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Safety-Critical Control for Systems with Impulsive Actuators and Dwell Time Constraints

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Abstract— **This paper presents extensions of control barrier function (CBF) and control Lyapunov function (CLF) theory to systems wherein all actuators cause impulsive changes to the state trajectory, and can only be used again after a minimum dwell time has elapsed. These rules define a hybrid system, wherein the controller must at each control cycle choose whether to remain on the current state flow or to jump to a new trajectory. We first derive a sufficient condition to render a specified set forward invariant using extensions of CBF theory. We then derive related conditions to ensure asymptotic stability in such systems, and apply both conditions online in an optimization-based control law with aperiodic impulses. We simulate both results on a spacecraft docking problem with multiple obstacles.**

I. Introduction

Control Barrier Functions (CBFs) [1] are a tool for designing control laws that render state trajectories always inside a specified set. Each CBF converts a set of allowable states, herein called the *CBF set*, to a set of allowable control inputs at every state in that set [2]. Any control input within this set will render the future state trajectory inside the CBF set. The controller thus has freedom to work towards other goals, such as convergence, as long as the control remains within the input set generated by the CBF. CBFs thus provide a computationally tractable solution to many nonlinear constrained control problems. While the original formulations of CBFs [3]–[5] considered continuous-time systems, subsequent authors have published numerous extensions to sampled systems [6]–[11], discrete-time systems [12], [13], and hybrid systems [14]–[17], among others. In this paper, we develop set invariance rules for a specific class of hybrid systems: systems with impulsive actuators that are only permitted to be used after a minimum *dwell time* has elapsed since their previous use. This models, for instance, a spacecraft with chemical thrusters.

Impulsive systems are a special class of hybrid systems, and there has been much work on stability of hybrid systems over the past two decades [18]–[23], and more recently work on set invariance [4], [17], [24] and CBFs [14]–[16], [25], [26] for hybrid systems. A hybrid system is a combination of a set of time intervals where a system *flows* according to a state differential equation called the *flow map*, and a set of times where the state *jumps* (changes instantaneously) according to an algebraic function called the *jump map*. Control may be applied along the flows, at the jumps (also called impulses), or both. In this letter, we study systems where control occurs only via jumps, and jumps occur only when control is applied, as is formalized in Section II. The work in [17], [24] show that a hybrid system renders a set forward invariant if 1) the flow map always lies within the tangent cone of the set, and 2) the image of the set through the jump map is a subset of the set. The authors in [4], [15], [16] then rewrite these conditions for CBFs and CBF sets. However, these two conditions have no way to incorporate a minimum dwell time constraint (equivalently, a minimum time between events [22]).

Recall that the problem of finding a CBF is equivalent to the problem of finding a controlled-invariant set [27]. For hybrid systems, this equivalency follows from, e.g., [15, Def. 3.6] and [26, Def. 5]. In this letter, due to the minimum dwell time constraint, rather than applying control to render the state inside such a controlled-invariant set, we must apply control to render the state into a set whose forward reachable set remains a subset of the CBF set at least until the dwell time has elapsed. This is an inherently different problem than that addressed by typical CBFs [1], [5] or by the hybrid CBFs in [4], [14]–[17], [24]–[26], and has more in common with margins for ensuring set invariance between samples under sampled controllers such as [6]–[9], [28], [29]. This paper applies the same concept of sampling margins as in [6], now modified for impulsive rather than zero-order-hold control, to guarantee set invariance under a minimum dwell time. Additionally, in Section III-E.1, we propose a variation of our method for reducing conservatism.

Finally, the addition of the minimum dwell time constraint also complicates stability. The work in [18] provides a formula for a maximum dwell time at which stability is still guaranteed, and similar stability certificates for specified dwell times are presented in [21]–[23]. However, all of these results are overly restrictive, because they all place weak assumptions (e.g., exponentially bounded divergence) on the flows in exchange for strong requirements (e.g., rapid exponential contractivity) on the jumps. This is sensible in general, since the jumps are controlled and the flows are uncontrolled, but the spacecraft community has long developed controllers with weaker assumptions on the jumps [30]–[32], though not with the desired minimum dwell time. Thus, building up from the minimum dwell time constraint, this letter presents conditions to

1) render a CBF set forward invariant subject to impulsive control with a minimum dwell time constraint, and
2) render the origin asymptotically stable subject to the same impulsive control and dwell time rules.

This paper is organized as follows. Section II presents the

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system model. Section III presents the set invariance strategy, the asymptotic stability strategy, and some mathematical tools. Section IV presents simulations of these methods on a satellite docking problem. Section V presents concluding remarks.

