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Design of Stabilizing Feedback Controllers for High-Order Nonholonomic Systems

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Abstract— **This paper presents a novel stabilizing control design strategy for driftless control-affine systems with an arbitrary degree of nonholonomy. The proposed approach combines a time-varying control component that generates motion in the direction of prescribed Lie brackets with a state-dependent component, ensuring the stability of the equilibrium. The coefficients of the state-dependent component are derived in such a way that the trajectories of the resulting closed-loop system approximate the gradient flow of a Lyapunov-like function. In the case of a quadratic Lyapunov function, this guarantees the exponential stability of the equilibrium. The usability of this approach is demonstrated on general two-input systems having the fourth degree of nonholonomy. The proposed stabilization scheme is illustrated with several examples.**

I. Introduction

In recent decades, an increasing body of research has been dedicated to control problems for nonholonomic systems. These systems are characterized by non-integrable state and velocity constraints, which significantly complicate the development of control strategies. In practice, such systems describe the motion of many important engineering objects, such as mobile robots, wheeled systems, autonomous underwater vehicles, robotic manipulators, rolling bodies, etc. [1]. A large number of fundamental results exist on the control of nonholonomic systems, with comprehensive reviews of the main approaches provided, e.g., in [2]–[5]. It is worth noting that, as demonstrated in Brockett's seminal work [6], nonholonomic systems cannot be stabilized using a smooth time-invariant feedback law. While they can be stabilized using a time-varying feedback law [7] or a discontinuous time-invariant control [8], to date, there is no universally applicable methodology for stabilizing control design for general nonholonomic systems.

A wide range of approaches for controlling general nonholonomic systems relies on Lie-algebraic techniques. An essential assumption in this context is that the system's vector fields, along with their iterated Lie brackets, span the entire tangent space at each point of the state space. Several authors have leveraged this assumption to devise time-periodic control laws, aiming to make the trajectories of nonholonomic systems approximate those of an extended system. For instance, an algorithm for computing time-periodic feedback controls to approximate collision-free paths was introduced in [9]. Other studies, such as [10], [11], have exploited an unbounded sequence of oscillating controls with unbounded frequencies to approximate the trajectories of the extended system. In [12], it was shown that the solutions of oscillating systems can be approximated by the solutions of an averaged one, provided that the control frequencies tend to infinity. An approach for generating admissible paths of nonholonomic systems via solutions of an auxiliary parabolic PDE was described in [13], and auxiliary energy reduction technique was applied to the motion planning problem with homotopy class constraints in [14]. The complexity issue for open-loop controls that generate motion in the direction of higher-order Lie brackets was addressed in [15], [16] for systems with two inputs. A significant number of publications provide stabilization strategies for three-dimensional nonholonomic systems and multidimensional chained-form systems, as seen in [17]–[22]. However, the stabilization of general classes of systems with arbitrary degrees of nonholonomy remains an open challenge.

In this paper, we address the stabilization problem for high-order nonholonomic systems. Our control design builds upon the concept introduced in [23] for the controllability rank condition without iterated Lie brackets. This concept was further extended in [5], [24], [25] to encompass second-degree nonholonomic systems, trajectory tracking, and obstacle avoidance problems. The primary contribution of the present paper lies in providing time-varying feedback functions that ensure the exponential stability of the given equilibrium for systems with an arbitrary degree of nonholonomy. The developed approach is versatile in that it can be combined with known open-loop controls, which steer the system's motion towards the direction of predetermined Lie brackets. Once such open-loop controls are provided, we combine them with state-dependent coefficients defined by the system's vector fields and their iterated Lie brackets. In particular, we illustrate the proposed scheme with general two-input systems having a degree of nonholonomy of 4. Additionally, this paper presents a modified algorithm for computing the state-dependent coefficients that simplifies control design compared to the approach in [5], [24], [25], [26]. A key aspect of this modification involves replacing the matrix of the system's vector fields and their Lie brackets with a new one featuring a sparse structure.

The rest of this paper is organized as follows. Section II contains the problem statement along with necessary notations and definitions. The main results are presented in Section III and proved in Section V. Section IV illustrates the stabilization of the kinematic model of a car with two off-hooked trailers and a model nilpotent system.

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II. Problem statement, notations, definitions

Consider a driftless control-affine system

$$\dot{x} = \sum_{k=1}^{m} u_k f_k(x), \quad x \in D \subset \mathbb{R}^n,\ u \in \mathbb{R}^m, \tag{1}$$

where $x = (x_1, ..., x_n)^\top$ is the state, $u = (u_1, ..., u_m)^\top$ is the control, $m < n$, $0 \in D$, and $f_k : D \to \mathbb{R}^n$ are smooth. We focus on *constructing a feedback law* $u = h(t, x)$*, which exponentially stabilizes the equilibrium* $x^* = 0$ *of* (1)*, i.e. ensures that the solutions* $x(t)$ *of the closed-loop system with* $x(0)$ *from a given neighborhood of* x^* *satisfy* $\|x(t)\| = O(e^{-\lambda t})$ *as* $t \to \infty$*, with some* $\lambda > 0$*.*

To classify systems (1), we follow the line of [27].

Definition 1: Let $\Delta_1 = \operatorname{span}\{f_1, \ldots, f_m\}$, $\Delta_i = \Delta_{i-1} + [\Delta_{i-1}, \Delta_{i-1}]$, where $[\Delta_{i-1}, \Delta_{i-1}] = \operatorname{span}\{[g, h] : g \in \Delta_{i-1}, h \in \Delta_{i-1}\}$ for $i = 2, 3, ...$ (here, and in the sequel, $[g, h](x) = \frac{\partial h(x)}{\partial x} g(x) - \frac{\partial g(x)}{\partial x} h(x)$ denotes the Lie bracket of g and h). A number $\nu \in \mathbb{N}$ is the *degree of nonholonomy* of (1) at $x \in D$, if $\Delta_{\nu-1}(x) \neq \Delta_\nu(x) = \Delta_{\nu+1}(x) = ...$.

