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# Feedback Optimization of Incentives for Distribution Grid Services

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*Abstract*— **Energy prices and net power injection limitations regulate the operations in distribution grids and typically ensure that operational constraints are met. Nevertheless, unexpected or prolonged abnormal events could undermine the grid's functioning. During contingencies, customers could contribute effectively to sustaining the network by providing services. This paper proposes an incentive mechanism that promotes users' active participation by essentially altering the energy pricing rule. The incentives are modeled via a linear function whose parameters can be computed by the system operator (SO) by solving an optimization problem. Feedback-based optimization algorithms are then proposed to seek optimal incentives by leveraging measurements from the grid, even in the case when the SO does not have a full grid and customer information. Numerical simulations on a standard testbed validate the proposed approach.**

## I. Introduction

The massive deployment of distributed energy resources (DERs) is dramatically changing distribution networks (DNs). *Prosumers*, i.e., entities that can be both producers and consumers of energy [1] will populate DNs and could provide services, e.g., by contributing to voltage profile improvements. Nevertheless, grid instabilities might arise if DERs are not properly managed.

*Literature Review:* Many works proposing control schemes for regulating net power injections in DNs assume that DERs apply power setpoints, possibly directly dispatched from the SO, aiming at the grid's well-being. However, prosumers may have priorities misaligned with those of the SO and refuse to cooperate. The work [2] treated the case in which the prosumer compliance is modeled with a Bernoulli distribution. SOs could leverage economic incentives like discounts on the energy price to encourage *rational* prosumers, i.e., aiming at maximizing their benefits, to provide grid services [3], [4] during abnormal operations, e.g., heat or cold waves [5]. The work [6] proposes an incentive-based mechanism facilitating the contribution of local flexible resources to the congestion management of DNs, fulfilling the SO's and prosumers' objectives. Authors of [7] develop an incentive scheme in which an aggregator coordinates several prosumers and determines the user payments by solving an asymmetric Nash bargaining model. A review of incentive mechanisms for DNs is provided in [3].

Market-based algorithms to incentivize DERs to provide services to the grid while maximizing their objectives and economic benefits were designed in the literature [8]–[10]. For example, customers may be incentivized to adjust the output powers of DERs in real-time to aid voltage regulation [11], control the aggregate network demand [12], and follow regulating signals [13]. The work [14] proposes a pricing mechanism for energy communities ensuring that operational constraints are satisfied and that the surplus of each community member is higher than the one under standalone settings. A trading scheme for increasing the exchange of electricity from prosumers to a distribution network meeting the network constraints is designed in [15].

*Statement of Contributions:* In this paper, we devise an incentive mechanism to promote the participation of prosumers in providing grid services. We assume that prosumers are subject to a Net Energy Metering (NEM) tariff design. NEM is a system that allows DERs owners to send excess energy back to the grid in exchange for credits on their utility bills. Under NEM 1.0, the system's first version, homeowners with solar panels could send excess energy back to the grid and receive credits at the retail rate [16]. The goal of the SO is to design optimal incentive functions that promote the satisfaction of operational constraints while minimizing the cost for the SO. The incentives make rational prosumers change their power demand to support grid operations by essentially altering the energy price and are designed so that the prosumers are not penalized or rewarded if they do not change their behavior. When the SO has full grid information, i.e., it knows the network topology, the power demands and generations, and the prosumer preferences, the incentives can be computed by solving an optimization problem. When instead partial information is available, we propose a feedback control framework. Power and voltage measurements compensate for the lack of information and are used to iteratively update the incentives until convergence to the optimal ones. We formally characterize the proposed framework under common choices of incentives and prosumer preferences, even though our approach can be applied when these are general differentiable functions, and provide numerical results over the standard IEEE 33 bus feeder.

*Notation:* Lower- (upper-) case boldface letters denote column vectors (matrices). The identity matrix, the vector of all ones, and the vector of all zeros are denoted by $\mathbf{I}$, $\mathbf{1}$, $\mathbf{0}$; the corresponding dimension will be clear from the context. The sets of real numbers and nonnegative real numbers are denoted as $\mathbb{R}$ and $\mathbb{R}^+$, respectively. The two norm of a matrix $\mathbf{A}$ is defined by $\|\mathbf{A}\| = \sqrt{\lambda_{\max}(\mathbf{A}^\top\mathbf{A})}$, where $\lambda_{\max}(\mathbf{A}^\top\mathbf{A})$ is the largest eigenvalue of $\mathbf{A}^\top\mathbf{A}$.