II. Preliminaries

Notations: Given a time domain $\mathcal{T} \subseteq \mathbb{R}$, spatial domain $\mathcal{X} \subseteq \mathbb{R}^n$, and function $\eta : \mathcal{T} \times \mathcal{X} \to \mathbb{R}$, denoted $\eta(t,x)$, let $\partial_t \eta$ denote the partial derivative with respect to t. Let $\nabla\eta$ denote the gradient row vector with respect to x. Let $\dot{\eta} = \partial_t \eta + \nabla\eta \dot{x}$ denote the total derivative of η in time. Let $\mathbb{N}$ denote the set of nonnegative integers. Let $\|\cdot\|$ denote the 2-norm. A continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is class-$\mathcal{K}_\infty$, denoted $\alpha \in \mathcal{K}_\infty$ if 1) $\alpha(0) = 0$, 2) α is strictly increasing, and 3) $\lim_{\lambda\to\infty} \alpha(\lambda) = \infty$. Let $\mathcal{K}_r$ denote the set of continuous functions $\beta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying 1) $\beta(0) = 0$, and 2) $\beta(\lambda) > 0$ for all $\lambda > 0$.

Model: Spacecraft with chemical thrusters are frequently modeled as evolving according to an ordinary differential equation (ODE) with impulsive jumps. When activated, the thruster subsystem causes an instantaneous change, called an impulse, to the spacecraft velocity, and then the system flows according to the ODE until the next impulse is applied. Control can only be applied at the impulses. We also assume the following two restrictions on impulses:

R-1) the controller is sampled with fixed period Δt and an impulse can only be applied at the sample times; and
R-2) the controller can only apply an impulse at least ΔT after the last impulse was applied, where $\Delta T > \Delta t$.

Let $\mathcal{T} \subseteq \mathbb{R}$ be a time domain, $\mathcal{X} \subseteq \mathbb{R}^n$ be the state space, and $\mathcal{U} \subseteq \mathbb{R}^m$ be the set of allowable controls. To encode R-1, let

$$\mathcal{D}_0 \triangleq \{t \in \mathcal{T} \mid t = t_0 + k\Delta t, k \in \mathbb{N}\} \tag{1a}$$

be the set of controller sample times originating from initial time $t_0 \in \mathcal{T}$. To encode R-2, let the additional state $\sigma \in \mathbb{R}_{\geq 0}$ encode the time since the last impulse was applied. A tuple (t,σ) is an *impulse opportunity* if $t \in \mathcal{D}_0$ and $\sigma \geq \Delta T$, or equivalently, if (t,σ) lies in the set of impulse opportunities

$$\mathcal{D} \triangleq \mathcal{D}_0 \times \{\sigma \in \mathbb{R}_{\geq 0} \mid \sigma \geq \Delta T\}\,. \tag{1b}$$

The control is thus a map $u : \mathcal{D} \times \mathcal{X} \to \mathcal{U}$ defined only at the set of impulse opportunities $\mathcal{D}$. The time ΔT is called the minimum *dwell time* between impulses [18]. Assume that $\Delta T = q\Delta t$ for some $q \in \mathbb{N}$.

We can thus model the spacecraft generally as

$$\begin{cases} \begin{cases} \dot{x} = f(t,x) \\ \dot{\sigma} = 1 \end{cases} & (t,\sigma) \notin \mathcal{D} \\ \begin{cases} x^+ = g(t,x,u) \\ \sigma^+ = \sigma \text{ if } u = 0 \\ \sigma^+ = 0 \text{ if } u \neq 0 \end{cases} & (t,\sigma) \in \mathcal{D} \end{cases} \tag{1c}$$

The system (1c) defines a hybrid system with flow set $\mathcal{C} \triangleq (\mathcal{T} \times \mathbb{R}_{\geq 0}) \setminus \mathcal{D}$, flow map $f : \mathcal{T} \times \mathcal{X} \to \mathbb{R}^n$, jump set $\mathcal{D}$, and jump map $g : \mathcal{T} \times \mathcal{X} \times \mathcal{U} \to \mathcal{X}$. We note that (1c) has time-dependent jumps, and therefore is also a timed automaton [33]. In this paper, we assume that the maps f and g are known and single-valued (rather than being differential inclusions), that $g(t,x,0) = x$ for all $t \in \mathcal{T}, x \in \mathcal{X}$, and that solutions to (1) exist and are unique for all $t \in \mathcal{T}$. Also assume that $\sigma(t_0) = \Delta T$ at the initial time t_0, so that the initial state tuple $(t_0, \sigma(t_0), x(t_0))$ is an impulse opportunity.

Note that at every impulse opportunity $(t,\sigma) \in \mathcal{D}$, the controller u can choose whether or not to apply an impulse, so impulses will generally be aperiodic, and may lack an average dwell time as in [18]. For brevity in Section III, given a control law $u : \mathcal{D} \times \mathcal{X} \to \mathcal{U}$, denote the set of impulse opportunities where the control law chooses to not apply an impulse as

$$\mathcal{Z}_{\text{coast}} \triangleq \{(t,\sigma,x) \in \mathcal{D} \times \mathcal{X} \mid u(t,\sigma,x) = 0\}\,. \tag{2}$$

The central problem addressed in Section III is as follows.