In the sequel, we assume $\dim(\Delta_\nu(x)) = n$ in D, so that (1) is controllable. Similar to [8], [23], we exploit the concept of sampling. For a given $\varepsilon > 0$, we define a partition π_ε of $[0, +\infty)$ into $T_j = [t_j, t_{j+1})$, $t_j = \varepsilon j$, $j = 0, 1, \ldots$.

Definition 2: Given a feedback $u = h(t, x)$, $h : [0, +\infty) \times D \to \mathbb{R}^m$, $\varepsilon > 0$, and $x^0 \in D$, a π_ε*-solution* of (1) corresponding to x^0 and $h(t, x)$ is a continuous function $x(t) \in D$ defined for $t \geq 0$ satisfying the initial condition $x(0) = x^0$ and the differential equations $\dot{x} = \sum_{k=1}^{m} h_k(t, x(t_j)) f_k(x)$, $t \in (t_j, t_{j+1})$, for each $j = 0, 1, 2, \ldots$.

Let us mention that sampled data techniques are widely used for controlling nonholonomic systems, see, e.g., [28], [29]. In our paper, we propose a hybrid approach in which the state-dependent control components remain constant over the ε-time interval and depend on the state of the system at $j\varepsilon$. Meanwhile, the time-dependent control components evolve continuously.

Notations: $\ell = \overline{1, \nu}$ – the integer ℓ varies from 1 to $\nu \in \mathbb{N}$; $B_\delta(x)$ and $\overline{B_\delta(x)}$ – the open and the closed δ-neighborhoods of an $x \in \mathbb{R}^n$ with $\delta > 0$, respectively;
$I_\ell = (i_1, \ldots, i_\ell)$ – ℓ-dimensional multi-index;
$f_{I_\ell}(x) = \left[f_{i_1}, \left[f_{i_2}, \left[\ldots \left[f_{i_{\ell-1}}, f_{i_\ell}\right]\right]\right]\right](x)$ – ℓ-th order iterated Lie bracket; for $I_1 \in \{1, ..., m\}$, f_{I_1} is a vector field of (1); $L^\infty([0, \epsilon]; R^m)$ – the class of essentially bounded measurable functions on $[0, \varepsilon]$ with values in $\mathbb{R}^m$.

III. Stabilizing controllers for systems with an arbitrary degree of nonholonomy

A. Control strategy

In this section, we propose a stabilizing control design methodology for a ν-th degree nonholonomic system (1), $\nu \geq 2$. Suppose that the vector fields of (1) satisfy the controllability rank condition, i.e. there are sets of multi-indices $S_\ell \subseteq \{1, ..., m\}^\ell$, $\ell = \overline{1, \nu}$, s. t. $\sum_{\ell=1}^{\nu} |S_\ell| = n$, and

$$\operatorname{span}\left\{f_{I_\ell}(x) \,|\, I_\ell \in S_\ell,\ \ell = \overline{1, \nu}\right\} = \mathbb{R}^n \text{ for all } x \in D. \tag{2}$$

Equivalently, condition (2) can be formulated as

Assumption 1: There are sets of multi-indices $S_\ell \subseteq \{1, ..., m\}^\ell$, $\ell = \overline{1, \nu}$, $|S_\ell| = n_\ell \in \mathbb{N}$, s. t. $\sum_{\ell=1}^{\nu} n_\ell = n$, and $\mathcal{F}(x) = \Big(f_{I_\ell}(x)\Big)_{I_\ell \in S_\ell, \ell = \overline{1, \nu}}$ is nonsingular matrix in D.

For the sake of convenience, we assume that (2) contains only left-iterated Lie brackets; however, this assumption is not restrictive since any iterated Lie bracket can be represented as a linear combination of left-iterated Lie brackets [30]. The main idea behind our control design approach is *to reduce the stabilization problem for* (1) *to constructing input functions that generate motion along the required Lie brackets.* Namely, consider a parameterized time-varying feedback law of the form $u = u^\varepsilon(t, x)$ with

$$u^\varepsilon(t, x) = \sum_{\ell=1}^{\nu} \varepsilon^{\frac{1}{\ell} - 1} \sum_{I_\ell \in S_\ell} \phi_{I_\ell}^{(\varepsilon)}(t, a_{I_\ell}(x)). \tag{3}$$

Here, $\varepsilon > 0$ is a parameter, $\phi_{I_\ell}^{(\varepsilon)} : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^m$ are time-varying components, and $a_{I_\ell} : D \to \mathbb{R}$ are state-dependent coefficients. The components of the vectors $\phi_{I_\ell}^{(\varepsilon)}$ are denoted by $\phi_{I_\ell}^{(\varepsilon,k)}$, i.e., for any $\alpha \in \mathbb{R}$, $\phi_{I_\ell}^{(\varepsilon)}(t, \alpha) = (\phi_{I_\ell}^{(\varepsilon,1)}(t, \alpha), ..., \phi_{I_\ell}^{(\varepsilon,m)}(t, \alpha))^\top \in \mathbb{R}^m$.

Our main goal is to prove that, under a proper choice of inputs $\phi_{I_\ell}^{(\varepsilon,k)}(t, \alpha)$, system (1) can be stabilized by controls (3) with the state-dependent coefficients $a_{I_\ell}(x)$ defined similarly to the approach of [5], [23], but for an arbitrary degree of nonholonomy.

B. Construction of state-dependent coefficients

Let us denote by $x(t; x^0, u(\cdot))$, $t \in [0, \varepsilon]$ the solution of system (1) corresponding to the initial condition $x(0) = x^0 \in D$ and admissible control $u(\cdot) \in L^\infty([0, \varepsilon]; \mathbb{R}^m)$.

