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Robust Control of Discrete-Time Systems with Coefficient Matrices Given by Polytopic Martingales

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***Abstract*— This paper is concerned with robust control of discrete-time linear stochastic systems with coefficient matrices given by polytopic martingales. To the best of our knowledge, this class of stochastic systems have not been dealt with as a target of control due to the absence of required theory. For such systems, we discuss the following two types of approaches for robust stabilization: One is the proposed stochastic control approach using the martingale property of the coefficient matrices, and the other is a deterministic control approach without using the information. Through theoretical and numerical comparisons of the two approaches, we demonstrate the effectiveness of the proposed stochastic control approach in the sense of conservativeness.**

I. Introduction

As a class of systems modeling real objects, linear parameter-varying (LPV) systems [1], [2] are popular. They can describe time variations of system properties by using time-varying parameters in the system representation. If the variation of the system properties is deterministic, then one of the most traditional approaches for controlling such systems is to use parameter-dependent Lyapunov inequalities [1], [3] (with the *deterministic* time-varying parameters). On the other hand, if the variation is essentially stochastic and if its probabilistic information can be obtained in advance, then the additional use of the information could contribute to improving the associated control performance. For example, in the cases of discrete-time linear systems, the independent and identically distributed (i.i.d.) property of time-varying parameters has been used in [4], [5], and the Markov property has been used in the context of Markov jump linear systems in [6], [7]. In both of these two examples, the use of the probabilistic information is known to lead to achieving less conservative results in analysis and synthesis, as long as the used information is correct.

To facilitate utilization of such probabilistic information in control problems, an earlier study [8] of the authors developed a framework for stability analysis of discrete-time linear systems with general stochastic dynamics; just for reference, we also discussed a nonlinear extension of this result in [9], although nonlinear systems are not in the scope of the present paper. As long as we deal with second-moment stability (i.e., mean square stability), stability analysis of the systems with any class of stochastic dynamics can be dealt with as a special case in this framework theoretically.

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By exploiting the results in [8], the present paper lays the first foundation of practical control theory for discrete-time linear stochastic systems with coefficient matrices given by polytopic martingales. In particular, we theoretically and numerically demonstrate that the use of the martingale property of the systems is effective in reducing the conservativeness of the associated analysis and synthesis. A discrete-time martingale is, roughly speaking, such a stochastic process that the conditional expected value of the next realization, given all the past realizations, is equal to the most recent realized value. In filtering theory, the next estimated value is often considered to be distributed with mean equal to the most recent estimated or filtered value. For example, in the ensemble Kalman filter (EnKF) [10], a finite number of ensemble members are distributed at each time step with a Gaussian distribution with mean equal to the most recent filtered value. This kind of technologies are expected to be compatible with the control theory with martingales.

Note that the earlier study [8] focused on stability of discrete-time linear systems having general stochastic dynamics with a purely mathematical motivation. That is, the purpose of the earlier study was to find universal essences in stability of the stochastic systems (involving the conventional discrete-time Markov jump systems), and not to provide a practical framework that can be applied to real control problems directly; see, e.g., the Lyapunov inequality (37) in [8], which is incompatible with numerical computations as it is. This standpoint is totally different from the present paper. We focus on the systems with dynamics determined by martingales and discuss numerically tractable LMI conditions for their control. We hope that such a work constitutes a point of departure for exploring a new field in control theory.

We use the following notation in this paper. The set of real numbers, that of positive real numbers, that of integers, and that of nonnegative integers are denoted by $\mathbf{R}$, $\mathbf{R}_+$, $\mathbf{Z}$, and $\mathbf{N}_0$, respectively. For $t \in \mathbf{Z}$, we define $\mathbf{Z}_+(t) := [t, \infty) \cap \mathbf{Z}$ and $\mathbf{Z}_-(t) := (-\infty, t] \cap \mathbf{Z}$. The set of n-dimensional real column vectors and that of $m \times n$ real matrices are denoted by $\mathbf{R}^n$ and $\mathbf{R}^{m\times n}$, respectively. The set of $n \times n$ symmetric matrices and that of $n \times n$ positive definite matrices are denoted by $\mathbf{S}^{n\times n}$ and $\mathbf{S}_+^{n\times n}$, respectively. The Borel σ-algebra on the set $(\cdot)$ is denoted by $\mathcal{B}(\cdot)$. The Euclidean norm is denoted by $\|\cdot\|$. For random variables s_1 and s_2, the expectation of s_1 and the conditional expectation of s_1 given s_2 are denoted by $E[s_1]$ and $E[s_1|s_2]$, respectively; this notation is used also for random matrices. For the real square matrix M, $\mathrm{He}(M) := M + M^T$, where M^T denotes the transpose of M.