Problem 1. *Given dynamics (1) and a set $\mathcal{S}_{safe}(t) \subset \mathcal{X}$, derive conditions on the control u that are sufficient to 1) guarantee $x(t)$ remains in $\mathcal{S}_{safe}(t), \forall t \in \mathcal{T}$, and 2) render the origin asymptotically stable, where we assume $0 \in \mathcal{S}_{safe}(t), \forall t \in \mathcal{T}$.*

The conditions arising from Problem 1 can then be enforced online using optimization-based control laws as is typical in the CBF literature [1, Sec. II-C]. Unlike [1], in this letter, we allow these optimizations to be nonlinear programs. Such programs are more computationally expensive than the quadratic programs in [1], but we assume that this cost is acceptable due to the long dwell time ΔT between impulses.

III. Impulsive Timed Control Barrier Functions and Control Lyapunov Functions

In this section, we first present some definitions and tools in Section III-A, before using these tools to address invariance of a subset of $\mathcal{S}_{\text{safe}}(t)$ in Section III-B. We then address stability of the origin in two parts in Sections III-C-III-D, and provide examples and additional tools in Section III-E.

A. Flows and Bounding Functions

In this letter, we will utilize predictions about the future state. Suppose that no jumps occur in some interval $[t,\tau] \subset \mathcal{T}$. Then define the flow operator $p : \mathcal{T} \times \mathcal{T} \times \mathcal{X} \to \mathcal{X}$ as

$$p(\tau,t,x) = y(\tau) \text{ where } \dot{y}(s) = f(s,y(s)),\ y(t) = x\,. \tag{3}$$

Next, we are interested in approximations of the future state. Given a scalar function $h : \mathcal{T} \times \mathcal{X} \to \mathbb{R}$, and an initial state (t,x), denote by $\psi_h : \mathcal{T}\times\mathcal{T}\times\mathcal{X} \to \mathbb{R}$ any function satisfying

$$\psi_h(\tau,t,x) \geq h(s,p(s,t,x)),\ \forall s \in [t,\tau]\,. \tag{4}$$

That is, ψ_h is an upper bound on the evolution of the function h for any interval $[t,\tau]$ during which there are no control impulses. Methods to find such a bounding function are described in [6]–[8], [11], [28], [29] and others, and thus are only briefly elaborated upon here in Section III-E. We note that [6]–[8], [11], [28], [29] all include a term that accounts for the effects of the control input $u \in \mathcal{U}$, whereas this term can be ignored here since f in (1) is independent of u.

B. Set Invariance

We first address the safety part of Problem 1. To apply the method of CBFs, we seek a function $h : \mathcal{T} \times \mathcal{X} \to \mathbb{R}$ such that the set

$$\mathcal{S}_h(t) \triangleq \{x \in \mathcal{X} \mid h(t,x) \le 0\} \tag{5}$$

satisfies $\mathcal{S}_h(t) \subseteq \mathcal{S}_{\text{safe}}(t), \forall t \in \mathcal{T}$. The definition of CBF [5, Def. 5] can be generalized to the system (1) as follows.

Definition 1. *Let ψ_h be as in (4). A continuous function $h : \mathcal{T} \times \mathcal{X} \to \mathbb{R}$ is an* Impulsive Timed Control Barrier Function (ITCBF) *for the system (1) if*

$$\inf_{u \in \mathcal{U}} \psi_h(t + \Delta T, t, g(t,x,u)) \le 0, \forall x \in \mathcal{S}_h(t), \forall t \in \mathcal{T}\,. \tag{6}$$

Note that 1) we relax [5, Def. 5] to no longer require differentiability of h, though differentiability is helpful when applying tools from [6]–[8], [11], [28], [29], and 2) condition (6) does not include a class-$\mathcal{K}$ function, as this is unnecessary in sampled controllers. The following theorem then provides sufficient conditions for forward invariance of $\mathcal{S}_h(t)$.

Theorem 1. *Given an ITCBF $h : \mathcal{T} \times \mathcal{X} \to \mathbb{R}$ for the system (1), let $\mathcal{S}_h(t)$ be as in (5), and ψ_h as in (4). Let $u : \mathcal{D} \times \mathcal{X} \to \mathcal{U}$ be a control law, and let $\mathcal{Z}_{coast}$ be as in (2). If u satisfies*

$$\psi_h(t + \Delta t, t, x) \le 0, \quad \forall (t, \sigma, x) \in (\mathcal{D} \times \mathcal{S}_h) \cap \mathcal{Z}_{coast}, \tag{7a}$$

$$\psi_h(t + \Delta T, t, y) \le 0, \quad \forall (t, \sigma, x) \in (\mathcal{D} \times \mathcal{S}_h) \setminus \mathcal{Z}_{coast}, \tag{7b}$$

where $y = g(t, x, u(t, \sigma, x))$, then u renders time-varying set $\mathcal{S}_h(t)$ forward invariant for all $t \in \mathcal{T}$.