Assume for a moment that there is a family of parameterized inputs $\phi_{I_\ell}^{(\varepsilon)}(t, \alpha)$, $I_\ell \in S_\ell$, $\varepsilon > 0$, $\alpha \in \mathbb{R}$, which satisfies:

P1) $\phi_{I_\ell}^{(\varepsilon)}(t, \alpha)$ are continuous in $(t, \alpha) \in \mathbb{R}^+ \times \mathbb{R}$, ε-periodic in t for each fixed α, and there are constants $C_{I_\ell} > 0$ such that $\|\phi_{I_\ell}^{(\varepsilon)}(t, \alpha)\| \leq C_{I_\ell} |\alpha|^{1/\ell}$ for all $t \geq 0$, $\alpha \in \mathbb{R}$;

P2) there exists a $\hat{\delta} > 0$ such that, for any column vector $\bar{\alpha} = (\alpha_{I_\ell})^T_{I_\ell \in S_\ell, \ell = \overline{1, \nu}} \in B_{\hat{\delta}}(0) \subset \mathbb{R}^n$, the solution $x(t; x^0, u(\cdot))$ of system (1) with $x^0 \in D$ and the control

$$u(t) = \sum_{\ell=1}^{\nu} \varepsilon^{\frac{1}{\ell} - 1} \sum_{I_\ell \in S_\ell} \phi_{I_\ell}^{(\varepsilon)}(t, \alpha_{I_\ell}) \tag{4}$$

is represented at time $t = \varepsilon$ as

$$x(t; x^0, u^\varepsilon(t, \bar{\alpha})) = x^0 + \varepsilon \mathcal{F}(x^0) \bar{\alpha} + \Omega(\varepsilon, x^0, \bar{\alpha}), \tag{5}$$

provided that $\varepsilon > 0$ is small enough, where the matrix $\mathcal{F}(x^0)$ is given in Assumption 1 and $\Omega(\varepsilon, x^0, \bar{\alpha}) \in \mathbb{R}^n$.

P3) there exist $\varpi > 0$, $\eta > 0$, and $\varepsilon_0 > 0$ such that the remainder of (5) satisfies the estimate $\|\Omega(\varepsilon, x^0, \bar{\alpha})\| \leq \varpi \varepsilon^{1+\eta} \|\bar{\alpha}\|$ for all $x^0 \in D$, $\bar{\alpha} \in B_{\hat{\delta}}(0) \subset \mathbb{R}^n$, $\varepsilon \in (0, \varepsilon_0]$.

We will demonstrate that the construction of control functions outlined above can be applied to develop explicit stabilizing control schemes for certain classes of nonholonomic systems.

Theorem 1: Assume that the vector fields $f_1, ..., f_m : D \to \mathbb{R}^n$ satisfy the rank condition (2) with some $\nu \geq 2$, D is a domain containing the point $0 \in \mathbb{R}^n$, and the vector functions $\phi_{I_\ell}^{(\varepsilon)} : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^m$ satisfy conditions P1)–P3). Let the feedback control be defined by (3) with the column vector $a(x) = (a_{I_\ell}(x))^\top_{I_\ell \in S_\ell, \ell=\overline{1,\nu}} \in \mathbb{R}^n$ obtained from

$$a(x) = -\gamma \mathcal{F}^{-1}(x)x, \ \gamma > 0. \tag{6}$$

Then for any $\delta > 0$ such that $\overline{B_\delta(0)} \in D$ and any $\bar{\gamma} \in \underline{(0, \gamma)}$, there exists an $\bar{\varepsilon} > 0$ such that, for any $x^0 = x(0) \in \overline{B_\delta(0)}$ and $\varepsilon \in (0, \bar{\varepsilon}]$, the corresponding π_ε-solution $x(t)$ of the closed-loop system (1), (3) satisfies the properties

$$\|x(t)\| = O(e^{-\bar{\gamma}t/\nu}), \ \|u^\varepsilon(t, x)\| = O(e^{-\bar{\gamma}t/\nu^2}) \text{ as } t \to \infty. \tag{7}$$

The proof of Theorem 1 is given in Section V.

Remark 1: In general, formulas (6) can be extended to $a(x) = -\gamma \mathcal{F}^{-1}(x) \nabla V(x)$, where V satisfies the estimates $\alpha_1 \|x - x^*\|^{2m} \leq V(x) \leq \alpha_2 \|x - x^*\|^{2m}$, $\beta_1 V(x)^{2-\frac{1}{m}} \leq \|\nabla V(x)\|^2 \leq \beta_2 V(x)^{2-\frac{1}{m}}$, $\left\|\frac{\partial^2 V(x)}{\partial x^2}\right\| \leq \mu V(x)^{1-\frac{1}{m}}$, with some $x^* \in D$ constants $\alpha_1, \alpha_2, \beta_1, \beta_2, \mu > 0$, and $m \geq 1$. Such $a(x)$ ensures an exponential decay rate of the π_ε solutions any given $x^* \in D$ for $m = 1$ and a polynomial decay rate for $m > 1$. This can be proved by exploiting the estimation techniques from [31].

C. Construction of time-dependent inputs

1) General strategy: Theorem 1 gives a constructive solution of the stabilization problem for general nonholonomic systems, provided that there exist time-varying inputs satisfying properties P1)–P3). For the construction of such inputs, some known results can be applied [5], [10], [11], [24], [32]. In this section, we discuss possible explicit approaches for the construction of time-varying inputs.

2) Nonholonomic systems with two inputs: The class of systems with two inputs is of special interest in the literature:

$$\dot{x} = u_1 f_1(x) + u_2 f_2(x), \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}^2. \tag{8}$$

There are several approaches devoted to the steering problem for (8), e.g., [15], [16]. Although the results of the above-mentioned papers do not yield a straightforward solution to the stabilization problem, they provide constructive ways for defining control inputs which generate the motion along a given Lie bracket. In this subsection, we demonstrate an implementation of such time-varying inputs into control strategies of the form (4).