II. Discrete-Time Linear Systems with Coefficient Matrices Given by Polytopic Martingales

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space, where Ω, $\mathcal{F}$ and P are a sample space, a σ-algebra and a probability measure, respectively. For a set $\boldsymbol{X}$, an $\boldsymbol{X}$-valued random variable X_0 is defined as a mapping $X_0 : (\Omega, \mathcal{F}) \to (\boldsymbol{X}, \mathcal{B}(\boldsymbol{X}))$, which is also denoted by $X_0 : \Omega \to \boldsymbol{X}$ for short. Similarly, an $\boldsymbol{X}$-valued stochastic process $X = (X_k)_{k \in \boldsymbol{T}}$ on the set $\boldsymbol{T}$ of time instants is defined as a mapping $X : \Omega \to \boldsymbol{X}^{\boldsymbol{T}}$, where $\boldsymbol{X}^{\boldsymbol{T}}$ is the set of all the $\boldsymbol{X}$-valued functions of $k \in \boldsymbol{T}$ that map $\boldsymbol{T}$ to $\boldsymbol{X}$; e.g., a Z-dimensional stochastic process on $\mathbf{Z}$ is defined as a mapping from Ω to $(\mathbf{R}^Z)^{\mathbf{Z}}$.

A. Discrete-Time Linear Systems with Stochastic Dynamics

Let us consider a discrete-time Z-dimensional stochastic process $\xi = (\xi_k)_{k \in \mathbf{Z}} : \Omega \to (\mathbf{R}^Z)^{\mathbf{Z}}$ and an associated discrete-time linear stochastic system

$$x_{k+1} = A(\xi_k) x_k \tag{1}$$

with a Borel-measurable matrix-valued function $A : \mathbf{R}^Z \to \mathbf{R}^{n \times n}$. The class of ξ as well as that of A in (1) will be confined later so that the coefficient matrix $A(\xi_k)$ becomes what we call a polytopic martingale.

In this paper, we denote the initial time instant by $k_0 \in \mathbf{Z}$, which is given arbitrarily, and are interested in the behavior of the state $x_k \in \mathbf{R}^n$ with $k \in \mathbf{Z}_+(k_0)$ for each fixed k_0. We suppose that the initial state $x_{k_0} \in \mathbf{R}^n$ of the system is given as a deterministic vector. In addition, we also suppose that the value $\hat{\xi}^{(k_0-1)-} \in \widehat{\boldsymbol{\Xi}}_{k_0}$ of the stochastic process $\xi^{(k_0-1)-} := (\xi_k)_{k \in \mathbf{Z}_-(k_0-1)}$, which is a subsequence of ξ, is given at k_0, where $\widehat{\boldsymbol{\Xi}}_{k_0}$ denotes the support of $\xi^{(k_0-1)-}$ (i.e., the set of values that $\xi^{(k_0-1)-}$ can take) . That is, $\hat{\xi}^{(k_0-1)-}$ corresponds to the realization of $\xi^{(k_0-1)-}$, which should be determined at time $k = k_0$. In our framework, this $\hat{\xi}^{(k_0-1)-}$ gives the initial condition for all the probability distributions associated with $\xi^{k_0+} := (\xi_k)_{k \in \mathbf{Z}_+(k_0)}$ (i.e., those from the initial time k_0). If ξ is assumed to be a Markov process, only the value of ξ_{k_0-1} is required for determining the distribution of ξ^{k_0+}. Since we do not use the Markov assumption, dealing only with ξ_{k_0-1} is insufficient, and the whole value of $\xi^{(k_0-1)-}$ is required. Fortunately, however, this will not be an obstacle in the arguments in this paper, which focuses on the systems with coefficients given by polytopic martingales. The details will be clearer later.

For notational simplicity, let $E_{k_0}[\cdot] := E[\cdot | \xi^{(k_0-1)-} = \hat{\xi}^{(k_0-1)-}]$ and $E_{k_0}[\cdot | \mathcal{F}_k] := E[\cdot | \xi^{(k_0-1)-} = \hat{\xi}^{(k_0-1)-}, \xi_{k_0}, \xi_{k_0+1}, \ldots, \xi_k]$, where $\mathcal{F}_k$ is the σ-algebra generated by $\xi_{k_0}, \xi_{k_0+1}, \ldots, \xi_k$. By definition, the latter conditional expectation can be seen as a random variable depending only on $\xi_{k_0}, \xi_{k_0+1}, \ldots, \xi_k$ with the initial condition $\xi^{(k_0-1)-} = \hat{\xi}^{(k_0-1)-}$. With this notation, we define second-moment exponential stability [11] of system (1) as follows.

Definition 1: The system (1) is said to be exponentially stable in the second moment if there exist $a \in \mathbf{R}_+$ and $\lambda \in (0,1)$ such that

$$E_{k_0}[\|x_k\|^2] \le a \|x_{k_0}\|^2 \lambda^{2(k-k_0)} \quad (\forall k \in \mathbf{Z}_+(k_0); \\ \forall x_{k_0} \in \mathbf{R}^n;\ \forall \hat{\xi}^{(k_0-1)-} \in \widehat{\boldsymbol{\Xi}}_{k_0};\ \forall k_0 \in \mathbf{Z}). \tag{2}$$

This definition applies to the system (1) with any classes of ξ and A. For more details, see [8]. Although we will introduce assumptions on ξ and A in the following subsection, they do not lead to simplification of this definition, in contrast to the well-known cases with Markovian assumptions. Hence, we use this definition as it is. If one consider some further specific situations, then k_0 in (2) may be fixed at 0, and $\hat{\xi}^{(k_0-1)-}$ may not be required through introducing a sort of initial distribution instead, although details are omitted.

B. Coefficient Matrices Given by Polytopic Martingales

Let us introduce the following assumptions on the stochastic process ξ and the function A for system (1).