Proof. Given $(t_0, \sigma(t_0)) \in \mathcal{D}$ and $x(t_0) \in \mathcal{S}_h(t_0)$, divide $\{t \in \mathcal{T} \mid t \ge t_0\}$ into a sequence of intervals $\mathcal{I}_k = [t_k, t_{k+1}]$, $k \in \mathbb{N}$, where $t_{k+1} > t_k$. Then a sufficient condition for u to render $\mathcal{S}_h(t)$ forward invariant for all future $t \in \mathcal{T}$ is for the following two properties to hold for every $k \in \mathbb{N}$: 1) u renders $x(t) \in \mathcal{S}_h(t)$ for all t in the interval $\mathcal{I}_k$, and 2) the endpoint t_{k+1} of $\mathcal{I}_k$ is an impulse opportunity. If $u = 0$, then condition (7a) implies that both properties hold for $t_{k+1} = t_k + \Delta t$. If $u \ne 0$, then condition (7b) implies that both properties hold for $t_{k+1} = t_k + \Delta T$. Thus, u renders $\mathcal{S}_h(t)$ forward invariant. ■

Thus, we have two conditions analogous to [5, Cor. 2] that render sets of the form (5) forward invariant subject to the impulsive dynamics (1). The remaining challenge is to determine functions h and ψ_h satisfying (6) and (4), respectively. We first discuss conditions for asymptotic stability before providing examples of h and ψ_h in Section III-E.

C. One-Step MPC Impulsive Stability

We now begin to address the stability part of Problem 1. There has been much work on stability of hybrid systems with continuous actuators [9], [19], [20], [34], impulsive actuators [21]–[23], or both [18]. In summary, given a Lyapunov function $V : \mathcal{T} \times \mathcal{X} \to \mathbb{R}_{\ge 0}$, the conditions [21, Eq. 5], [22, Eq. 8], and [18, Eq. 4b] state that if $V(t, g(t,x,u)) \le cV(t,x)$ for $c \in (0,1)$, then for sufficiently frequent jumps, the origin of the system (1c) is exponentially stable. These conditions can be readily applied to stabilize (1) using periodic impulses. However, when the dwell time ΔT is large, a more efficient strategy may be to examine the predicted value of the Lyapunov function after ΔT has elapsed rather than immediately after the impulse. To this end, consider the following lemma.

Assumption 1. *Let $V : \mathcal{T} \times \mathcal{X} \to \mathbb{R}_{\ge 0}$ be a continuously differentiable function satisfying*

$$\alpha_1(\|x\|) \le V(t,x) \le \alpha_2(\|x\|) \tag{8}$$

for all $x \in \mathcal{X}$ and all $t \in \mathcal{T}$ for two functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$.

Lemma 1. *Let Assumption 1 hold. Assume that there exists $\alpha_3 \in \mathcal{K}_\infty$ such that f in (1) satisfies $\|f(t,x)\| \le \alpha_3(\|x\|)$ for all $t \in \mathcal{T}$ and $x \in \mathcal{X}$. Let p be as in (3). Let $u : \mathcal{D} \times \mathcal{X} \to \mathcal{U}$ be a control law, and denote $\mathcal{Z}_1 \equiv \mathcal{Z}_{coast}$ as in (2) and $\mathcal{Z}_2 = (\mathcal{D} \times \mathcal{X}) \setminus \mathcal{Z}_1$. For the system (1), if u satisfies*

$$V(t + \Delta t, p(t + \Delta t, t, x)) \le V(t,x), \ \forall (\cdot) \in \mathcal{Z}_1, \tag{9a}$$

$$V(t + \Delta T, p(t + \Delta T, t, y)) \le V(t,x), \ \forall (\cdot) \in \mathcal{Z}_2, \tag{9b}$$

where $(\cdot) = (t, \sigma, x)$ and $y = g(t, x, u(t, \sigma, x))$, then u renders the origin uniformly stable as in [35, Def. 4.4].

Proof. See [36, Lemma 1]. ■

Lemma 1 differs from [18], [21], [22] in three ways. First, (9) provides conditions on the future state, which is explicitly computed using (3), rather than the present state. Second, these predictions allow us to avoid explicitly checking for upper bounds on the growth of V during flows, as is required in [21], [22]. Third, Lemma 1 allows for aperiodic impulses, as long as (9) are checked at their respective frequencies.

We refer to (9) as a "one-step Model Predictive Control (MPC)" strategy. That is, to evaluate (9), we input the control u at a single (i.e. "one-step") time instance, make a prediction using (3), and then check a condition on V, analogous to checking constraints in an MPC optimization. Note that encoding (9) into an optimization problem could be computationally expensive, since checking (9) entails computing the solution to a differential equation during every iteration of the optimization. In Section IV, we assume that this cost is acceptable, or that we have an analytic form for the solution, as is the case for many spacecraft orbits.