To simplify presentation, we restrict this brief paper to the consideration of left-iterated Lie brackets of length $\nu \leq 4$. It is clear that the constant controls

$$\phi_1^{(\varepsilon,1)}(t, \alpha) = \phi_2^{(\varepsilon,2)}(t, \alpha) = \alpha, \ \phi_1^{(\varepsilon,2)}(t, \alpha) = \phi_2^{(\varepsilon,1)}(t, \alpha) = 0 \tag{9}$$

can be used to steer (8) along the vector fields f_I, $I \in \{1, 2\}$:

$$x(\varepsilon; x^0, \phi_1^{(\varepsilon)}(\cdot, \alpha)) = x^0 + \varepsilon\alpha f_1(x^0) + O(\varepsilon^2),$$
$$x(\varepsilon; x^0, \phi_2^{(\varepsilon)}(\cdot, \alpha)) = x^0 + \varepsilon\alpha f_2(x^0) + O(\varepsilon^2) \text{ for small } \varepsilon > 0.$$

Moreover, by computing the Chen–Fliess expansion with

$$\begin{aligned} &\phi_{12}^{(\varepsilon,1)}(t, \alpha) = \sqrt{4\kappa_{12}\pi|\alpha|} \cos(2\kappa_{12}\pi t/\varepsilon), \\ &\phi_{12}^{(\varepsilon,2)}(t, \alpha) = \sqrt{4\kappa_{12}\pi|\alpha|} \operatorname{sign}(\alpha) \sin(2\kappa_{12}\pi t/\varepsilon), \end{aligned} \tag{10}$$

$$\begin{aligned} &\phi_{112}^{(\varepsilon,1)}(t, \alpha) = (4\kappa_{112}\pi)^{\frac{2}{3}} |\alpha|^{\frac{1}{3}} \cos(2\kappa_{112}\pi t/\varepsilon), \\ &\phi_{112}^{(\varepsilon,2)}(t, \alpha) = -2(4\kappa_{112}\pi)^{\frac{2}{3}} |\alpha|^{\frac{1}{3}} \operatorname{sign}(\alpha) \cos(4\kappa_{112}\pi t/\varepsilon), \end{aligned} \tag{11}$$

$$\begin{aligned} &\phi_{1112}^{(\varepsilon,1)}(t, \alpha) = 2(2\kappa_{1112}\pi)^{\frac{3}{4}} |\alpha|^{\frac{1}{4}} \cos(2\kappa_{1112}\pi t/\varepsilon), \\ &\phi_{1112}^{(\varepsilon,2)}(t, \alpha) = -6(2\kappa_{1112}\pi)^{\frac{3}{4}} |\alpha|^{\frac{1}{4}} \operatorname{sign}(\alpha) \sin(6\kappa_{1112}\pi t/\varepsilon), \end{aligned} \tag{12}$$

we obtain, for small $\varepsilon > 0$:

$$x(\varepsilon; x^0, \varepsilon^{-\frac{1}{2}} \phi_{12}^{(\varepsilon)}(\cdot, \alpha)) = x^0 + \varepsilon\alpha[f_1, f_2](x^0) + O(\varepsilon^{\frac{3}{2}}),$$
$$x(\varepsilon; x^0, \varepsilon^{-\frac{2}{3}} \phi_{112}^{(\varepsilon)}(\cdot, \alpha)) = x^0 + \varepsilon\alpha[f_1, [f_1, f_2]](x^0) + O(\varepsilon^{\frac{4}{3}}),$$
$$x(\varepsilon; x^0, \varepsilon^{-\frac{3}{4}} \phi_{1112}^{(\varepsilon)}(\cdot, \alpha)) = x^0 + \varepsilon\alpha[f_1, [f_1, [f_1, f_2]]](x^0) + O(\varepsilon^{\frac{5}{4}}).$$

Here, κ_{12}, κ_{112}, and κ_{1112} are arbitrary nonzero integers. To ensure P3), we need the non-resonance assumption.

Assumption 2: There are no resonances of order up to 4 in each of the following tuples: $(\kappa_{12}, \ \kappa_{112}, \kappa_{1112})$, $(\kappa_{12}, \ 2\kappa_{112}, \kappa_{1112})$, $(\kappa_{12}, \ \kappa_{112}, 3\kappa_{1112})$, and $(\kappa_{12}, \ 2\kappa_{112}, 3\kappa_{1112})$.

Theorem 1 directly implies the following result.

Theorem 2: Let the rank condition (2) hold with $D = \mathbb{R}^5$, $\nu = 4$, $S_1 = \{1, 2\}$, $S_2 = \{(1, 2)\}$, $S_3 = \{(1, 1, 2)\}$, $S_4 = \{(1, 1, 1, 2)\}$:

$$\operatorname{rank}\{f_1, f_2, [f_1, f_2], [f_1, [f_1, f_2]], [f_1, [f_1, [f_1, f_2]]]\} = 5. \tag{13}$$

Suppose that the controls u_1, u_2 are defined by formulas (3) with $\nu = 4$ and $a(x) = -\gamma x$, where the time-varying inputs $\phi_1^{(\varepsilon)}(t, \alpha)$, $\phi_{12}^{(\varepsilon)}(t, \alpha)$, $\phi_{112}^{(\varepsilon)}(t, \alpha)$, $\phi_{1112}^{(\varepsilon)}(t, \alpha)$ are defined by (9), (10), (11), (12), respectively, and Assumption 2 is satisfied. Then the assertion of Theorem 1 holds.

D. Simplified formulas for state-dependent coefficients

Suppose that, in each set of columns f_{I_ℓ} of the matrix $\mathcal{F}(x)$ from Assumption 1, there is an $n_\ell \times n_\ell$ block, $n_\ell = |S_\ell|$, with nonzero determinant. Without loss of generality, assume that these blocks are located on the main diagonal of $\mathcal{F}(x)$, which can always be achieved by an appropriate change of variables. Then, we introduce the modified rank condition.