Assumption 1: For each $k_0 \in \mathbf{Z}$ and every $\hat{\xi}^{(k_0-1)-} \in \widehat{\boldsymbol{\Xi}}_{k_0}$, the stochastic process ξ satisfies the following conditions.

1a) For each $k \in \mathbf{Z}_+(k_0)$, ξ_k is $\mathcal{F}_k$-measurable (this is automatically satisfied by the present definition of $\mathcal{F}_k$).

1b) For each $k \in \mathbf{Z}_+(k_0)$, $E_{k_0}[\|\xi_k\|] < \infty$ (this is automatically satisfied by the following condition 2).

1c) For each $k \in \mathbf{Z}_+(k_0)$,

$$E_{k_0}[\xi_{k+1} | \mathcal{F}_k] = \xi_k \text{ a.s.,} \tag{3}$$

where a.s. means "almost surely".

2) The support of ξ_{k_0} is a subset of or given by

$$\mathbf{E}^Z := \left\{ \theta \in \mathbf{R}^Z \,\middle|\, \theta^{(i)} \ge 0 \ (i = 1, \ldots, Z),\ \sum_{i=1}^{Z} \theta^{(i)} = 1 \right\}, \tag{4}$$

where $\theta = [\theta^{(1)}, \ldots, \theta^{(Z)}]^T$.

Assumption 2: The function $A : \mathbf{R}^Z \to \mathbf{R}^{n \times n}$ is given by

$$A(\theta) = \sum_{i=1}^{Z} \theta^{(i)} A^{(i)} \quad (\theta \in \mathbf{E}^Z) \tag{5}$$

with deterministic matrices $A^{(i)} \in \mathbf{R}^{n \times n}$ $(i = 1, \ldots, Z)$.

These assumptions are also referred to in [8] as a specific example of prospective assumptions. The condition 1 in Assumption 1 is nothing but the definition of martingales [12]. Under these assumptions, not only ξ but also the time sequence of $A(\xi_k)$ becomes a martingale; this can be confirmed as follows, where $\xi_k^{(i)}$ denotes the i-th entry of ξ_k.

$$\begin{aligned} E_{k_0}[A(\xi_{k+1}) | \mathcal{F}_k] &= E_{k_0}\left[\sum_{i=1}^{Z} \xi_{k+1}^{(i)} A^{(i)} \,\middle|\, \mathcal{F}_k \right] \\ &= \sum_{i=1}^{Z} E_{k_0}\left[\xi_{k+1}^{(i)} | \mathcal{F}_k \right] A^{(i)} \\ &= \sum_{i=1}^{Z} \xi_k^{(i)} A^{(i)} \text{ a.s.} \\ &= A(\xi_k) \text{ a.s.} \end{aligned} \tag{6}$$

In particular, $A(\xi_k)$ takes a value only in the polytope $\{A(\theta) : \theta \in \mathbf{E}^Z\}$. Hence, we call the time sequence of this $A(\xi_k)$ a polytopic martingale.

In this paper, we discuss stability analysis and synthesis for the systems with coefficient matrices given by polytopic martingales. The following section first discusses LMI-based stability analysis.

III. Robust Stability Analysis

By using the stochastic control approach in [8], the present stability problem can be tackled. Since an LMI condition for such stability analysis has been already obtained in [8], this section first revisits this earlier result briefly. Then, to facilitate understanding of the meaning of such a stochastic control based result, this section also discusses another LMI condition that can be derived with the conventional deterministic control approach. These arguments lead to showing that the information that ξ is a martingale is indeed useful for reducing conservativeness in the associated analysis.

A. LMI Condition Derived with Stochastic Control Approach

In [8], Lyapunov inequality conditions are shown for the system (1) with general $A(\xi_k)$. Those results, together with the S-variable approach in [13], lead us to the following theorem (for more details, see Subsection VI.C in [8]).

Theorem 1: Suppose that Assumptions 1 and 2 are satisfied. The system (1) is exponentially stable in the second moment if there exist $\lambda \in (0,1)$, $P^{(i)} \in \mathbf{S}_+^{n\times n} (i = 1, \ldots, Z)$ and $F_1, F_2 \in \mathbf{R}^{n\times n}$ such that

$$\begin{bmatrix} \lambda^2 P^{(i)} & 0 \\ 0 & -P^{(i)} \end{bmatrix} + \mathrm{He}\left(\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \begin{bmatrix} A^{(i)} & I \end{bmatrix} \right) > 0 \quad (i = 1, \ldots, Z). \tag{7}$$

The inequality (7) is a standard finite-dimensional LMI for a fixed λ, and thus is numerically tractable. As we can see, the initial condition $\xi^{(k_0-1)-} = \hat{\xi}^{(k_0-1)-}$ is also needless to be dealt with in this LMI. Let $\widetilde{\Xi}$ denote the set of the stochastic processes ξ (i.e., the set of mappings $\xi : \Omega \to (\mathbf{R}^Z)^{\mathbf{Z}}$) satisfying Assumption 1. Since the inequality (7) involves no ξ_k, the corresponding sufficient stability condition in Theorem 1 ensures stability under any of the processes ξ in $\widetilde{\Xi}$. This means that Theorem 1 gives not a mere stability condition but a robust stability condition for system (1) satisfying Assumptions 1 and 2.