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## II. Grid Modeling

We model a low voltage[1] DN with $N+1$ buses with an undirected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$, where nodes $\mathcal{N} = \{0, 1, \ldots, N\}$ are associated with the electrical buses and whose edges represent the electric lines. The substation, labeled as 0, is modeled as an ideal voltage generator (the slack bus) imposing the nominal voltage of 1 p.u. Each bus, except the substation, is assumed to be a prosumer [1]. Prosumer $n$ can generate the active power $r_n \in \mathbb{R}_+$ potentially exploiting behind-the-meter DERs. Also, prosumer $n$ has an active and reactive power demands $d_n \in \mathbb{R}_+$ and $q_n \in \mathbb{R}$. The net active power injection is

$$p_n = r_n - d_n. \tag{1}$$

Net powers take positive (negative) values, i.e., $p_n, q_n \geq 0$ ($p_n, q_n \leq 0$) when they are *injected into* (*absorbed from*) the grid. When $p_n \geq 0$, $n$ behaves like a generator; when $p_n \leq 0$, $n$ behaves like a load. Let $\mathbf{d} \in \mathbb{R}^N$ and $\mathbf{r} \in \mathbb{R}^N$ collect all the demands and DER outputs. Potentially, each prosumer $n$ may have some flexibility in the net power injection, i.e.,

$$p_n \in [\underline{p}_n, \overline{p}_n], \quad n = 1, \ldots, N. \tag{2}$$

If $n$'s power injection is non flexible load, e.g., $n$ hosts a critical load, then $\underline{p}_n = \overline{p}_n$. The model (2) potentially captures load limitations enforced to not compromise the network's operation, e.g., dynamic operating envelopes [17]. The limitation (2) is then equivalent to

$$d_n \in \mathcal{D}_n = [\underline{d}_n, \overline{d}_n], \quad n = 1, \ldots, N. \tag{3}$$

Denote by $v_n \in \mathbb{R}$ the voltage magnitude at bus $n \in \mathcal{N}$, and let the vector $\mathbf{v} \in \mathbb{R}^N$ collect the voltage magnitudes of buses $1, \ldots, N$. Voltage magnitudes are nonlinear functions of the power injections; however first-order Taylor expansion of the power flow equation yields [1]

$$\mathbf{v} = \mathbf{R}\mathbf{p} + \mathbf{X}\mathbf{q} + \boldsymbol{\omega} \tag{4}$$

where $\mathbf{R} \in \mathbb{R}_+^{N \times N}$ and $\mathbf{X} \in \mathbb{R}_+^{N \times N}$ are symmetric and positive definite matrices [1] and $\boldsymbol{\omega} \in \mathbb{R}_+^N$. $\mathbf{R}$ and $\mathbf{X}$ represent the sensitivity of the voltage magnitudes w.r.t. net power injection variations.

## III. Incentives for Grid Services

According to the NEM 1.0, prosumer $n$ net power injection is charged following the rule

$$\gamma(p_n) = -\pi p_n + \pi_0$$

where $\pi > 0$ is the retail rate and $\pi_0$ captures non-volumetric surcharges, e.g., the connection charge [16]. When the prosumer net consumes (produces), the first term in $\gamma(p_n)$ is positive (negative), meaning that $n$ is charged (remunerated). Without loss of generality, we assume that the coefficients $\pi, \pi_0$ are the same for all the prosumers and fixed. Indeed, the price coefficients are defined in the contract between the utility and the customers and are usually updated once every several months or a few years.

The *surplus* of customer $n$ is the difference between the comfort and the payment from consumption

$$\begin{aligned} \hat{S}_n(d_n, r_n) &= U_n(d_n) - \gamma(p_n) \\ &= U_n(d_n) - \pi d_n + \pi r_n - \pi_0 \end{aligned} \tag{5}$$

where we used (1). The utility of consumption $U_n(d_n)$ is assumed to be strictly concave and continuously differentiable with a marginal utility function $\nabla U_n$. We denote the inverse marginal utility by $f_n := (\nabla U_n)^{-1}, \forall n \in \mathcal{N}$.

We assume that each prosumer $n$ acts *rationally*, i.e., aims to maximize its surplus. That is, $n$ sets its power demand as the solution of the following *prosumer optimization problem*

$$\hat{d}_n = \arg \max_{d_n \in \mathcal{D}_n} \hat{S}_n(d_n, r_n) \tag{6}$$

The optimal demand can be easily computed as

$$\hat{d}_n = [f_n(\pi)]_{\mathcal{D}_n}. \tag{7}$$

where $[\cdot]_{\mathcal{D}_n}$ denotes the projection onto the set $\mathcal{D}_n$.

Even though (2) typically ensures that the network operates correctly, unexpected or abnormal events, like sudden generation drops or heat and cold waves, might affect the network operations. The SO could then ask the prosumers to provide grid services to avoid grid damages and instabilities and compensate them by means of incentives captured by continuously differentiable functions $g_n(d_n, \xi_n)$ parameterized in $\xi_n$. For each $n \in \mathcal{N}$ and $\xi_n$, the incentive computed at the nominal consumption $\hat{d}_n$ should be zero, i.e.,

$$g_n(\hat{d}_n, \xi_n) = 0. \tag{8}$$

Equation (8) ensures that an agent is not charged or remunerated if it does not provide ancillary services, i.e., if it keeps is demand at $\hat{d}_n$. The function parameters for all the prosumers are collected in the vector $\boldsymbol{\xi} = \begin{bmatrix} \xi_1 & \ldots & \xi_N \end{bmatrix}^\top$. The incentive $g_n(d_n, \xi_n)$ essentially shapes prosumer $n$ surplus (5), which becomes