Assumption 3: There exist $S_\ell \subset \{1, \ldots, m\}^\ell$, $\ell = \overline{1, \nu}$, $|S_\ell| = n_\ell \in \mathbb{N}$, such that $\sum\limits_{\ell=1}^{\nu} n_\ell = n$, and the following $n_\ell \times n_\ell$ matrices are nonsingular in D: $\tilde{\mathcal{F}}_\ell(x) = \left(f_{I_\ell}^\ell(x)\right)_{I_\ell \in S_\ell}$, where the vectors $f_{I_\ell}^\ell : \mathbb{R}^n \to \mathbb{R}^{n_\ell}$ are defined as $f_{I_\ell}^\ell(x) = \left(f_{I_\ell, n_{\ell-1}+1}(x), \ldots, f_{I_\ell, n_{\ell-1}+n_\ell}(x)\right)^\top$, $\ell = \overline{1, \nu}$, with $f_{I_\ell, j}$ standing for the j-th element of f_{I_ℓ}, $n_0 := 0$.

Assumption 3 allows us to exploit the matrix

$$\tilde{\mathcal{F}}(x) = \begin{pmatrix} \tilde{\mathcal{F}}_1(x) & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ldots \\ 0 & 0 & 0 & \tilde{\mathcal{F}}_\nu(x) \end{pmatrix}, \tag{14}$$

which is simpler to invert than the matrix $\mathcal{F}(x)$.

This property may be particularly advantageous for high-dimensional systems, where employing matrix $\mathcal{F}(x)$ results

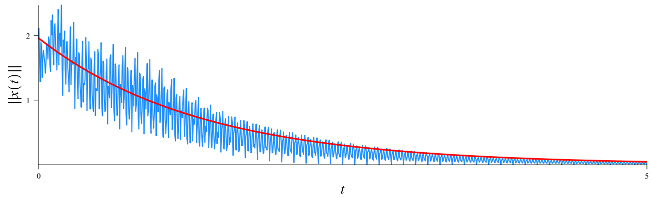

Fig. 1. Time plots of the norm of the solution (blue) of system (16), (17), (18) and the function $\|x(0)\|e^{-\gamma t/4}$ (red).

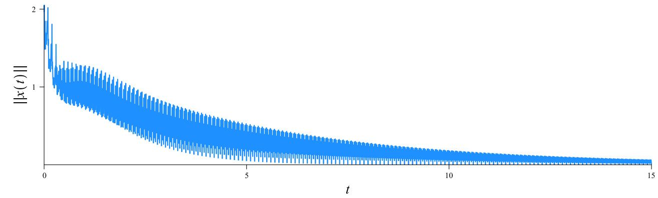

Fig. 2. Time plots of the norm of the solution of system (16), (17), (15) with the sparse matrix $\tilde{\mathcal{F}}$ given by (19).

in unwieldy expressions for $a(x)$. The principal conclusion of this section is that, under certain mild additional assumptions, the vector of state-dependent coefficients can be expressed as

$$a(x) = -\gamma \tilde{\mathcal{F}}^{-1}(x)x,\ \gamma > 0. \tag{15}$$

Theorem 3: Assume that $f_1, ..., f_m : D \to \mathbb{R}^n$ satisfy Assumption 3 with some $\nu \geq 2$, D is a domain containing the point $0 \in \mathbb{R}^n$, and the vector functions $\phi^{(\varepsilon)}_{I_\ell} : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^m$ satisfy conditions P1)–P3). Suppose also that

P4) $\|\mathcal{F}(x) - \tilde{\mathcal{F}}(x)\| \leq c_\Delta \|x\|$ for all $x \in D$, with some $c_\Delta \geq 0$.

Let the feedback control be defined by (3) with $a(x)$ defined by (15). Then there exist $\delta > 0$ and $\bar{\varepsilon} > 0$ such that, for any $x^0 = x(0) \in \overline{B_\delta(0)}$ and $\varepsilon \in (0, \bar{\varepsilon}]$, the corresponding π_ε-solution $x(t)$ of the closed-loop system (1), (3) satisfies (7). The proof of Theorem 3 is outlined in Section V-B.

IV. Examples

A. Car with two trailers model

Consider the kinematic model of a car with two off-hooked trailers presented, e.g., in [26]:

$$\dot{x} = u_1 f_1(x) + u_2 f_2(x), \quad x \in \mathbb{R}^5,\ u \in \mathbb{R}^2, \tag{16}$$

$$f_1(x) = (\cos x_3, \sin x_3, 0, \sin(x_3 - x_4), f_{15}(x))^\top,$$
$$f_2(x) = (0, 0, 1, -d_0 \cos(x_3 - x_4), f_{25}(x))^\top,$$
$$f_{15}(x) = \sin(x_3 - x_5) + (d_1 + 1)\sin(x_4 - x_3)\cos(x_4 - x_5),$$
$$f_{25}(x) = d_0(d_1 + 1)\cos(x_3 - x_4)\cos(x_4 - x_5) - d_0 \cos(x_3 - x_5).$$