Let us temporarily consider the case that $\xi_k = \theta$ $(\forall k \in \mathbf{Z})$ for a deterministic constant vector $\theta \in \mathbf{E}^Z$. Then, the corresponding deterministic process ξ satisfies Assumption 1, regardless of the value of $\theta \in \mathbf{E}^Z$. This observation implies that the LMI in Theorem 1 also gives a robust stability condition for deterministic time-invariant systems with polytopic uncertainties described by $\theta \in \mathbf{E}^Z$. This is actually not surprising because the LMI (7) (with some minor modifications) itself is originally known as such a deterministic robust stability condition [13]; a special case of this LMI condition is shown in [14]. In particular, the LMI for the time-invariant setting is recognized to be the least conservative among the existing LMIs in the field of deterministic robust control at this moment. This in turn implies that our Theorem 1 can be interpreted as extending this known best record by considering stochastic situations. By using the information that ξ is a martingale, the conventional constraint on ξ to be deterministic and time-invariant for using the LMI (7) is alleviated in our approach (a similar comment also applies to the synthesis discussed later).

B. LMI Condition Derived with Deterministic Control Approach

We next derive another inequality condition by using the conventional deterministic control approach. The core idea in this subsection is quite simple; we view the martingale ξ as a deterministic time-varying process (i.e., disregard the information about the stochastic property of ξ), and consider ensuring robust stability deterministically. Through comparison with the condition obtained with such an idea, we clarify the role of the information that ξ is a martingale in robust control.

Let us consider the Z-dimensional deterministic process $\boldsymbol{\theta} = (\theta_k)_{k\in\mathbf{Z}}$ belonging to

$$\mathcal{E}^Z := \{(\theta_k)_{k\in\mathbf{Z}} : \theta_k \in \mathbf{E}^Z \ (\forall k \in \mathbf{Z})\}, \tag{8}$$

and the associated system

$$v_{k+1} = A(\theta_k) v_k. \tag{9}$$

This system is nothing but system (1) without the information that ξ is a martingale. If we view $\boldsymbol{\theta}$ as a sample path of ξ, then the system (9) generates the corresponding sample path of x_k in (1). For such a deterministic system, we define exponential stability as follows.

Definition 2: Suppose that $\boldsymbol{\theta} \in \mathcal{E}^Z$ is given. The system (9) is said to be exponentially stable if there exist $a \in \mathbf{R}_+$ and $\lambda \in (0,1)$ such that

$$\|v_k\|^2 \le a\|v_{k_0}\|^2 \lambda^{2(k-k_0)} \quad (\forall k \in \mathbf{Z}_+(k_0); \forall v_{k_0} \in \mathbf{R}^n; \forall k_0 \in \mathbf{Z}). \tag{10}$$

This definition is completely consistent with Definition 1; if $\xi = \boldsymbol{\theta}$ then the expectation need not be considered and Definition 1 immediately reduces to Definition 2 for $x_k = v_k$. For these two stability notions of different systems, the following theorem holds.

Theorem 2: Suppose that $A^{(i)} \in \mathbf{R}^{n\times n}$ $(i = 1, \ldots, Z)$ are given, and Assumptions 1 and 2 are satisfied. For given $a \in \mathbf{R}_+$ and $\lambda \in (0,1)$, the following condition 2 is a sufficient condition for condition 1.

1) The stochastic system (1) satisfies (2) for all $\xi \in \widetilde{\Xi}$ (i.e., robustly exponentially stable in the second moment).
2) The deterministic system (9) satisfies (10) for all $\boldsymbol{\theta} \in \mathcal{E}^Z$ (i.e., robustly exponentially stable).

Proof: Take an arbitrary $\xi \in \widetilde{\Xi}$. By Assumption 1, any sample path $\xi(\omega_0)$ $(\omega_0 \in \Omega)$ of this ξ belongs to $\mathcal{E}^Z$. This, together with condition 2, leads to

$$\|x_k\|^2 \le a\|x_{k_0}\|^2 \lambda^{2(k-k_0)} \text{ a.s.} \quad (\forall k \in \mathbf{Z}_+(k_0); \forall x_{k_0} \in \mathbf{R}^n; \forall k_0 \in \mathbf{Z}) \tag{11}$$

for the stochastic system (1). Taking conditional expectations for both sides of this inequality leads to (2). These arguments apply to each $\xi \in \hat{\Xi}$, and hence, the proof is completed. ■

This theorem implies that robust stability of the stochastic system (1) may be evaluated through analyzing that of the deterministic system (9). Since an LMI condition with the same level of conservativeness as (7) (i.e., the least conservative LMI condition for the present robust stability problem at this moment) has not been explicitly shown in the literature in the time-varying setting, we provide a brief proof for it. For a basic concept of deterministic control in this direction, see, e.g., [3], [13].

First, the following theorem would be almost standard in the field of deterministic control [15].

Theorem 3: Suppose that $\boldsymbol{\theta} \in \mathcal{E}^Z$ is given, and Assumption 2 is satisfied. The following two conditions are equivalent.