D. Impulsive Stability via Restriction to Stable Flows

Motivated by fuel efficiency, a strategy in aerospace systems (e.g. [30]) is to allow a system to coast uncontrolled until a control impulse is necessary to continue stabilization. In this subsection, we implement this strategy subject to constraints R-1, R-2 via a specialization of Lemma 1. In technical terms, given a Lyapunov function V as in (8), we seek to render the set

$$\mathcal{S}_v(t) \triangleq \{x \in \mathcal{X} \mid v(t,x) \le 0\} \tag{10}$$

forward invariant, where, for readability, we denote

$$v(t,x) \equiv \dot{V}(t,x) = \partial_t V(t,x) + \nabla V(t,x) f(t,x)\,. \tag{11}$$

This is possible under dynamics (1) if $v : \mathcal{T} \times \mathcal{X} \to \mathbb{R}$ is also an ITCBF as in Definition 1. Let ψ_v be an upper bound for v analogous to ψ_h in (4). In the following theorem, we provide new conditions to establish stability using such a coasting strategy. However, if $x(t_0) \notin \mathcal{S}_v(t_0)$, then these conditions will not initially apply, so we instead fall back on the "one-step MPC" strategy in (9). Divide the state space into two sets: 1) $\mathcal{Z}_1 \cup \mathcal{Z}_2$, where the controller enforces (9), and 2) $\mathcal{Z}_3 \cup \mathcal{Z}_4$, where the controller enforces the new conditions (12).

Theorem 2. *Let Assumption 1 hold. Assume that there exists $\alpha_3 \in \mathcal{K}_\infty$ such that f in (1) satisfies $\|f(t,x)\| \le \alpha_3(\|x\|)$ for all $t \in \mathcal{T}$ and $x \in \mathcal{X}$. Let v be as in (11), ψ_v be as in (4), and p be as in (3). Let $\mathcal{Z}_1$, $\mathcal{Z}_2$, $\mathcal{Z}_3$, and $\mathcal{Z}_4$ be four disjoint sets such that $\mathcal{Z}_1 \cup \mathcal{Z}_3 = \mathcal{Z}_{coast}$ in (2), and $\mathcal{Z}_2 \cup \mathcal{Z}_4 = (\mathcal{D} \times \mathcal{X}) \setminus \mathcal{Z}_{coast}$. Then for the system (1), any control law $u : \mathcal{D} \times \mathcal{X} \to \mathcal{U}$ satisfying (9) and all of the following*

$$\psi_v(t+\Delta t, t, x) \le 0, \qquad \forall (t,\sigma,x) \in \mathcal{Z}_3, \quad \text{(12a)}$$
$$\psi_v(t+\Delta T, t, g(t,x,u(t,\sigma,x))) \le 0, \forall (t,\sigma,x) \in \mathcal{Z}_4, \quad \text{(12b)}$$
$$V(t, g(t,x,u(t,\sigma,x))) \le V(t,x), \qquad \forall (t,\sigma,x) \in \mathcal{Z}_4, \quad \text{(12c)}$$

will render the origin uniformly stable as in [35, Def. 4.4].

Proof. Let $(t_k, \sigma(t_k)) \in \mathcal{D}$ be an impulse opportunity, and let $(t_{k+1}, \sigma(t_{k+1})) \in \mathcal{D}$ be the next impulse opportunity. For brevity, denote $z_k = (t_k, \sigma(t_k), x(t_k))$. First, Lemma 1 implies that if $z_k \in \mathcal{Z}_1 \cup \mathcal{Z}_2$, then $V(t_{k+1}, x(t_{k+1})) \le V(t_k, x(t_k))$.

Next, if $z_k \in \mathcal{Z}_3 \cup \mathcal{Z}_4$, conditions (12a)-(12c) similarly imply that $V(t_{k+1}, x(t_{k+1})) \le V(t_k, x(t_k))$. Specifically, if $z_k \in \mathcal{Z}_3 \subseteq \mathcal{Z}_{\text{coast}}$, then no impulse is applied, and (12a) implies that $V(t, x(t))$ is nonincreasing along the flow f for all $t \in [t_k, t_k + \Delta t)$ until the next impulse opportunity at $t_{k+1} = t_k + \Delta t$. Next, if $z_k \in \mathcal{Z}_4$, then a nonzero impulse is applied, (12c) implies that V is nonincreasing during the impulse, and (12b) implies that $V(t, x(t))$ is nonincreasing along the flow f for all $t \in (t_k, t_k + \Delta T)$ until the next impulse opportunity at $t_{k+1} = t_k + \Delta T$. Thus, $V(t_{k+1}, x(t_{k+1})) \le V(t_k, x(t_k))$ for all $(t_k, \sigma(t_k)) \in \mathcal{D}$, so the origin is uniformly stable by the same argument as Lemma 1. ■