$$S_n(d_n, r_n, \xi_n) = \hat{S}_n(d_n, r_n) + g_n(d_n, \xi_n) \tag{9}$$

and the solution of the new prosumer optimization problem

$$d_n^*(\xi_n) = \arg \max_{d_n \in \mathcal{D}_n} S_n(d_n, r_n, \xi_n) \tag{10}$$

is favorable for grid operations, see Figure 1. The SO's goal is to find the $\boldsymbol{\xi}$ that minimize the cost of sustaining the distribution grid while ensuring that operational constraints are met, i.e., to solve the *incentive optimization problem*[2]

$$\boldsymbol{\xi}^* := \arg \min_{\boldsymbol{\xi}} \sum_n g_n(d_n^*(\xi_n), \xi_n) - \pi d_n^*(\xi_n) - \pi_0 + \pi r_n \tag{11a}$$

$$\text{s.t.} \quad \underline{\mathbf{v}} \leq \mathbf{v} \leq \overline{\mathbf{v}} \tag{11b}$$

$$\underline{p}_0 \leq p_0 \leq \overline{p}_0 \tag{11c}$$

$$\mathbf{d}^*(\boldsymbol{\xi}) \geq \mathbf{0} \tag{11d}$$

[1]The proposed methods are suitable for applications in both low-voltage and medium-voltage DNs. However, to keep the notation light, we will focus hereafter on single-phase low-voltage networks.

[2]Though in the following we will consider problem (11), in principle our approach is suitable and can be easily extended also when the problem of interest include other constraints, e.g., line flow limits. Also, constraints on $\boldsymbol{\xi}$ could be added to comply with possible regulatory frameworks.

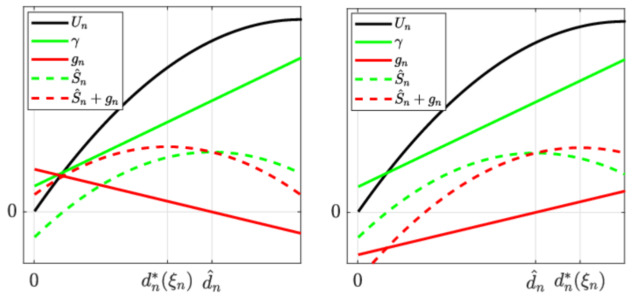


Fig. 1. The incentive function shapes prosumer $n$ surplus. Here, the utility of consumption is quadratic and the incentive function is linear. When $\xi_n$ is negative, the demand is reduced, see the left panel. When $\xi_n$ is positive, the demand increases, see the left panel. Heed that $g_n(\hat{d}_n, \xi_n)$ equals zero, meaning that no remuneration is given to $n$ if it does not provide services.

where the vector $\mathbf{d}^*(\boldsymbol{\xi})$ collects all the $d_n(\xi_n)$'s. The constraint (11b) captures voltage operational constraints; whereas (11c) enforces the power exchange with the external network to be within a desired interval, possibly modeling the case in which the grid is required to behave as a Virtual Power Plant. Finally, (11d) guarantees that the loads remain nonnegative. The interactions between the utility company (leader) and prosumers (followers) is a Stackelberg game [18], where the players select the optimal strategy by solving the optimization problems (10) and (11).

## IV. Feedback Algorithms for the Computation of the Optimal Incentives

The optimal incentive $\boldsymbol{\xi}^*$ can be computed by directly solving problem (11) when the SO has full network information, i.e., it knows the grid parameters $\mathbf{R}$, the power demands $\mathbf{d}$ and $\mathbf{q}$, the DER power outputs $\mathbf{r}$, the user preferences $U_n$'s, and the incentive functions $g_n$'s. However, such a scenario of perfect grid information is unusual in distribution networks, for instance, because of a lack of real-time metering infrastructure. Hence, we propose the following feedback optimization algorithms in which the missing information is compensated by *measurements* and problem (11) is solved iteratively. To that aim, it is convenient to introduce the Lagrangian of (11)

$$
\begin{aligned}
\mathcal{L}(\boldsymbol{\xi}, \overline{\boldsymbol{\lambda}}, \underline{\boldsymbol{\lambda}}, \boldsymbol{\nu}, \overline{\mu}, \underline{\mu}) &= g(\mathbf{d}^*(\boldsymbol{\xi}), \boldsymbol{\xi}) - \pi \mathbf{1}^\top \mathbf{d}^*(\boldsymbol{\xi}) + c' \\
&+ \overline{\boldsymbol{\lambda}}^\top (\mathbf{v} - \overline{\mathbf{v}}) - \underline{\boldsymbol{\lambda}}^\top (\mathbf{v} - \underline{\mathbf{v}}) - \boldsymbol{\nu}^\top \mathbf{d}^*(\boldsymbol{\xi}) \\
&+ \overline{\mu}(p_0 - \overline{p}_0) - \underline{\mu}(p_0 - \underline{p}_0). \qquad (12)
\end{aligned}
$$

where $c' = \sum_n (\pi r_n - \pi_0)$ and $g$ is the sum of all the $g_n$. Collect the Lagrange multipliers in

$$
\boldsymbol{\theta} := \begin{bmatrix} \overline{\boldsymbol{\lambda}}^\top & \underline{\boldsymbol{\lambda}}^\top & \overline{\mu}^\top & \underline{\mu}^\top & \boldsymbol{\nu}^\top \end{bmatrix}^\top, \quad \boldsymbol{\theta} \in \mathbb{R}_+^{3N+2}.
$$