1) The deterministic system (9) is exponentially stable.
2) There exist $\underline{\epsilon}, \overline{\epsilon} \in \mathbf{R}_+$, $\lambda \in (0,1)$ and $P : \mathbf{R}^Z \to \mathbf{S}^{n\times n}$ such that

$$P(\theta_k) \geq \underline{\epsilon} I, \tag{12}$$

$$P(\theta_k) \leq \overline{\epsilon} I, \tag{13}$$

$$\lambda^2 P(\theta_k) - A(\theta_k)^T P(\theta_{k+1}) A(\theta_k) > 0 \quad (\forall k \in \mathbf{Z}_+(k_0); \forall k_0 \in \mathbf{Z}). \tag{14}$$

Since the direct use of the inequality condition in this theorem is numerically difficult, we confine the mapping P to the form

$$P(\theta) = \sum_{i=1}^{Z} \theta^{(i)} P^{(i)} \quad (\theta \in \mathbf{E}^Z) \tag{15}$$

with vertices $P^{(i)} \in \mathbf{S}^{n\times n}$ $(i = 1, \ldots, Z)$. Then, by using the S-variable technique [13], we obtain the following theorem.

Theorem 4: Suppose that Assumption 2 is satisfied. For given $\lambda \in (0,1)$, the following condition 2 is a sufficient condition for condition 1.

1) There exist $\underline{\epsilon}, \overline{\epsilon} \in \mathbf{R}_+$ and $P : \mathbf{R}^Z \to \mathbf{S}^{n\times n}$ satisfying (12)–(14) for all $\boldsymbol{\theta} \in \mathcal{E}^Z$.
2) There exist $P^{(i)} \in \mathbf{S}_+^{n\times n}$ $(i = 1, \ldots, Z)$ and $F_1, F_2 \in \mathbf{R}^{n\times n}$ such that

$$\begin{bmatrix} \lambda^2 P^{(i)} & 0 \\ 0 & -P^{(j)} \end{bmatrix} + \mathrm{He}\left(\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \begin{bmatrix} A^{(i)} & I \end{bmatrix} \right) > 0 \quad (i, j = 1, \ldots, Z). \tag{16}$$

Proof: Take an arbitrary $\boldsymbol{\theta} \in \mathcal{E}^Z$. Multiplying both sides of (16) by $\theta_k^{(i)} \theta_{k+1}^{(j)}$ and summing up the results with respect to $i, j = 1, \ldots, Z$ lead us to

$$\begin{bmatrix} \lambda^2 P(\theta_k) & 0 \\ 0 & -P(\theta_{k+1}) \end{bmatrix} + \mathrm{He}\left(\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \begin{bmatrix} A(\theta_k) & I \end{bmatrix} \right) > 0 \quad (\forall k \in \mathbf{Z}_+(k_0); \forall k_0 \in \mathbf{Z}) \tag{17}$$

with P given by (15). Pre- and post-multiplying $\begin{bmatrix} I & -A(\theta_k)^T \end{bmatrix}$ and its transpose on (17) further leads to (14). These arguments apply to each $\boldsymbol{\theta} \in \mathcal{E}^Z$, while (15) and $P^{(i)} \in \mathbf{S}_+^{n\times n}$ $(i = 1, \ldots, Z)$ naturally ensure the existence of $\boldsymbol{\theta}$-independent $\underline{\epsilon}, \overline{\epsilon} \in \mathbf{R}_+$ satisfying (12) and (13). This completes the proof. ■

Theorems 2–4 imply that robust stability of the stochastic system (1) can be analyzed by searching for the solution of (16). In particular, the λ in (16) gives an upper bound of the minimal λ satisfying (2), as is the case with that in (7). The inequality (16) is nothing but the LMI condition derived with the deterministic control approach.

Compared to (7), the simultaneous LMI (16) consists of a Z times as large number of LMIs. In particular, the Z LMIs constituting (7) are nothing but those in (16) for $i = j$. This implies that at least the analysis based on Theorem 1 is no more conservative than that based on Theorems 2–4. In practice, this difference may actually cause a non-zero gap in the sense of conservativeness. This suggests the usefulness of the information that ξ is a martingale in the present robust stochastic control.

IV. Robust State-Feedback Stabilization

This section extends the two conditions for analysis shown in the preceding section toward state-feedback controller synthesis. As is the case with analysis, the stochastic control approach leads us to a less conservative result than the deterministic control approach also in this synthesis. To confirm the gap between the two approaches, however, not only the former but the latter approaches are discussed in parallel.

A. Synthesis Problem

Let us consider the open-loop plant

$$x_{k+1} = A_{\mathrm{op}}(\xi_k) x_k + B_{\mathrm{op}}(\xi_k) u_k \tag{18}$$

with Borel-measurable matrix-valued functions $A_{\mathrm{op}} : \mathbf{R}^Z \to \mathbf{R}^{n\times n}$ and $B_{\mathrm{op}} : \mathbf{R}^Z \to \mathbf{R}^{n\times m}$. We assume that ξ satisfies Assumption 1 and A_{op} and B_{op} satisfy an assumption similar to Assumption 2, which is denoted by Assumption 2' (the vertices are denoted by $A_{\mathrm{op}}^{(i)} \in \mathbf{R}^{n\times n}, B_{\mathrm{op}}^{(i)} \in \mathbf{R}^{n\times m}$ $(i = 1, \ldots, Z)$). For such a stochastic plant, we consider the state-feedback controller

$$u_k = K x_k \tag{19}$$

with a time-invariant static gain $K \in \mathbf{R}^{m\times n}$. Then, the closed-loop system is given by (1) with (5) and

$$A^{(i)} = A_{\mathrm{op}}^{(i)} + B_{\mathrm{op}}^{(i)} K \quad (i = 1, \ldots, Z). \tag{20}$$

This section tackles the problem of designing a gain $K \in \mathbf{R}^{m\times n}$ that robustly stabilizes this closed-loop system in the sense of second-moment exponential stability.