Here, (x_1, x_2) represents the coordinates of the center of the car, x_3, x_4, and x_5 represent the angles between the horizontal line and longitudinal axes of the car, the first, and the second trailers, respectively, u_1 denotes the driving velocity, and u_2 stands for the steering velocity of the front axle, and d_0, d_1 are length parameters. Stabilizing controls for (16) were proposed in [26] through a transformation to privileged coordinates and the construction of a nilpotent quasihomogeneous approximate system. In this section, we demonstrate the exponential stabilizability of system (16) using the controls outlined in Theorems 2 and 3. It is noteworthy that, unlike the results presented in [26], this approach does not require any specific (and sometimes cumbersome) transformations. In this case, condition (2) holds in a neighborhood of the origin with $\nu = 4$, $S_1 = \{1, 2\}$, $S_2 = \{(1, 2)\}$, $S_3 = \{(1, 1, 2)\}$, $S_4 = \{(1, 1, 1, 2)\}$, and the 5×5 matrix $\mathcal{F}(x)$ has the form

$$\begin{pmatrix} \cos x_3 & 0 & \sin x_3 & 0 & 0 \\ \sin x_3 & 0 & -\cos x_3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \sin(x_3 - x_4) & -d_0 \cos(x_3 - x_4) & f_{34}(x) & f_{44}(x) & f_{54}(x) \\ f_{15}(x) & f_{25}(x) & f_{35}(x) & f_{45}(x) & f_{55}(x) \end{pmatrix},$$

where the expressions for $f_{ij}(x)$ can be found in [26]. We apply the functions $\phi^{(\varepsilon)}_1$, $\phi^{(\varepsilon)}_2$, $\phi^{(\varepsilon)}_{12}$, $\phi^{(\varepsilon)}_{112}$, $\phi^{(\varepsilon)}_{1112}$ given by (9)–(12) to satisfy conditions P1)–P3). Then, the control design scheme proposed in Theorem s 1 and 2 results in the following time-varying feedback law of the form (3):

$$\begin{aligned} u^\varepsilon(t, x) =& \phi^{(\varepsilon)}_1(t, a_1(x)) + \phi^{(\varepsilon)}_2(t, a_2(x)) + \varepsilon^{-\frac{1}{2}} \phi^{(\varepsilon)}_{12}(t, a_{12}(x)) \\ &+ \varepsilon^{-\frac{2}{3}} \phi^{(\varepsilon)}_{112}(t, a_{112}(x)) + \varepsilon^{-\frac{3}{4}} \phi^{(\varepsilon)}_{1112}(t, a_{1112}(x)), \end{aligned} \tag{17}$$

$$a(x) = -\gamma \mathcal{F}^{-1}(x)(x_1, x_2, x_3, x_4, x_5)^\top,\ \gamma > 0. \tag{18}$$

According to Theorem 1, the solutions of system (16), (17), (18) exponentially tend to 0 with the decay rate $\|x(t)\| = O\left(e^{-\bar{\gamma} t/4}\right)$ as $t \to \infty$, where $\bar{\gamma}$ can be made arbitrary close to γ by choosing a small enough ε. Fig. 1 illustrates the behavior of solutions of system (16), (17), (18). In particular, it depicts time-plots of the norm of the solutions together with the graph of $\|x(0)\|e^{-\gamma t/4}$. The results of numerical simulations illustrate the exponential convergence of the norm to 0, confirming the theoretical estimate provided by Theorem 1. For this example, we take the mechanical parameters $d_0 = d_1 = 0.1$, and the control parameters $\varepsilon = 0.1$, $\gamma = 3$, $\kappa_{1112} = 5, \kappa_{112} = 6, \kappa_{12} = 8$, with the initial condition $x(0) = \left(1, -1, \frac{\pi}{4}, \frac{\pi}{4}, -\frac{\pi}{4}\right)^\top$. Let us underline that the parameters κ_{12}, κ_{112}, κ_{1112} are chosen according to the non-resonance Assumption 2. To illustrate the results of Section III-D, consider the matrix

$$\tilde{\mathcal{F}}(x) = \begin{pmatrix} \cos x_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\cos x_3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & f_{44}(x) & 0 \\ 0 & 0 & 0 & 0 & f_{55}(x) \end{pmatrix}. \tag{19}$$

The matrix $\tilde{\mathcal{F}}(x)$ can be represented in the block-diagonal form (14) with the change of variables $\tilde{x}_1 = x_1$, $\tilde{x}_2 = x_3$, $\tilde{x}_3 = x_2$, $\tilde{x}_4 = x_4$, $\tilde{x}_5 = x_5$. According to Theorem 3, we can take controls of the form (16) with $a(\tilde{x}) = -\gamma \mathcal{F}^{-1}(\tilde{x})(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5)^\top$. Fig. 2 presents the time plot of the norm of the solution of system (16), (17) with $\tilde{\mathcal{F}}(x)$ and the same parameters as before. It should be noted that the solution exhibits exponential convergence, albeit slower compared to functions (18). Nevertheless, the main advantage of the proposed modification is simpler computations, which are especially beneficial for high-dimensional systems.

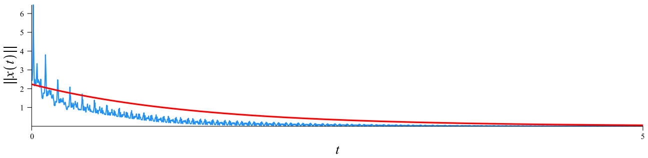

Fig. 3. Time plot of the norm of the solution of system (20), (21) (blue) and the function $\|x(0)\|e^{-\gamma t/4}$ (red).

B. Nilpotent system with the growth vector $r = (2, 3, 5, 6)$

Consider the nonlinear system presented in [15]:

$$\dot{x} = u_1 f_1(x) + u_2 f_2(x), \quad x \in \mathbb{R}^6,\ u \in \mathbb{R}^2, \tag{20}$$
$$f_1(x) = (1, 0, -x_2, -x_1x_2, -x_2^2, (x_1^2 - x_2^2)x_2)^\top,$$
$$f_2(x) = (0, 1, x_1, x_1^2, x_1x_2, (x_2^2 - x_1^2)x_1)^\top.$$