*1) A Dual Ascent Method and a Primal-Dual Method:* First, consider the case in which the SO does not have available real-time information about behind-the-meter generation $r_n$ and reactive power demand $q_n$ of prosumers $1, \ldots, N$. A dual ascent algorithm solving (23) reads

$$
\begin{aligned}
\boldsymbol{\xi}(t+1) &= \arg\min_{\boldsymbol{\xi}} \mathcal{L}(\boldsymbol{\xi}(t), \boldsymbol{\theta}(t)) && (13a) \\
\overline{\boldsymbol{\lambda}}(t+1) &= \left[\overline{\boldsymbol{\lambda}}(t) + \epsilon(\mathbf{v}(t) - \overline{\mathbf{v}})\right]_{\mathbb{R}_+^N} && (13b) \\
\underline{\boldsymbol{\lambda}}(t+1) &= \left[\underline{\boldsymbol{\lambda}}(t) + \epsilon(\underline{\mathbf{v}} - \mathbf{v}(t))\right]_{\mathbb{R}_+^N} && (13c) \\
\overline{\mu}(t+1) &= \left[\overline{\mu}(t) + \epsilon(p_0(t) - \overline{p}_0)\right]_{\mathbb{R}_+} && (13d) \\
\underline{\mu}(t+1) &= \left[\underline{\mu}(t) + \epsilon(\underline{p}_0 - p_0(t))\right]_{\mathbb{R}_+} && (13e) \\
\boldsymbol{\nu}(t+1) &= \left[\boldsymbol{\nu}(t) + \epsilon \mathbf{d}^*(t)\right]_{\mathbb{R}_+^N} && (13f)
\end{aligned}
$$

The incentive parameters are iteratively updated in (13) until convergence to their optimum values. The values of $\mathbf{v}(t)$ and $p_0(t)$ are directly measured.

If the minimization in (13a) cannot be easily performed, the next first-order primal-dual method can be pursued

$$
\begin{aligned}
\boldsymbol{\xi}(t+1) &= \boldsymbol{\xi}(t) - \epsilon\Big(\nabla \mathbf{d}^*(t) \nabla_d g(\mathbf{d}^*(t), \boldsymbol{\xi}(t)) \\
&+ \nabla p_0(t)\left(\overline{\mu}(t) - \underline{\mu}(t)\right) - \nabla \mathbf{d}^*(t)\boldsymbol{\nu}(t) - \pi \nabla \mathbf{d}^*(t)\mathbf{1}\Big) \\
&+ \nabla_{\xi} g(\mathbf{d}^*(t), \boldsymbol{\xi}(t)) + \nabla \mathbf{v}(t)\left(\overline{\boldsymbol{\lambda}}(t) - \underline{\boldsymbol{\lambda}}(t)\right) && (14a) \\
&(13b) - (13f) && (14b)
\end{aligned}
$$

Note that to implement the primal update (14a), one needs the measurement of the demand $\mathbf{d}^*(t)$, the sensitivity of the demand to the incentive signal given by the gradient matrix $\nabla \mathbf{d}^*(t)$, and the sensitivity of the power-flow model to the incentives given by the gradients $\nabla \mathbf{v}(t)$ and $\nabla p_0(t)$. These matrices can be computed knowing the prosumer utility functions (the $U_n$'s) and the network model, or estimated from historical data (e.g., from previous demand response events).

*2) A Zero-Order Feedback-Based Methods:* Second, consider the most extreme case when also the sensitivity matrices above are unknown and the SO has available only demand and voltage measurements. We propose to use a zero-order method to seek saddle points of (12) similar to, e.g., [19]. In particular, we employ a double-evaluation approach for approximating the gradient of the Lagrangian:

$$
\widehat{\nabla} \mathcal{L}(t) := \frac{\boldsymbol{\zeta}(t)}{2\sigma}\left[\mathcal{L}\left(\widehat{\boldsymbol{\xi}}_+(t), \boldsymbol{\theta}(t)\right) - \mathcal{L}\left(\widehat{\boldsymbol{\xi}}_-(t), \boldsymbol{\theta}(t)\right)\right] \quad (15)
$$

where *perturbed* incentives $\widehat{\boldsymbol{\xi}}_+(t)$ and $\widehat{\boldsymbol{\xi}}_-(t)$ are applied to the system with $\widehat{\boldsymbol{\xi}}_\pm(t) := \boldsymbol{\xi}(t) \pm \sigma \boldsymbol{\zeta}(t)$. Here, $\sigma > 0$ is a parameter that controls the magnitude of perturbation, and $\boldsymbol{\zeta}(t) \in \mathbb{R}^N$ is a perturbation signal which can be either chosen as a random or deterministic process. In Section VI, we show an application in which $\boldsymbol{\zeta}(t)$ is a random signal. With approximation (15) at hand, the zero-order method is

$$
\begin{aligned}
\boldsymbol{\xi}(t+1) &= \boldsymbol{\xi}(t) - \epsilon \widehat{\nabla} \mathcal{L}(t) && (16a) \\
&(13b) - (13f). && (16b)
\end{aligned}
$$

Observe that (16a) can be implemented in a complete model-free fashion provided that the measurements of demand, voltages, and aggregate power are available.