B. Synthesis-Oriented LMI Condition Derived with Stochastic Control Approach

As a natural extension of Theorem 1, we have the following Theorem.

Theorem 5: Suppose that Assumptions 1 and 2' are satisfied. There exists $K \in \mathbf{R}^{m\times n}$ such that the closed-loop system consisting of the plant (18) and the controller (19) is exponentially stable in the second moment robustly with respect to $\widetilde{\Xi}$ if for given $G_0 \in \mathbf{R}^{n\times n}$, there exist $\lambda \in (0,1)$, $X^{(i)} \in \mathbf{S}_+^{n\times n}$ $(i=1,\ldots,Z)$, $Y \in \mathbf{R}^{m\times n}$ and $S \in \mathbf{R}^{n\times n}$ such that

$$\begin{aligned}&\begin{bmatrix}\lambda^2 X^{(i)} & 0\\ 0 & -X^{(i)}\end{bmatrix}\\ &+\mathrm{He}\left(\begin{bmatrix}G_0^T\\ I\end{bmatrix}\begin{bmatrix}A_{\mathrm{op}}^{(i)}S^T + B_{\mathrm{op}}^{(i)}Y & S^T\end{bmatrix}\right) > 0\\ &(i=1,\ldots,Z).\end{aligned} \tag{21}$$

In particular, S is nonsingular, and $K = YS^{-T}$ is one such stabilizing gain.

Proof: Since S is nonsingular by the lower right block of (21) and $X^{(i)} > 0$, the congruence transformation using $\begin{bmatrix}S^{-1} & 0\\ 0 & S^{-1}\end{bmatrix}$ and the change of variables $F_2 = S^{-1}$, $P^{(i)} = S^{-1}X^{(i)}S^{-T}$ and $K = YS^{-T}$ applied to (21) lead us to

$$\begin{aligned}&\begin{bmatrix}\lambda^2 P^{(i)} & 0\\ 0 & -P^{(i)}\end{bmatrix} + \mathrm{He}\left(\begin{bmatrix}S^{-1}G_0^T\\ F_2\end{bmatrix}\begin{bmatrix}A^{(i)} & I\end{bmatrix}\right) > 0\\ &(i=1,\ldots,Z)\end{aligned} \tag{22}$$

with (20). Hence, for $F_1 = S^{-1}G_0^T$ with given G_0, (7) holds. This, together with Theorem 1, completes the proof. ■

In the above theorem, G_0 is viewed as a constant matrix to be given, so that (22) becomes an LMI. It is known in the literature of deterministic control that this G_0 has to be Schur stable, and the same comment also applies to the present stochastic control approach; this can be confirmed through pre- and post-multiplying $\begin{bmatrix}I & -G_0^T\end{bmatrix}$ and its transpose on (21), which leads to a Lyapunov inequality with the A matrix given by G_0. The most simple and reasonable selection of a Schur stable G_0 is $G_0 = 0$. Then, we immediately obtain the following corollary from Theorem 5.

Corollary 1: Suppose that Assumptions 1 and 2' are satisfied. There exists $K \in \mathbf{R}^{m\times n}$ such that the closed-loop system consisting of the plant (18) and the controller (19) is exponentially stable in the second moment robustly with respect to $\widetilde{\Xi}$ if there exist $\lambda \in (0,1)$, $X^{(i)} \in \mathbf{S}_+^{n\times n}$ $(i = 1,\ldots,Z)$, $Y \in \mathbf{R}^{m\times n}$ and $S \in \mathbf{R}^{n\times n}$ such that

$$\begin{bmatrix}\lambda^2 X^{(i)} & *\\ A_{\mathrm{op}}^{(i)}S^T + B_{\mathrm{op}}^{(i)}Y & S+S^T - X^{(i)}\end{bmatrix} > 0 \quad (i=1,\ldots,Z), \tag{23}$$

where $*$ denotes the transpose of the lower left block in the matrix. In particular, S is nonsingular, and $K = YS^{-T}$ is one such stabilizing gain.

In both Theorem 5 and Corollary 1, the synthesis of a stabilizing gain can be performed[1] by solving an LMI for a fixed $\lambda \in (0,1)$.

C. Synthesis-Oriented LMI Condition Derived with Deterministic Control Approach

As is the case with the stochastic control approach in the preceding subsection, the LMI (16) can be extended toward the present synthesis by using the S-variable technique. Although we explicitly show only a deterministic control counterpart of Corollary 1 in the following, a similar result can be immediately obtained also for Theorem 5.

