure) set to a point $(x, 0_Y)$, where x is an unstable equilibrium of (7). However, such behavior is precluded by the hyperbolicity of all unstable equilibria of (7). This can be relaxed to the requirement that all unstable equilibria of (7) are isolated and have at least one eigenvalue with positive real part, similar to [10]. Then, the argument proceeds similarly, but will rely on the center-stable manifold theorem instead. Similarly, the hyperbolicity assumption on 0_X can be relaxed at the cost of local exponential stability.

Corollary 1 (Upper Triangular System). *Consider an upper triangular system on $X_1 \times X_2 \times \dots \times X_n$ given by*

$$\dot{x}_1 = f_1(x_1, x_2, \dots, x_n), \quad (17a)$$

$$\dot{x}_2 = f_2(x_2, \dots, x_n), \quad (17b)$$

$$\begin{aligned} &\vdots \\ \dot{x}_n &= f_n(x_n), \quad (17c) \end{aligned}$$

where for all $i = 1, 2, \dots, n$, the unforced system

$$\dot{x}_i = f_i(x_i, 0_{i+1}, 0_{i+2}, \dots, 0_n) \quad (18)$$

is almost globally asymptotically stable with respect to $0_i \in X_i$ and its chain recurrent set contains only hyperbolic equilibria. Then, the upper triangular system (17a)-(17c) is almost globally asymptotically stable and locally exponentially stable with respect to $(0_1, 0_2, \dots, 0_n)$ as long as all forward trajectories of (17a)-(17c) are bounded.

Proof. The claim follows by induction. In particular, the claim is trivial for $n = 1$, and assuming it holds for $n = k - 1$, the claim for $n = k$ follows by Theorem 1 with (17a) as the outer subsystem (6a) and (17b)-(17c) as the inner subsystem (6b), i.e. $x = x_1$ and $y = (x_2, x_3, \dots, x_n)$. \blacksquare

III. HYPOTHESES OF THE MAIN RESULTS

We now explore the hypotheses of Theorem 1 and Corollary 1 in greater detail, showing how they can be verified.

A. Gradient-Like Systems

Systems with no chain recurrent points besides equilibria are often called “gradient-like”, and the following fact shows that such a property is often easily verified. We cannot include the proof due to space, but very similar notions are discussed in [10, Sec. IV], [17, Cor. 2.4], and [12, Sec. 7.12].

Fact 2. *If $\mathcal{E}(\Phi)$ consists of isolated points and there is a proper¹, continuous function $V : M \rightarrow \mathbb{R}$ that is decreasing² along nonequilibrium trajectories, then $\mathcal{R}(\Phi) = \mathcal{E}(\Phi)$.*

Remark 3. Fact 2 implies that Theorem 1 also holds if the assumption that (7) is almost globally asymptotically stable and gradient-like is replaced by the existence of a Lyapunov function for (7) around 0_X which is decreasing along all nonequilibrium trajectories. Some authors call this a *strict* Lyapunov function [12], [17], but the control community tends to reserve this term for Lyapunov functions with strictly negative derivative along nonequilibrium trajectories [18].

Remark 4. Two important classes of systems to which Fact 2 applies are as follows. It can be shown that for a Riemannian manifold (Q, κ) , a strict Rayleigh dissipation ν , and a proper Morse function $V : Q \rightarrow [0, \infty)$ with a unique minimum at $0_Q \in Q$, both the gradient dynamics on Q given by

$$\dot{q} = -\text{grad}_\kappa V(q) \quad (19)$$

and Euler-Lagrange dynamics on TQ given by³

$$\overset{\kappa}{\nabla} \dot{q} \dot{q} = -\text{grad}_\kappa V(q) - \kappa^\sharp \circ \nu^\flat(\dot{q}) \quad (20)$$

are almost globally asymptotically stable and locally exponentially stable around $0_Q \in Q$ and $0_{TQ} = (0_Q, 0) \in TQ$ respectively, and moreover all chain recurrent points of both systems are hyperbolic equilibria. An early, influential analysis of the previous stability properties is [20], while a

¹A function $V : M \rightarrow \mathbb{R}$ is *proper* if it has compact sublevel sets, which morally generalizes the notion of “radially unbounded” functions on \mathbb{R}^n .