*Remark 1:* Algorithms (13), (14), and (16) have a feedback control implementation. Indeed, voltage and

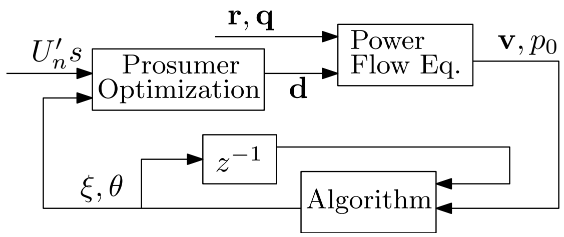


Fig. 2. Block scheme representation of the feedback control system.

power measurements enable the Lagrange multipliers updates (13b)–(13f) when the values of $\mathbf{v}$ and $p_0$ cannot be directly computed via the power flow equations because of lack of information. A schematic representation of the overall closed-loop system is given in Figure 2. In our setup, the system's state consists of the primal and the dual variables; the renewable generations, the prosumer utility functions $U_n$'s, and the reactive power demand act as a system's input.

## V. Convexification of the Incentive Optimization Problem and Stability Analysis

An SO can use the methods described in Section IV to solve the incentive problem (11), which is possibly nonconvex. In the following, we introduce commonly adopted choices of functions and parameters resulting in a convex problem that allows for explicit algorithm formulation and stability analysis.

*a) Quadratic utility functions:* similar to what is commonly done in the literature, e.g., see [18], [20], consider the quadratic prosumer utility functions

$$U_n(d_n) = -\frac{\alpha_n}{2} d_n^2 + \beta_n d_n, \quad \alpha_n \in \mathbb{R}^+, \beta_n \geq \pi. \tag{17}$$

*b) Unbounded power demand:* the prosumers can choose the $d_n$'s to be an arbitrary nonnegative number, i.e., we disregard (2). The projection in (7) would just complicate the notation hereafter without adding anything conceptually and can be performed easily in practical applications.

*c) Linear incentive functions:* we consider linear incentive functions of the form [21]

$$g_n(d_n, \xi_n) = \xi_n(d_n - \hat{d}_n)$$

Heed that a prosumer is not charged or remunerated if it does not change its power demand, i.e., $g_n(\hat{d}_n, \xi_n) = 0$.

*d) Approximated power exchange:* We consider the linearized power flow equation (4). Also we neglect the power losses and approximate the power delivered to the distribution network through the substation as

$$p_0 = -\sum_n p_n = \mathbf{1}^\top(\mathbf{d} - \mathbf{r}).$$

Together with the approximation (4), the former equation yields the convex optimization problem reported in the following. The SO could however in principle solve an optimization problem considering the true power flow equations.

The former choices yield the next quantities. The *nominal* (i.e., the one in the absence of incentives) demand and the net power injection for prosumer $n$ obtained by solving the prosumer optimization problem can be written as

$$\hat{d}_n := \frac{\beta_n - \pi}{\alpha_n}, \qquad \hat{p}_n := r_n - \hat{d}_n.$$

It is clear that the demand is a decreasing function of the energy price. The surplus (9) becomes

$$S_n(d_n, \xi_n, r_n) = -\frac{\alpha_n}{2} d_n^2 + \beta_n d_n - \pi d_n + \pi r_n - \pi_0 + \xi_n(d_n - \hat{d}_n)$$

and the optimal consumption (7) for prosumer $n$ is

$$d_n^*(\xi_n) = \frac{\beta_n - \pi + \xi_n}{\alpha_n} = \hat{d}_n + \frac{\xi_n}{\alpha_n}.$$

The new surplus maximizer is a linear perturbation of the one without incentives. Collect all the optimal consumptions with or without incentives in the vectors $\mathbf{d}^*(\boldsymbol{\xi})$ and $\hat{\mathbf{d}}$. Then,

$$\hat{\mathbf{d}} = \mathbf{A}\boldsymbol{\beta} - \pi\mathbf{A}\mathbf{1} \tag{18}$$

$$\mathbf{d}^*(\boldsymbol{\xi}) = \hat{\mathbf{d}} + \mathbf{A}\boldsymbol{\xi} \tag{19}$$

with $\mathbf{A} := \operatorname{diag}(\mathbf{a})$, $\mathbf{a} := \begin{bmatrix} \frac{1}{\alpha_1} & \dots & \frac{1}{\alpha_N} \end{bmatrix}^\top$, $\mathbf{A}$ positive definite. Constraint (11d) can be reformulated in terms of $\boldsymbol{\xi}$ as

$$\boldsymbol{\xi} \geq \pi\mathbf{1} - \boldsymbol{\beta}, \qquad \boldsymbol{\beta} := \begin{bmatrix} \beta_1 & \beta_2 & \dots & \beta_N \end{bmatrix}^\top. \tag{20}$$