For all $x \in \mathbb{R}^6$, system (20) satisfies the rank condition (2):

$$\operatorname{span}\{f_1(x), f_2(x), [f_1, f_2](x), [f_1, [f_1, f_2]](x), [f_2, [f_2, f_1]](x), [f_1, [f_1, [f_1, f_2]]](x)\} = \mathbb{R}^6,$$

so that $\nu = 4$, $S_1 = \{1, 2\}$, $S_2 = \{(1, 2)\}$, $S_3 = \{(1, 1, 2), (2, 2, 1)\}$, $S_4 = \{(1, 1, 1, 2)\}$. Applying the construction of Theorem 1, we obtain the following feedback law for system (20):

$$\begin{aligned} u^\varepsilon(t, x) =& \phi_1^{(\varepsilon)}(t, a_1(x)) + \phi_2^{(\varepsilon)}(t, a_2(x)) + \varepsilon^{-\frac{1}{2}}\phi_{12}^{(\varepsilon)}(t, a_{12}(x)) \\ &+ \varepsilon^{-\frac{2}{3}}\phi_{112}^{(\varepsilon)}(t, a_{112}(x)) + \varepsilon^{-\frac{2}{3}}\phi_{221}^{(\varepsilon)}(t, a_{221}(x)) \\ &+ \varepsilon^{-\frac{3}{4}}\phi_{1112}^{(\varepsilon)}(t, a_{1112}(x)), \end{aligned} \tag{21}$$

where $\phi_1^{(\varepsilon)}$, $\phi_2^{(\varepsilon)}$, $\phi_{12}^{(\varepsilon)}$, $\phi_{112}^{(\varepsilon)}$, $\phi_{1112}^{(\varepsilon)}$ are defined in (9)–(12), the function $\phi_{221}^{(\varepsilon)}$ is obtained from (11) by interchange of indices $1 \leftrightarrow 2$, and $a(x) = (a_1(x), a_2(x), a_{12}(x), a_{112}(x), a_{221}(x), a_{1112}(x))^\top$, $a_1(x) = -\gamma x_1$, $a_2(x) = -\gamma x_2$, $a_{12}(x) = -\frac{\gamma}{2}x_3$, $a_{112}(x) = \gamma\left(\frac{1}{2}x_1x_3 - \frac{1}{3}x_4\right)$, $a_{221}(x) = \gamma\left(\frac{1}{3}x_5 - \frac{1}{2}x_2x_3\right)$, $a_{1112}(x) = -\frac{\gamma}{24}\left(6(x_1^2 - x_2^2)x_3 - 8x_1x_4 + 8x_2x_5 - 3x_6\right)$.

Fig. 3 shows the time plot of the norm of the solution $x(t)$ to system (20), (21) for $x(0) = (1, -1, 1, -1, 1, -1)^\top$ and parameters $\gamma = 3$, $\varepsilon = 0.1$, $\kappa_{12} = 3$, $\kappa_{112} = 1$, $\kappa_{221} = 5$, $\kappa_{1112} = 13$. The plot of the function $\|x(0)\|e^{-\gamma t/4}$ is shown in red. The results of numerical simulations illustrate the accuracy of the decay rate estimate from Theorem 1.

V. Proof of the main result

A. Proof of Theorem 1

For a given $\delta > 0$ such that $D_0 = \overline{B_\delta(0)} \subset D$, we define M_f, $M_{\mathcal{F}}$, $L > 0$, such that, for all $x \in D_0$, $i = \overline{1, m}$, $\|\mathcal{F}^{-1}(x)\| \le M_{\mathcal{F}}$, $\left\|\frac{\partial f_i(x)}{\partial x}\right\| \le L$, $\|f_i(x)\| \le M_f$. The above constants exist under the assumptions of Theorem 1 due to continuity of the corresponding maps. To simplify the presentation, we denote the solution of system (1) with the initial condition x^0 and control $u^\varepsilon(t, x^0)$ at time t as $x(t)$, i.e $x(t) := x(t; x^0, u(\cdot))$.

In the first step, we estimate the norm of the control (3) and of the vector $x(t)$ on $[0, \varepsilon]$, for any $\varepsilon > 0$. With this purpose, consider the function $U(x^0) := \sum_{\ell=1}^{\nu} \varepsilon^{\frac{1}{\ell}-1} \sum_{I_\ell \in S_\ell} \|\phi_{I_\ell}^{(\varepsilon)}(t, a_{I_\ell}(x^0))\|$, which is defined for any $x^0 \in D$. From $P1$) and the Hölder inequality,

$$U(x^0) \le \sum_{\ell=1}^{\nu} \varepsilon^{\frac{1}{\ell}-1}\|a(x^0)\|^{1/\ell}C_\ell, \tag{22}$$

where $C_\ell = \sqrt{m}\Big(\sum_{I_\ell \in S_\ell}(C_{I_\ell})^{\frac{2\ell}{2\ell-1}}\Big)^{1-\frac{1}{2\ell}} > 0$. Thus, for all $t \ge 0$, $\varepsilon > 0$, $x^0 \in D$,

$$\|u^\varepsilon(t, x^0)\| \le \sum_{\ell=1}^{\nu} \varepsilon^{\frac{1}{\ell}-1}\|a(x^0)\|^{1/\ell}C_\ell. \tag{23}$$

Let $d > 0$ be the distance from the set D_0 to the boundary of D, and let $\varepsilon_1 > 0$ be any number satisfying the inequality

$$M_f\sqrt{m}\sum_{\ell=1}^{\nu} \varepsilon_1^{\frac{1}{\ell}}\|a(x)\|^{1/\ell}C_\ell < d \text{ for all } x \in D_0. \tag{24}$$

If $D = \mathbb{R}^n$, $\varepsilon_1 > 0$ can be arbitrary. Then, for all $t \in [0, \varepsilon]$,

$$\begin{aligned} \|x(t) - x^0\| &\le \int_0^t \sum_{k=1}^{m} \|f_k(x(t))\||u_k^\varepsilon(x^0, t)|dt \\ &\le M_f\sqrt{m}\sum_{\ell=1}^{\nu} \varepsilon^{\frac{1}{\ell}}\|a(x^0)\|^{1/\ell}C_\ell < d. \end{aligned} \tag{25}$$

This means that, for any $x^0 \in D_0$ and $\varepsilon \in (0, \varepsilon_1]$, the solution $x(t)$ of system (1) with control $u^\varepsilon(t, x^0)$ of the form (3) satisfies the property $x(t) \in D$ for all $t \in [0, \varepsilon_1]$.