²A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is *decreasing* if $f(t_2) < f(t_1)$ whenever $t_1 < t_2$. Note that this does *not* imply $\dot{f}(t) < 0$ for all t , e.g. $t \mapsto -t^3$.

³The maps $\kappa^\flat, \nu^\flat : TQ \rightarrow T^*Q$ and $\kappa^\sharp, \nu^\sharp : T^*Q \rightarrow TQ$ are the *musical isomorphisms* with respect to the Riemannian metrics μ and ν [19].

detailed modern treatment can be found in [19, Chap. 6]. The chain recurrence claim is immediate by Fact 2 and the observation that the “potential” V and the “total energy”

$$W : (q, \dot{q}) \mapsto V(q) + \frac{1}{2} \kappa(\dot{q}, \dot{q}), \quad (21)$$

are respectively decreasing along all nonequilibrium trajectories of (19) (cf. gradient descent) and (20) [19, Prop. 4.66].

B. Boundedness of Forward Trajectories

Forward boundedness is guaranteed when the outer subsystem evolves on a compact manifold. To use Theorem 1 to certify the stability of a cascade evolving on a noncompact manifold (e.g. \mathbb{R}^n or any tangent bundle), we require compositional criteria for forward boundedness. In this section, we give growth rate criteria suitable for our geometric setting on the “interconnection term” Σ_h and on a Lyapunov function for the unforced outer subsystem Σ_x . The result is analogous to (and inspired by) [4, Thm. 4.7], which certifies forward boundedness in \mathbb{R}^n using the standard Euclidean norm. In a Riemannian manifold (X, μ) , we use instead the Riemannian distance and the dual norms on each tangent and cotangent space induced by the metric. We denote both norms by $\|\cdot\|_\mu$.

Theorem 2 (Forward Boundedness of a Cascade). *Consider a cascade on $X \times Y$ given by*

$$\dot{x} = f(x, y), \quad (22a)$$

$$\dot{y} = g(y). \quad (22b)$$

Suppose the following conditions on the subsystems hold:

Σ_y : For (22b), $0_Y \in Y$ is a stable hyperbolic equilibrium.

Σ_x : $W : X \rightarrow \mathbb{R}_{\geq 0}$ is a proper Lyapunov function for

$$\dot{x} = f(x, 0_Y) \quad (23)$$

such that for some constants $\lambda \geq 0, d_0 \geq 1$,

$$\|dW_x\|_\mu \text{dist}(0_X, x) \leq \lambda W(x) \quad (24)$$

for all $(x, y) \in \{x \in M : \text{dist}(0_X, x) \geq d_0\} \times \mathcal{B}(0_Y)$.

Σ_h : For some continuous maps $\alpha, \beta : \mathcal{B}(0_Y) \rightarrow \mathbb{R}_{\geq 0}$ that are vanishing and differentiable at 0_Y , the interconnection term $h : (x, y) \mapsto f(x, y) - f(x, 0_Y)$ satisfies

$$\|h(x, y)\|_\mu \leq \alpha(y) \text{dist}(0_X, x) + \beta(y). \quad (25)$$

Then, the trajectory of (22a)-(22b) through any initial condition $(x_0, y_0) \in X \times \mathcal{B}(0_Y)$ is bounded for all forward time.

Proof. Since W is a proper Lyapunov function for (23), the forward trajectory through any initial condition of the form $(x_0, 0_Y)$ is bounded, so it suffices to consider initial conditions (x_0, y_0) with $y_0 \neq 0_Y$. Fix $(x_0, y_0) \in X \times \mathcal{B}(0_Y)$ with $y_0 \neq 0_Y$ and let $(x(t), y(t))$ denote its forward trajectory.