Under $\mathbf{d}^*(\boldsymbol{\xi})$, the power delivered to the DN is

$$p_0(\boldsymbol{\xi}, \mathbf{r}) = \mathbf{1}^\top\mathbf{A}\boldsymbol{\xi} + \mathbf{1}^\top\hat{\mathbf{d}} - \mathbf{1}^\top\mathbf{r}. \tag{21}$$

Also, the remuneration due to the prosumers, i.e., the sum of the incentives, for their services is quadratic in $\boldsymbol{\xi}$. Indeed:

$$\sum_n g_n(d_n^*(\xi_n), \xi_n) - \pi d_n^*(\xi_n) + \pi r_n - \pi_0 = \\ = \sum_n \frac{\xi_n^2}{\alpha_n} - \frac{\pi}{\alpha_n}\xi_n - \gamma(\hat{p}_n) = \boldsymbol{\xi}^\top\mathbf{A}\boldsymbol{\xi} + \mathbf{b}^\top\boldsymbol{\xi} + c$$

where $\mathbf{b} := -\pi\mathbf{A}\mathbf{1}$, $c := -\sum_n \gamma(\hat{p}_n)$. Using (4) and (19), the voltage magnitudes become a function of the incentive

$$\mathbf{v}(\boldsymbol{\xi}, \mathbf{r}) = -\mathbf{R}\mathbf{A}\boldsymbol{\xi} + \mathbf{R}\mathbf{r} + \hat{\mathbf{v}} \tag{22}$$

with $\hat{\mathbf{v}} := \mathbf{X}\mathbf{q} - \mathbf{R}\hat{\mathbf{d}} + \boldsymbol{\omega}$. Equations (18) – (22) can be used to approximate (11) with the strictly convex problem

$$\boldsymbol{\xi}^* = \arg\min_{\boldsymbol{\xi}} \boldsymbol{\xi}^\top\mathbf{A}\boldsymbol{\xi} + \mathbf{b}^\top\boldsymbol{\xi} + c \tag{23a}$$

$$\text{s.t. } (20) - (21) - (22)$$

$$\underline{\mathbf{v}} \leq \mathbf{v} \leq \overline{\mathbf{v}} \tag{23b}$$

$$\underline{p}_0 \leq p_0 \leq \overline{p}_0 \tag{23c}$$

We will hereafter assume that the feasible set described by equations (20), (21), (22), (23b), and (23c) is *non empty*. Hence, problem (23) admits a unique minimizer. Defining

$$\boldsymbol{\Phi} := \begin{bmatrix} -\mathbf{A}\mathbf{R} & \mathbf{A}\mathbf{R} & \mathbf{A}\mathbf{1} & -\mathbf{A}\mathbf{1} & -\mathbf{I} \end{bmatrix}^\top, \ \boldsymbol{\Phi} \in \mathbb{R}^{(3N+2)\times N}$$

$$\boldsymbol{\phi} = \begin{bmatrix} (\hat{\mathbf{v}} - \overline{\mathbf{v}})^\top & (\underline{\mathbf{v}} - \hat{\mathbf{v}})^\top & \hat{\mathbf{d}}^\top\mathbf{1} - \overline{p}_0 \\ \underline{p}_0 - \hat{\mathbf{d}}^\top\mathbf{1} & \pi\mathbf{1}^\top - \boldsymbol{\beta} \end{bmatrix}^\top, \quad \boldsymbol{\phi} \in \mathbb{R}^{3N+2}$$

problem (23) can be rewritten as

$$\boldsymbol{\xi}^* = \arg\min_{\boldsymbol{\xi}} \boldsymbol{\xi}^\top \mathbf{A}\boldsymbol{\xi} + \mathbf{b}^\top\boldsymbol{\xi} + c \tag{24a}$$

$$\text{s.t.} \quad \boldsymbol{\Phi}\boldsymbol{\xi} + \boldsymbol{\phi} \leq \mathbf{0} \tag{24b}$$

and its Lagrangian (cf. (12)) is

$$\mathcal{L}(\boldsymbol{\xi}, \boldsymbol{\theta}) = \boldsymbol{\xi}^\top \mathbf{A}\boldsymbol{\xi} + \mathbf{b}^\top\boldsymbol{\xi} + c + \boldsymbol{\theta}^\top(\boldsymbol{\Phi}\boldsymbol{\xi} + \boldsymbol{\phi}). \tag{25}$$

We can use (25) to derive the equations of the dual ascent algorithm (13) or the primal-dual strategy (14) under the assumpion introduced earlier. The next result provides a condition for the convergence of algorithm (13) for the special case in which the renewable generation $\mathbf{r}$ is constant. The stability characterization of the primal-dual algorithm (14) and of the zero-order method (16) is left to future research.