Theorem 6: Suppose that Assumptions 1 and 2' are satisfied. There exists $K \in \mathbf{R}^{m\times n}$ such that the closed-loop system consisting of the plant (18) and the controller (19) is exponentially stable in the second moment robustly with respect to $\widetilde{\Xi}$ if there exist $\lambda \in (0,1)$, $X^{(i)} \in \mathbf{S}_+^{n\times n}$ $(i = 1,\ldots,Z)$, $Y \in \mathbf{R}^{m\times n}$ and $S \in \mathbf{R}^{n\times n}$ such that

$$\begin{aligned}&\begin{bmatrix}\lambda^2 X^{(i)} & *\\ A_{\mathrm{op}}^{(i)}S^T + B_{\mathrm{op}}^{(i)}Y & S+S^T - X^{(j)}\end{bmatrix} > 0\\ &(i,j=1,\ldots,Z).\end{aligned} \tag{24}$$

In particular, S is nonsingular, and $K = YS^{-T}$ is one such stabilizing gain.

Proof: The techniques used for Theorem 5 and Corollary 1, together with Theorems 2–4, complete the proof. ■

As is the case with analysis in Section III, (24) is a sufficient condition for (23), since the LMIs in (23) constitute only a part of those in (24) for $i = j$. Hence, the synthesis based on Corollary 1 is expected to be less conservative than that based on Theorem 6 for the present stochastic control problem. The existence of a gap between these two approaches is numerically confirmed in the following section.

V. Numerical Example

This section compares the discussed two approaches of control with a numerical example. Specifically, we numerically demonstrate the fact that the synthesis based on Corollary 1 is less conservative than that based on Theorem 6.

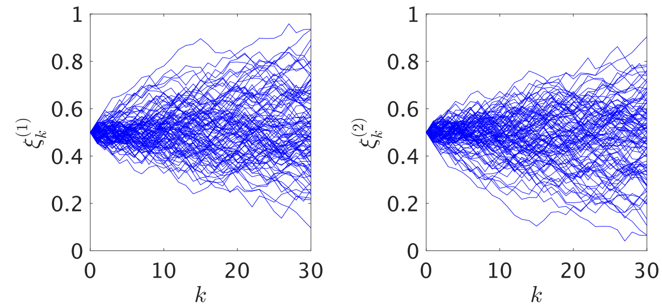

Fig. 1. 100 sample paths of ξ used for simulations.

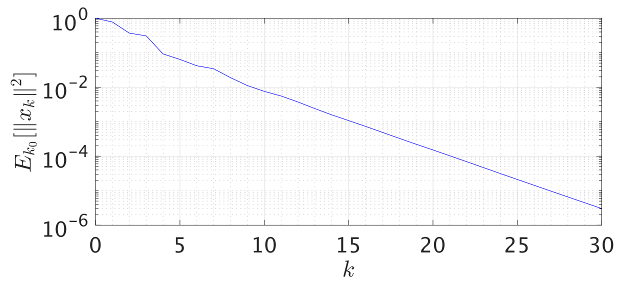

Fig. 2. Estimate of $E_{k_0}[\|x_k\|^2]$ calculated with 100 sample paths of ξ.

[1]Since the conditions in Theorem 5 and Corollary 1 involve the meaning of the test of stabilizability, it is not required to be additionally performed before the synthesis. Further characterization of stabilizability and controllability for the present system would be an interesting direction of studies.

A. Synthesis with Two Approaches

Let us consider the open-loop plant (18) satisfying Assumptions 1 and 2' with the coefficient matrices whose vertices are given by

$$A_{\mathrm{op}}^{(1)} = \begin{bmatrix} 0.73 & 0.98 & 0.73 \\ -0.49 & 0.37 & 0.61 \\ 0.25 & -0.49 & 1.2 \end{bmatrix}, \quad B_{\mathrm{op}}^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$
$$A_{\mathrm{op}}^{(2)} = \begin{bmatrix} 0.73 & 0.61 & 0.73 \\ -0.49 & -0.49 & 0.98 \\ 0 & -0.73 & 0.25 \end{bmatrix}, \quad B_{\mathrm{op}}^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \tag{25}$$

This plant is obviously not robustly stable since $A_{\mathrm{op}}^{(1)}$ is not Schur stable. For such a plant, we first tried to design a stabilizing state-feedback gain K based on Theorem 6. MATLAB, YALMIP [16] and SDPT3 [17] were used for this numerical computation, and λ was minimized through a bisection method with respect to λ^2. However, the minimal λ did not become less than 1, and we failed to stabilize the closed-loop system by this approach.

We next considered using Corollary 1. The same computation environment was used. Then, we obtained the solution

$$X^{(1)} = 10^4 \begin{bmatrix} 2.9177 & -1.7000 & -2.3522 \\ -1.7000 & 4.4589 & 1.7622 \\ -2.3522 & 1.7622 & 2.2453 \end{bmatrix},$$
$$X^{(2)} = 10^4 \begin{bmatrix} 1.5894 & 1.0302 & -0.9057 \\ 1.0302 & 5.4722 & 0.3396 \\ -0.9057 & 0.3396 & 0.9657 \end{bmatrix} \tag{26}$$

achieving the minimal value $\lambda = 0.8718$. The corresponding gain was $K = \begin{bmatrix} -0.4218 & -0.4477 & -1.1197 \end{bmatrix}$. Since this λ is less than 1, the closed-loop system with the above gain is ensured to be stable by our result. For reference, we minimized λ with respect to (24) again under the use of $X^{(1)}$ and $X^{(2)}$ in (26). Then, we obtained $\lambda = 1.0776$ as the minimal value. Since the synthesis based on Corollary 1 uses the information that ξ is a martingale, these results numerically demonstrate the effect of using such information in control problems.