Compared to [18], [21], [22], Theorem 2 imposes stricter conditions on the flows (12a)-(12b) in order to allow relaxed conditions on the jumps (12c) and the jump times. In [21], [22], it is assumed that the flows are destabilizing and jumps are exponentially stabilizing, whereas Theorem 2 says that if we can restrict the flow (12a)-(12b) to the set in (10) where $\dot{V} \le 0$, as is often possible in practice, then the jump (12c) only needs to be stabilizing, not exponentially stabilizing. This coasting strategy can reduce control usage compared to the exponentially stabilizing impulses in [21], [22], and is distinct from the coasting strategy in [30] because of the explicit inclusion of a minimum time between impulses. Note that (12a)-(12b) are identical to (7a)-(7b), so a controller as in Theorem 2 will further render $\mathcal{S}_v$ in (10) forward invariant if $x(t_0) \in \mathcal{S}_v(t_0)$ and $\mathcal{Z}_3 \cup \mathcal{Z}_4 = \mathcal{D} \times \mathcal{S}_v$. Finally, we present a result on asymptotic stability that we will use in Section IV.

Corollary 1. *Let the conditions of Theorem 2 hold. If there exists $\beta_1, \beta_2 \in \mathcal{K}_r$ and $\Delta T_{max} \in \mathbb{R}_{>0}$ such that 1) (13a)-(13b) hold and 2) either 2a) (13c)-(13d) hold or 2b) (13e)-(13f) hold*

$$V(t+\Delta t, p(t+\Delta t, t, x)) - w \le -\beta_2(w), \quad \forall(\cdot) \in \mathcal{Z}_1 \text{ (13a)}$$
$$V(t+\Delta T, p(t+\Delta T, t, y)) - w \le -\beta_2(w), \forall(\cdot) \in \mathcal{Z}_2 \text{ (13b)}$$
$$\psi_v(t+\Delta t, t, x) \le -\beta_1(w), \qquad \forall(\cdot) \in \mathcal{Z}_3 \text{ (13c)}$$
$$\psi_v(t+\Delta T, t, y) \le -\beta_1(w), \qquad \forall(\cdot) \in \mathcal{Z}_4 \text{ (13d)}$$
$$V(t,y) - w \le -\beta_2(w), \qquad \forall(\cdot) \in \mathcal{Z}_4 \text{ (13e)}$$
$$\sigma \ge \Delta T_{max} \implies u(t,\sigma,x) \ne 0, \qquad \forall(\cdot) \in \mathcal{D} \times \mathcal{X} \text{ (13f)}$$

where $(\cdot) = (t,\sigma,x)$, $y = g(t,x,u(t,\sigma,x))$, $w = V(t,x)$, then the origin is uniformly asymptotically stable [35, Def. 4.4].

Proof. See [36, Corollary 1]. ■

That is, if the Lyapunov function V is nonincreasing as in Theorem 2, and either the flows (13c)-(13d) or the jumps (13e)-(13f) cause V to strictly decrease, then the origin is asymptotically stable. Again, we provide alternative "one-step MPC" conditions (13a)-(13b) in case (13c)-(13f) cannot be satisfied because $x(t) \notin \mathcal{S}_v(t)$. If we further assume that β_1 and β_2 are linear functions, then the conditions in Corollary 1 become special cases of [18, Thm. 1].

E. Examples of Bounding Functions

In this subsection, we discuss in more detail how to develop ψ_h and ψ_v to use in the preceding theorems. Suppose second order dynamics such that $x = [r^{\mathrm{T}}, \dot{r}^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^n$ for flow dynamics $\ddot{r} = f_r(x)$. First, an obstacle avoidance constraint can be written using the following form of CBF h [37]:

$$\kappa(t,x) = \rho - \|r - r_0(t)\| \quad \text{(14a)}$$
$$h(t,x) = \kappa(t,x) + \gamma \dot{\kappa}(t,x) \quad \text{(14b)}$$
$$\psi_h(t+\delta, t, x) = \max\big\{h(t,x), \kappa(t,x) + (\gamma+\delta)\dot{\kappa}(t,x) + \left(\tfrac{1}{2}\delta^2 + \gamma\delta\right)\ddot{\kappa}_{\max}\big\} \quad \text{(14c)}$$

where $\rho \in \mathbb{R}_{>0}$ is the obstacle radius, $\gamma \in \mathbb{R}_{>0}$ is a constant, $r_0 : \mathcal{T} \to \mathbb{R}^{n/2}$ is the center of the obstacle, and $\ddot{\kappa}_{\max} \in \mathbb{R}_{\ge 0}$ is an upper bound on the possible values of $\ddot{\kappa}$ between t and $t+\delta$. We use formula (14c) for the bound ψ_h because κ in (14a) is not thrice differentiable, so we cannot make use of any higher order derivatives. Next, the rate of change of a Lyapunov function $V(t,x)$ can be upper bounded as

$$\psi_v(t+\delta, t, x) = \dot{V}(t,x) + \max\{0, \ddot{V}(t,x)\}\delta + \tfrac{1}{2}\dddot{V}_{\max}\delta^2 \text{(15)}$$

where $\dddot{V}_{\max} \in \mathbb{R}_{\ge 0}$ is an upper bound on the values of $\dddot{V}$.