*Proposition 5.1:* The dual ascent control scheme (13) is globally uniformly asymptotically stable if

$$\epsilon < 4\|\boldsymbol{\Phi}^\top\mathbf{A}^{-1}\boldsymbol{\Phi}\|^{-1}. \tag{26}$$

*Proof:* The minimizer w.r.t. the primal variable of the Lagrangian (25), which is

$$\boldsymbol{\xi}(\boldsymbol{\theta}) = -\frac{\mathbf{A}^{-1}}{2}(\mathbf{b} + \boldsymbol{\Phi}^\top\boldsymbol{\theta})$$

can be used to obtain the dual problem $\max_{\boldsymbol{\theta}\in\mathbb{R}_+^{3N+2}} \mathbf{h}(\boldsymbol{\theta})$, where

$$\mathbf{h}(\boldsymbol{\theta}) = \boldsymbol{\theta}^\top\left(\boldsymbol{\phi} - \frac{\boldsymbol{\Phi}\mathbf{A}^{-1}\mathbf{b}}{2}\right) - \boldsymbol{\theta}^\top\frac{\boldsymbol{\Phi}\mathbf{A}^{-1}\boldsymbol{\Phi}^\top}{4}\boldsymbol{\theta} - \frac{\mathbf{b}^\top\mathbf{A}^{-1}\mathbf{b}}{4}.$$

The former problem has zero duality gap with (23) because the Slater's conditions hold true [22]. The gradient of $\mathbf{h}$ is

$$\nabla\mathbf{h}(\boldsymbol{\theta}) = \frac{\boldsymbol{\Phi}\mathbf{A}^{-1}\boldsymbol{\Phi}^\top}{2}\boldsymbol{\theta} - \left(\frac{\boldsymbol{\Phi}\mathbf{A}^{-1}\mathbf{b}}{2} - \boldsymbol{\phi}\right)$$

and the dual ascent algorithm (13) becomes

$$\boldsymbol{\theta}(t+1) = \mathbf{f}(\boldsymbol{\theta}(t)), \quad \mathbf{f}(\boldsymbol{\theta}) = \left[\boldsymbol{\theta} + \epsilon\nabla\mathbf{h}(\boldsymbol{\theta})\right]_{\mathbb{R}_+^{3N+2}}.$$

By recalling that the projection is a nonexpansive operator, the map $\mathbf{f}$ is a contraction under condition (26). Indeed,

$$\begin{aligned}
&\|\mathbf{f}(\boldsymbol{\theta}) - \mathbf{f}(\boldsymbol{\theta}')\| = \\
&= \left\|\left[\boldsymbol{\theta} + \epsilon\nabla\mathbf{h}(\boldsymbol{\theta})\right]_{\mathbb{R}_+^{3N+2}} - \left[\boldsymbol{\theta}' + \epsilon\nabla\mathbf{h}(\boldsymbol{\theta}')\right]_{\mathbb{R}_+^{3N+2}}\right\| \\
&\leq \left\|\left(\mathbf{I} - \epsilon\frac{\boldsymbol{\Phi}\mathbf{A}^{-1}\boldsymbol{\Phi}^\top}{2}\right)(\boldsymbol{\theta} - \boldsymbol{\theta}')\right\| \\
&\leq \left\|\mathbf{I} - \epsilon\frac{\boldsymbol{\Phi}\mathbf{A}^{-1}\boldsymbol{\Phi}^\top}{2}\right\|\|\boldsymbol{\theta} - \boldsymbol{\theta}'\|
\end{aligned}$$

Now, we need to show that $\exists k \in \,]0,1[$ such that

$$\left\|\mathbf{I} - \epsilon\frac{\boldsymbol{\Phi}\mathbf{A}^{-1}\boldsymbol{\Phi}^\top}{2}\right\| \leq 1 - k \tag{27}$$

Denote by $\lambda_{\min}$ and $\lambda_{\max}$ the minimum and the maximum eigenvalues of $\boldsymbol{\Phi}\mathbf{A}^{-1}\boldsymbol{\Phi}^\top$. We have that $\lambda_{\max} = \boldsymbol{\Phi}\mathbf{A}^{-1}\boldsymbol{\Phi}^\top$. Being $\mathbf{A}$ positive definite, $\boldsymbol{\Phi}\mathbf{A}^{-1}\boldsymbol{\Phi}^\top$ is positive semidefinite. Also, since $\boldsymbol{\Phi}$ is full column rank, $\boldsymbol{\Phi}\mathbf{A}^{-1}\boldsymbol{\Phi}^\top$ is full rank and $\lambda_{\min} > 0$. Noting that $\left\|\mathbf{I} - \epsilon\frac{\boldsymbol{\Phi}\mathbf{A}^{-1}\boldsymbol{\Phi}^\top}{2}\right\| = \max\{1 - \frac{\epsilon}{2}\lambda_{\min}, \frac{\epsilon}{2}\lambda_{\max} - 1\}$, equation (27) is equivalent to the system

$$\begin{cases} k \leq \epsilon\frac{\lambda_{\min}}{2} \\ k \leq 2 - \epsilon\frac{\lambda_{\max}}{2}. \end{cases} \tag{28}$$

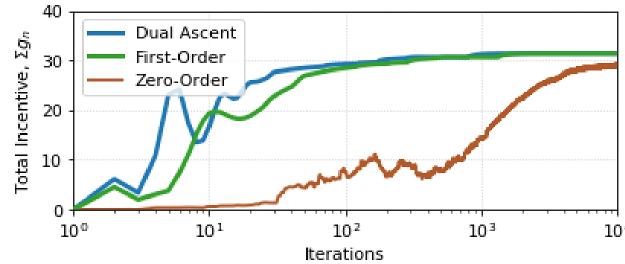


Fig. 3. Total incentive of customers vs. the number of iterations.