STEP 1. *There exist positive constants A and ω such that $\alpha(y(t)) + \beta(y(t)) \leq Ae^{-\omega t}$ for all $t \geq 0$.*

Proof. Let $d(t) := \text{dist}(0_Y, y(t)) > 0$. Since 0_Y is hyperbolic, there exist $C_0, \omega > 0$ such that, for all $t \geq 0$,

$$d(t) \leq C_0 e^{-\omega t}. \quad (26)$$

Next, since α and β are vanishing and differentiable at 0_Y , a local coordinate calculation (using uniform equivalence of continuous Riemannian metrics over compact sets) shows

$$\limsup_{t \rightarrow \infty} \frac{\alpha(y(t)) + \beta(y(t))}{d(t)} < \infty. \quad (27)$$

The quotient in the previous limit is a continuous function of t and thus is bounded for all $t \geq 0$ by some $C_1 > 0$. With (26), this yields the desired bound with $A := C_0 C_1$. \blacktriangledown

STEP 2. *$W(x(t))$ is bounded for all $t \geq 0$.*

Proof. Since W is a Lyapunov function for (23), we have

$$\dot{W} \leq dW_x(h(x, y)) \leq \|dW_x\|_\mu \|h(x, y)\|_\mu \quad (28)$$

$$\leq \|dW_x\|_\mu (\alpha(y) \text{dist}(0_X, x) + \beta(y)), \quad (29)$$

from (25). Hence, whenever $\text{dist}(0_X, x) \geq d_0 \geq 1$, we have

$$\dot{W} \leq \|dW_x\|_\mu \text{dist}(0_X, x) (\alpha(y) + \beta(y)). \quad (30)$$

Define $W_0 = \sup_{\{x : \text{dist}(0_X, x) \leq d_0\}} W(x)$ and consider any $t_2 \geq t_1 \geq 0$ where $W(x([t_1, t_2])) \subseteq [W_0, \infty)$. Then for all $t \in [t_1, t_2]$, (24), (30), and the conclusion of Step 1 imply

$$\frac{d}{dt} W(x(t)) \leq \lambda A e^{-\omega t} W(x(t)). \quad (31)$$

By Grönwall’s inequality, we obtain the bound

$$W(x(t_2)) \leq e^{\int_{t_1}^{t_2} \lambda A e^{-\omega t} dt} W(x(t_1)) \leq e^{\frac{\lambda A}{\omega}} W(x(t_1)). \quad (32)$$

This implies that for all $t \geq 0$,

$$W(x(t)) \leq C := e^{\frac{\lambda A}{\omega}} \max\{W_0, W(x(0))\}. \quad \blacktriangledown$$

Thus, since W is proper and 0_Y is attractive, it follows that $(x(t), y(t))$ is bounded for all $t \geq 0$. \blacksquare

A similar approach can be iterated (as in Corollary 1) to certify forward boundedness of an upper triangular system.

IV. APPLICATION OF THE RESULTS

We now revisit the motivating example (2a)-(2b). It is easily verified that (2b) takes the form of the Euler-Lagrange dynamics (20) for the kinetic energy metric and Rayleigh dissipation $\kappa = \nu = d\phi \otimes d\phi$ and the Morse potential function $V : \mathbb{S}^1 \rightarrow \mathbb{R}, \phi \mapsto 1 - \cos \phi$. Thus by Remark 4, (2b) is almost globally asymptotically stable and locally exponentially stable with respect to $y = (\phi, \dot{\phi}) = (0, 0)$, and moreover its chain recurrent set consists solely of hyperbolic equilibria. Clearly, the same is true for $x = (\theta, \dot{\theta}) = (0, 0)$ with respect to the restriction of (2a) to $y = (\phi, \dot{\phi}) = (0, 0)$.