1) Decreasing Conservatism: Note that the upper bounds derived in [6]–[8], [11], [28], [29] and implemented above were intended for relatively short horizon times $\tau - t$. For very large horizon times, these upper bounds can become overly conservative. We can optionally decrease this conservatism by breaking the interval $\tau - t$ into $n_\psi \in \mathbb{N}$ smaller intervals. To this end, let $\delta = (\tau - t)/n_\psi$ and $\tau_j = t + j\delta$, and for a scalar function $h : \mathcal{T} \times \mathcal{X} \to \mathbb{R}$, replace ψ_h as above with $\psi_h^* : \mathcal{T} \times \mathcal{X} \to \mathbb{R}^{n_\psi}$ with elements defined as

$$[\psi_h^*(\tau, t, x)]_j = \psi_h(\tau_j, \tau_{j-1}, p(\tau_{j-1}, t, x)) \quad (16)$$

for $j = 1, \cdots, n_\psi$. That is, ψ_h^* makes n_ψ exact state predictions using p in (3), which could be more expensive to compute, and bounds the evolution between these predictions using the original ψ_h function. This division is analogous to MPC with a control horizon of 1, a prediction horizon of n_ψ, and a discretization margin encoded in ψ_h. In the above work, all statements of the form $\psi_a(\cdot) \leq 0$, where a is h or v, can be equivalently replaced by $\psi_a^*(\cdot) \leq 0$ elementwise. We will demonstrate the utility of this strategy in Section IV.

IV. Simulations

We validate the above methods by simulating an impulsive system representative of spacecraft docking in low Earth orbit. Let $\mathcal{X} = \mathbb{R}^4$, $\mathcal{U} = \mathbb{R}^2$, let $\mu \in \mathbb{R}_{>0}$ be constant, and let

$$f(\cdot) = \begin{bmatrix} x_3 \\ x_4 \\ -\mu x_1/(x_1^2 + x_2^2)^{3/2} \\ -\mu x_2/(x_1^2 + x_2^2)^{3/2} \end{bmatrix}, \; g(\cdot) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 + u_1 \\ x_4 + u_2 \end{bmatrix}. \quad (17)$$

Let there be four CBFs h_i of the form (14b) for various obstacles $r_i(t) \in \mathbb{R}^2$, with ψ_{h_i} as in (14c). Let there be an additional constraint $\kappa_5(t, x) = (r - r_5)^{\mathrm{T}}(\dot{r}_5/\|\dot{r}_5\|) \leq 0$ with associated CBF h_5 also as in (14b). That is, κ_5 encodes that the controlled satellite r must always lie behind an uncontrolled target satellite $r_5(t) \in \mathbb{R}^2$. Let $x_t(t) = [r_5(t)^{\mathrm{T}}, \dot{r}_5(t)^{\mathrm{T}}]^{\mathrm{T}}$. We choose a Lyapunov function $V(t, x) = (x - x_t(t))^{\mathrm{T}} P (x - x_t(t))$ and approximation ψ_v^* as in (15) and (16). Let $\gamma_1, \gamma_2 \in \mathbb{R}_{\geq 0}$ and $J \in \mathbb{R}_{>0}$ be constants. The chosen control law is

$$u = \begin{cases} 0 & \psi_v(\cdot) \leq \gamma_1 V(t, x) \text{ and } \psi_{h_i}(\cdot) \leq 0,\ i \in \mathcal{I} \\ u^* & \text{else} \end{cases} \quad (18a)$$

where $(\cdot) = (t + \Delta t, t, x)$, $\mathcal{I} = \{1, 2, 3, 4, 5\}$, and u^* is

$$u^* = \arg\min_{u \in \mathbb{R}^2} u^{\mathrm{T}} u + J d^2 \quad (18b)$$

$$\text{s.t. } \psi_v^*(t + \Delta T, t, g(t, x, u)) \leq \gamma_1 V(t, x) + d \quad (18c)$$

$$V(t, g(t, x, u)) \leq \gamma_2 V(t, x) + d \quad (18d)$$

$$\psi_{h_i}(t + \Delta T, t, g(t, x, u)) \leq 0,\ i \in \mathcal{I}. \quad (18e)$$

We assume that the optimization (18) is always feasible, though we note that this is difficult to guarantee when there are multiple CBFs [2], [27], [38]. We simulated (18) using various choices of ΔT, and then repeated these simulations with ψ_h in (18e) replaced with ψ_h^* as in (16) with $n_{\psi_h} = 10$. The resultant trajectories, converted to Hill's frame for visualization, are shown in Fig. 1, and full results are shown in the video below[1]. A comparison to a trajectory pre-planner is also shown in Fig. 1, and details on select trajectories are shown in Figs. 2-3. All simulation code and parameters can also be found below[2].