B. Simulation

This subsection briefly introduces the behavior of the designed closed-loop system. For simulation, we set $k_0 = 0$ and $x_0 = [1, 0, 0]^T$. In addition, as a ξ satisfying Assumption 1, we consider the process whose distribution is described with two-sided truncated normal distributions, whose details are omitted due to limited space. The sample paths of this martingale ξ are shown in Fig. 1.

With these sample paths, we calculated the time evolution of $E_{k_0}[\|x_k\|^2]$ for the closed-loop system with the expectation replaced by the sample mean. Then, we obtained the result in Fig. 2. As we can see in this figure, the stabilization was achieved successfully. The decay rate of $E_{k_0}[\|x_k\|^2]$ calculated with the data at $k = 10$ and 30 was $\lambda_{\mathrm{est}} = 0.8223$, which is smaller than the minimal value $\lambda = 0.8718$ obtained at the synthesis stage. This also validates the fact that the minimal λ satisfying our LMIs gives an upper bound of the decay rate of the second moment; since our synthesis ensures the worst case performance with respect to $\widetilde{\Xi}$, the gap between the above two values is affected by the choice of the martingale used in the simulations.

VI. Conclusions

In this paper, we discussed robust stability analysis and synthesis for discrete-time linear systems with coefficients given by polytopic martingales. Through comparing the proposed stochastic control approach with a deterministic control approach, we theoretically and numerically demonstrated that the use of the martingale property in control is effective for reducing the associated conservativeness.

Further developments of associated control theory toward observer synthesis (including the investigation of whether the separation principal holds), gain-scheduled control and robust H_2 control would be possible future works. In addition, since the used system representation itself is considered to be new, not only such theoretical developments but also practical demonstration of their usefulness is considered to be also important.

References

[1] P. Apkarian, P. Gahinet, and G. Becker, "Self-scheduled H_∞ control of linear parameter-varying systems: A design example," *Automatica*, vol. 31, no. 9, pp. 1251–1261, 1995.

[2] J. Mohammadpour and C. W. Scherer, *Control of linear parameter varying systems with applications*. Springer Science & Business Media, 2012.

[3] J. Daafouz and J. Bernussou, "Parameter dependent Lyapunov functions for discrete time systems with time varying parametric uncertainties," *Systems & Control Letters*, vol. 43, no. 5, pp. 355–359, 2001.

[4] W. L. De Koning, "Compensatability and optimal compensation of systems with white parameters," *IEEE Transactions on Automatic Control*, vol. 37, no. 5, pp. 579–588, 1992.

[5] Y. Hosoe and T. Hagiwara, "Equivalent stability notions, Lyapunov inequality, and its application in discrete-time linear systems with stochastic dynamics determined by an i.i.d. process," *IEEE Transactions on Automatic Control*, vol. 64, no. 11, pp. 4764–4771, 2019.

[6] O. L. V. Costa, M. D. Fragoso, and R. P. Marques, *Discrete-Time Markov Jump Linear Systems*. London, UK: Springer-Verlag, 2005.

[7] O. L. V. Costa and D. Z. Figueiredo, "Stochastic stability of jump discrete-time linear systems with Markov chain in a general Borel space," *IEEE Transactions on Automatic Control*, vol. 59, no. 1, pp. 223–227, 2014.

[8] Y. Hosoe and T. Hagiwara, "On second-moment stability of discrete-time linear systems with general stochastic dynamics," *IEEE Transactions on Automatic Control*, vol. 67, no. 2, pp. 795–809, 2022.

[9] Y. Kawano and Y. Hosoe, "Contraction analysis of discrete-time stochastic systems," arXiv:2106.05635, 2021.

[10] G. Evensen, *Data Assimilation: The Ensemble Kalman Filter*, 2nd ed. Berlin Heidelberg, Germany: Springer-Verlag, 2009.

[11] F. Kozin, "A survey of stability of stochastic systems," *Automatica*, vol. 5, no. 1, pp. 95–112, 1969.

[12] A. Klenke, *Probability Theory: A Comprehensive Course*, 2nd ed. London, UK: Springer-Verlag, 2014.

[13] Y. Ebihara, D. Peaucelle, and D. Arzelier, *S-Variable Approach to LMI-Based Robust Control*. London, UK: Springer-Verlag, 2015.

[14] M. C. de Oliveira, J. Bernussou, and J. C. Geromel, "A new discrete-time robust stability condition," *Systems & Control Letters*, vol. 37, no. 4, pp. 261–265, 1999.

[15] M. Vidyasagar, *Nonlinear Systems Analysis*, 2nd ed. Philadelphia, PA, USA: SIAM, 2002.

[16] J. Löfberg, "YALMIP: A toolbox for modeling and optimization in MATLAB," in *Proc. 2004 IEEE International Symposium on Computer Aided Control Systems Design*, 2004, pp. 284–289.

[17] R. H. Tütüncü, K. C. Toh, and M. J. Todd, "Solving semidefinite-quadratic-linear programs using SDPT3," *Mathematical Programming Series B*, vol. 95, no. 2, pp. 189–217, 2003.