Let us analyze property $P2$) with $\bar{\alpha} = a(x^0)$ defined by (6). Together with $P3$), this implies that

$$\begin{aligned} \|x(\varepsilon)\| &= \|x^0 + \varepsilon\mathcal{F}(x^0)a(x^0) + \Omega(\varepsilon, x^0, a(x^0))\| \\ &= \|x^0(1 - \varepsilon\gamma) + \Omega(\varepsilon, x^0, \bar{\alpha})\| \\ &\le \|x^0(1 - \varepsilon\gamma)\| + \varpi\gamma M_{\mathcal{F}}\varepsilon^{1+\eta}\|x^0\|. \end{aligned}$$

Given any $\bar{\gamma} \in (0, \gamma)$, let ε_2 be such that $\varepsilon_2 < \min\left\{\varepsilon_1, \frac{1}{\gamma}, \frac{1}{\varpi M_{\mathcal{F}}}\left(1 - \frac{\bar{\gamma}}{\gamma}\right)\right\}$. Then, for any $\varepsilon \in (0, \varepsilon_2]$, $\|x(\varepsilon)\| \le \|x^0\|(1 - \bar{\gamma}\varepsilon)$. Repeating the above argumentation for $x(j\varepsilon) \in D_0$ for $j = 1, 2, \dots$, we conclude that $x(t) \in D$ for all $t \ge 0$, and $\|x(j\varepsilon)\| \le \|x^0\|(1 - \bar{\gamma}\varepsilon)^j \le \|x^0\|e^{-\bar{\gamma}j\varepsilon}$ for any $j \in \mathbb{N}$. Together with estimate (25) this implies that, for any $j \in \mathbb{N}$, $t \in [j, (j+1)\varepsilon]$, $\|x(t)\| \le \|x^0\|e^{-\bar{\gamma}j\varepsilon} + M_f\sqrt{m}U(x(j\varepsilon))\varepsilon$. From (22), $U(x(j\varepsilon))\varepsilon \le \sum_{\ell=1}^{\nu} \varepsilon^{\frac{1}{\ell}}\|a(x(j\varepsilon))\|^{1/\ell}C_\ell \le C_u\|x(j\varepsilon)\|^{\frac{1}{\nu}} \le C_u\|x^0\|^{\frac{1}{\nu}}e^{-\frac{\bar{\gamma}j\varepsilon}{\nu}}$, where $C_u = \sum_{\ell=1}^{\nu}(\delta)^{\frac{1}{\ell}-\frac{1}{\nu}}M_{\mathcal{F}}^{\frac{1}{\ell}}C_\ell$. Thus, for $t \to \infty$,

$$\|x(t)\| = O\left(e^{-\bar{\gamma}t/\nu}\right) \text{ and } \|u^\varepsilon(t, x)\| = O\left(e^{-\bar{\gamma}t/\nu^2}\right),$$

so that the assertion of Theorem 1 holds with $\bar{\varepsilon} = \varepsilon_2$.

B. Proof of Theorem 3

This result is proved similarly to Theorem 1. The main difference concerns estimating the remainder Ω in representation (5). In this case, we have $\|x(\varepsilon)\| = \|x^0 + \varepsilon\tilde{\mathcal{F}}(x^0)a(x^0) + \varepsilon(\mathcal{F}(x^0) - \tilde{\mathcal{F}}(x^0))a(x^0) + \Omega(\varepsilon, x^0, a(x^0))\| \le \|x^0\|(1 - \varepsilon\gamma) + c_\Delta\gamma\varepsilon\|x^0\|^2 + \varpi\gamma M_{\mathcal{F}}\varepsilon^{1+\eta}\|x^0\|$.

Assuming $0 < \delta < \frac{1}{c_\Delta}\left(1 - \frac{\hat{\gamma}}{\gamma}\right)$ with any $\hat{\gamma} \in (0, \gamma)$, we get $\|x(\varepsilon)\| \leq \|x^0\|(1 - \varepsilon\hat{\gamma}) + \varpi\gamma M_{\mathcal{F}}\varepsilon^{1+\eta}\|x^0\|$. The remainder of the proof proceeds along the same line as the proof of Theorem 1.

VI. Conclusions

The main theoretical contribution of this work, summarized in Theorem 1, provides an explicit design for time-varying controllers that exponentially stabilize the equilibrium of a general driftless control-affine system. Unlike previous papers [5], [23]–[26], the controllability rank condition (2) incorporates Lie brackets of arbitrary length, making the proposed control strategies suitable for systems with higher degrees of nonholonomy. Our general construction is adapted for a specific class of two-input systems with iterated Lie brackets up to length 4, as detailed in Theorem 2. A significant refinement involves a control design scheme using a sparse matrix, which encodes the dominant components of the vector fields from the controllability conditions, leading to a simplified formula for control coefficients (15) in Theorem 3. Numerical simulations, presented for a car model with off-hooked trailers in Section IV, confirm the efficiency of both design schemes – using the original vector fields and the sparse matrix – as shown in Figs. 1 and 2. Consequently, the simplified design from Theorem 3 shows potential benefits for future applications to large-scale systems by exploiting the sparsity of the corresponding matrix $\tilde{\mathcal{F}}$. The exponential envelopes, illustrated in Figs. 1 and 3, demonstrate the agreement of theoretical decay rate estimates (7) with numerical simulation, allowing for the tuning of the gain parameter γ in equations (6) and (15) to achieve the desired exponential behavior of the closed-loop system with an appropriate choice of the small parameter ε. It is essential to note that all proofs and numerical simulations are conducted for π_ε-solutions of the closed-loop system. A thorough analysis of the effects of sampling, along with the question of whether the proposed control design methodology ensures the stabilization of classical solutions, is left for future research.

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