By Theorem 1, for almost global asymptotic stability it will suffice to show forward boundedness, which we accomplish using the total energy (21). The natural choice of metric on the tangent bundle $X = T\mathbb{S}^1$ is the Sasaki metric [21] i.e. $\tilde{\kappa} = d\theta \otimes d\theta + d\dot{\theta} \otimes d\dot{\theta}$. Then, considering (without loss of generality) the range of angles $\theta \in [-\pi, \pi)$, we have

$$\underbrace{\sqrt{\sin^2 \theta + \dot{\theta}^2}}_{\|dW_x\|_{\tilde{\kappa}}} \underbrace{\sqrt{\theta^2 + \dot{\theta}^2}}_{\text{dist}(0_X, x)} \leq \theta^2 + \dot{\theta}^2 \leq \underbrace{\frac{\pi^2}{2}}_{\lambda} \underbrace{(1 - \cos \theta + \frac{\dot{\theta}^2}{2})}_{W(x)},$$

since $|\sin \theta| \leq |\theta|$ and $\theta^2 \leq \frac{\pi^2}{2}(1 - \cos \theta)$ for $\theta \in [-\pi, \pi)$, verifying that (24) holds. Furthermore, we compute

$$\underbrace{(1 - \cos 2\phi)(\sin \theta + \dot{\theta})}_{\|h(x,y)\|_{\tilde{\kappa}}} \leq \underbrace{(1 - \cos 2\phi)\sqrt{2}}_{\alpha(y)} \underbrace{\sqrt{\theta^2 + \dot{\theta}^2}}_{\text{dist}(0_X, x)}, \quad (33)$$

so (25) holds as well. Thus it follows by Theorem 2 that all forward trajectories of (2a)-(2b) with $y = (\phi, \dot{\phi})$ starting in the basin of attraction of (2b) are bounded, and so the system is almost globally asymptotically stable and locally exponentially stable with respect to $(0, 0, 0, 0) \in T\mathbb{T}^2$.

V. DISCUSSION

The disturbance robustness of systems with some similar properties, and the connection to gradient-like systems, was discussed in [10, Sec. IV]. However, those results (when combined with [9]) can only certify the stability of a cascade if the outer subsystem is almost globally input to state stable, requiring also “ultimate boundedness” with respect to *any* bounded disturbance, absent from our motivating example.

Our main results show that an upper triangular system consisting of almost globally asymptotically stable, gradient-like subsystems with no degenerate equilibria is itself almost globally asymptotically stable if all forward trajectories are bounded. Since *globally* asymptotically stable systems are gradient-like (with a single chain recurrent point, i.e. the stable equilibrium), the result is analogous to the fact that a cascade of globally asymptotically stable subsystems is globally asymptotically stable as long as all trajectories are bounded [4, Prop. 4.1]. Our second theorem generalizes a classic compositional method of verifying forward boundedness using growth rate criteria on the interconnection term and a Lyapunov function for the unforced outer subsystem. Unfortunately, the Riemannian distance function used in our condition can be difficult to compute explicitly in complex examples, but even loose upper and lower bounds on this distance could potentially be used to verify the inequalities.

Gradient-like dynamics are common in the closed loop subsystems of geometric controllers [2], [5]. Indeed, since cascades of mechanical systems with suitable dissipation and potential enjoy the required stability and chain recurrence properties, we see promising directions for the constructive synthesis of cascaded geometric controllers with almost global asymptotic stability for underactuated robotic systems, e.g. those possessing a geometric flat output (such as quadrotors and aerial manipulators) [22], which enjoy a cascade-like structure where the evolution of the system in the shape space is uniquely determined by the evolution in the symmetry group. Indeed, for a reference trajectory with constant acceleration, the closed loop error dynamics of the geometric quadrotor controller proposed in [2] take the form (1a)-(1b) and the subsystems are dissipative mechanical systems, but the translational subsystem is *not* almost globally input to state stable with respect to the rotational subsystem state.

VI. CONCLUSION

In this work, we present sufficient conditions for the almost global asymptotic stability of a cascade in which the

subsystems are only almost globally asymptotically stable. The result is extended inductively to upper triangular systems of arbitrary size. The approach relies on the forward boundedness of trajectories (which can be verified by growth rate criteria on the interconnection term and on a Lyapunov function for the unforced outer subsystem) and the absence of chain recurrent points other than hyperbolic equilibria in the unforced outer subsystem. The results are analogous to classic results for cascades of globally asymptotically stable systems. The compositional nature of the criteria facilitates stability verification for arbitrarily complex cascades, so long as the subsystems enjoy certain fundamental properties.

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