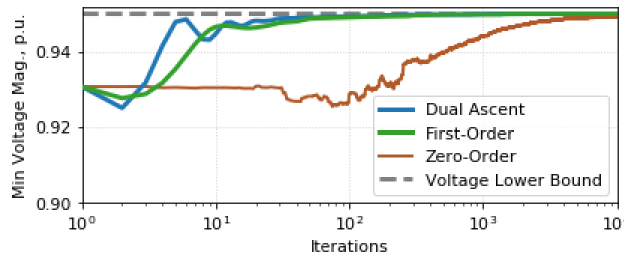


Fig. 4. Minimum nodal voltage magnitude vs. the number of iterations.

If (26) holds true, (27) is satisfied with $k > 0$ and meeting (28). The global asymptotic stability then follows. ■

## VI. Numerical Illustration

Here, we validate the incentive mechanism and the feedback-based optimization algorithms from Section IV on a realistic distribution feeder. The IEEE 33-bus radial distribution network [23] was simulated using PandaPower with 32 loads chosen randomly from 114 apartments sourced from the UMass Trace Repository [24] to be placed at each of the 32 load nodes. A normalized retail price $\pi$ was set at 1.0 and the prosumer quadratic utility function coefficients $\alpha_n$ were chosen uniformly at random between 0.3 and 3.0. A solar farm was connected to bus 31 with a capacity of 6 times its default node load size. The voltages are required to be in the range $[0.95 \text{ p.u.}, 1.05 \text{ p.u.}]$. Virtual power plant bounds of $\pm 0.2$ MW were placed around the power going into the feeder.

In the first set of numerical simulations, we considered a static case in which the utility function parameters, the reactive power demands, and the generations were fixed. The solar farm was disconnected causing some of the nodal voltage magnitudes to drop below 0.95 p.u. and initiating the incentive mechanism. The parameters of the algorithms were set to: $\epsilon = 0.5$ for the dual ascent; $\epsilon = 0.3$ for the first-order algorithm; $\sigma = 0.02$ and $\epsilon = 0.05$ for the zero-order method. A vector of uniform random variables between -1 and 1 was chosen for $\boldsymbol{\zeta}(t)$. The proposed algorithms are compared by showing their total incentive, minimum nodal voltage magnitude, and feeder power versus the number of iterations, in Figures 3, 4, and 5, respectively. As expected, the more information we have about the prosumers, the faster we can approach an optimal $\boldsymbol{\xi}$. Dual ascent utilizes complete knowledge of the prosumer utility functions to converge the fastest, while the first-order algorithm utilizes only the prosumer sensitivities to incentives to converge at a slightly slower rate. However, the zero-order algorithm has no knowledge of the prosumers and requires exploration to

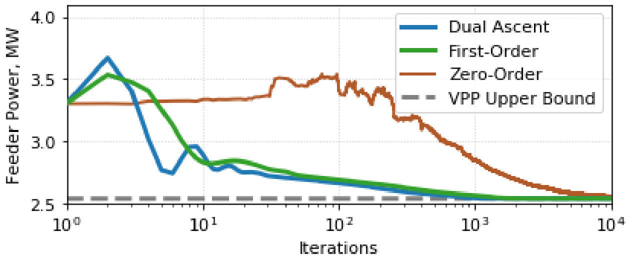


Fig. 5. Feeder power vs. the number of iterations.

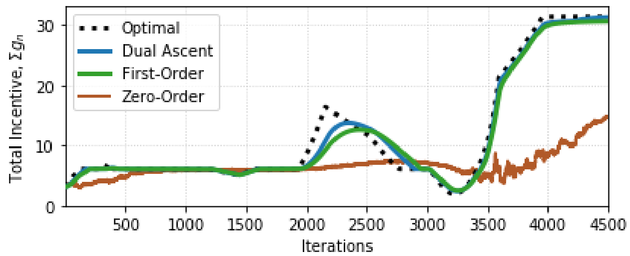


Fig. 6. Total incentive of customers vs. the number of iterations in the time-varying case.

slowly find effective values of $\xi$ with respect to the voltage and virtual power plant bounds.

In the second set of simulations, we tested the algorithms in a time-varying setting. The solar farm output is chosen according to the ISO New England aggregated solar production in the Western Massachusetts Zone on 9/19/2016 [25]. The incentive mechanism makes the prosumers adapt to the solar farm's volatility and ensures the fulfillment of the voltage and power constraints. Figure 6 shows the total incentive trajectories. The algorithms track the optimal incentives. Again, the dual ascent provides the best performance, followed by the primal-dual and then by the zero-order method.

## VII. Conclusion

We have presented an incentive mechanism that, by essentially changing the energy price, makes rational users change their demand and provide grid services, e.g., voltage and power regulation. The incentives are described here with affine functions. The function parameters that achieve the desired grid performance and minimize the cost for the SO can be computed by solving an optimization problem. When the problem cannot be directly solved because some grid/customer information is not available, we devised feedback control algorithms that iteratively update the incentives until convergence to the optimum. Future research directions include studying the convergence properties of our algorithms in time-varying cases and considering nonlinear incentive functions.

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