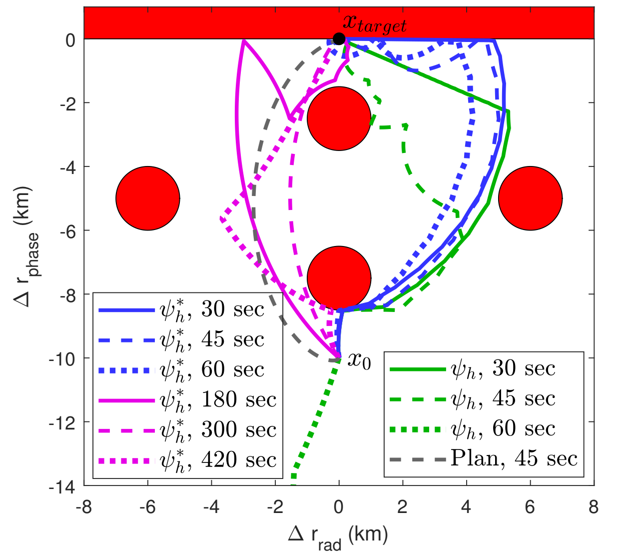

Fig. 1: Trajectories of (1) and (17) subject to the control (18)

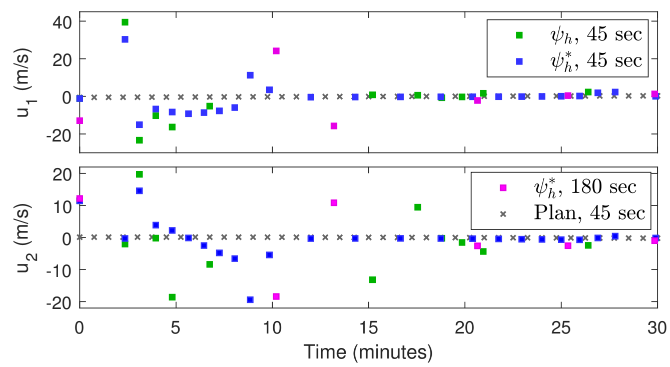

Fig. 2: Control inputs along selected trajectories in Fig. 1

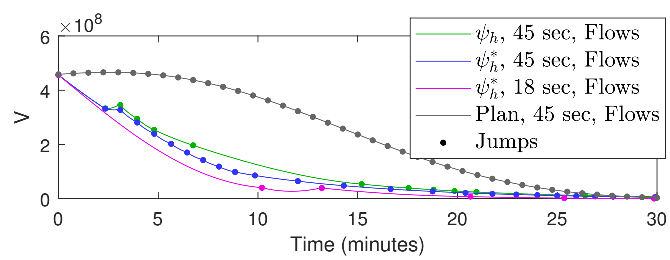

Fig. 3: Lyapunov function along selected trajectories in Fig. 1

All of the simulations in Fig. 1 remained safe, and eight of the nine trajectories converged to the target. The trajectory using ψ_h with $\Delta T = 60$ was so conservative that it immediately turned away from the target, whereas trajectories using ψ_h^* still converge with much larger ΔT, though the rate of convergence is slow for $\Delta T \geq 420$. This is because ψ_h^* implements (14c) with a smaller, less conservative, δ than ψ_h alone. That said, this decreased conservatism came at an average computational cost per control cycle, for $\Delta T = 45$, of 0.22 s using ψ_h^* and 0.022 s using ψ_h, both run on

[1] https://youtu.be/_o-FAGbvfgg

[2] https://github.com/jbreeden-um/phd-code/tree/main/2023/LCSS%20Impulsive%20Control

a 3.5 GHz CPU. The total fuel consumption varied from 188 m/s ($\Delta T = 30$ with ψ_h) to 18.2 m/s ($\Delta T = 300$ with ψ_h^*). For comparison, the pre-planned trajectory consumed between 12.2 m/s and 13.9 m/s depending on the choice of ΔT. This improvement is expected since (18) only considers T seconds of the trajectory at a time, whereas a pre-planner can optimize over longer sequences.

V. Conclusions

We have developed a methodology for extending the provable set invariance guarantees provided by CBFs to systems with impulsive actuators subject to a minimum dwell time constraint, and for ensuring asymptotic stability in the same systems. We encoded the resulting conditions in an optimization-based control law, which was successful in a simulated spacecraft docking. The conditions presented are generally nonlinear in the control input, thus leading to controllers that are nonlinear programs. We showed how one can reduce the conservatism of these controllers, in exchange for greater computational cost, by dividing the safety prediction horizon into multiple intervals using an MPC-like strategy. Future research directions might consider extensions to systems with disturbances, methods to further decrease conservatism, or the use of ITCBFs with optimal trajectory planning.

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