IEEE Control Systems Letters paper presented at
2023 62nd IEEE Conference on Decision and Control (CDC)
December 13-15, 2023. Marina Bay Sands, Singapore

Data Informativity for Lyapunov Equations

Ikumi Banno[1], Shun-ichi Azuma[2], Ryo Ariizumi[3], Toru Asai[1], and Jun-ichi Imura[4]

***Abstract*— Recently, the novel framework for data-driven analysis and control, called data informativity, was proposed. This notion represents whether the given data contain sufficient information to solve a problem or not. However, data informativity for solving a Lyapunov equation has never been addressed before. This letter characterizes the data informativity for the Lyapunov equations in the form of $AP + PA^\top = -Q$, where A and Q are square matrices and P is an unknown matrix. First, we clarify the relationship between the unique solution to the Lyapunov equation and the controllable subspace of a system. Second, based on this result, we provide a necessary and sufficient condition for the data informativity, which is characterized by the possibility of a certain matrix decomposition of Q, called the data-basis decomposition. Finally, we present a direct data-driven method for solving the Lyapunov equation based on our data informativity condition. This method has a potential to compute the solution even if the data do not contain sufficient information to identify the system.**

I. INTRODUCTION

In the field of system analysis and control, direct data-driven methods [1]–[12] have recently attracted attention, which directly analyze/control the system from measurement data bypassing the process of system identification [13]. The direct data-driven framework has a potential to analyze/control the system even when system identification cannot be applied due to the insufficiency of the measurement data. However, most of the above methods require rich data that allow us to identify the system.

In 2020, the novel framework for data-driven analysis and control, called *data informativity*, was proposed [1]. The data informativity represents whether given data contain sufficient information to solve a problem or not. So far, algebraic conditions equivalent to the data informativity for several problems have been derived, including system identification, stability, observability, state feedback stabilization, and sub-optimal LQR [1]–[4]. These results have revealed that some problems can be solved even if we do not have sufficient data to identify the system.

*This work was supported by JST Moonshot R&D Grant Number JPMJMS2021 and Nagoya University Interdisciplinary Frontier Fellowship JPMJFS2120.

[1]I. Banno and T. Asai are with the Graduate School of Engineering, Nagoya University; Furo-cho, Chikusa-ku, Nagoya 464-8603, Japan banno.ikumi.v2@s.mail.nagoya-u.ac.jp; asai@nuem.nagoya-u.ac.jp

[2]S. Azuma is with the Graduate School of Informatics, Kyoto University; Yoshida-honmachi, Sakyo-ku, Kyoto 606-8501, Japan sazuma@i.kyoto-u.ac.jp

[3]R. Ariizumi is with the Graduate School of Engineering, Tokyo University of Agriculture and Technology, 2-24-16, Naka-cho, Koganei, Tokyo 184-8588, Japan ryoariizumi@go.tuat.ac.jp

[4]J. Imura is with the School of Engineering, Tokyo Institute of Technology; Oh-Okayama, Meguro-ku, Tokyo 152-8552, Japan imura@sc.e.titech.ac.jp

Now, we are interested in solving the Lyapunov equations $AP+PA^\top = -Q$ (where A and Q are constant matrices and P is an unknown matrix) without explicit information about A, but with a state trajectory of the system $\dot{x}(t) = Ax(t)$. Solving this problem is important for the following reasons: First, Lyapunov equations play a key role in analyzing and designing dynamical systems, including stability, controllability, and observability analysis and stabilizing controller design [14]. Second, a solution to this problem could provide a unified framework for several data-driven tasks for the system analysis and design. In this background, data-driven methods for solving a Lyapunov equation have been already presented in [5], [6], which require rich data that allow us to identify the system. However, we still have a potential to compute the solutions from the data even when the system identification cannot be applied. Furthermore, data informativity for solving the Lyapunov equation has never been addressed.

Therefore, this letter characterizes the data informativity for solving the Lyapunov equation. First, as a preliminary step to the problem, we clarify the relationship between the unique solution to the Lyapunov equation and the controllable subspaces of a system, which is characterized by a matrix decomposition of Q. Second, by using this result, we show that a dataset is informative for the Lyapunov equation (i.e., the data contain sufficient information to uniquely determine the solution to the Lyapunov equation) if and only if a certain matrix decomposition of Q, called *data-basis decomposition*, is possible. Finally, we present a data-driven method for solving the Lyapunov equation based on our data informativity condition.

This letter is organized as follows. In Section II, we formulate the data informativity for the Lyapunov equation and the problem to be considered. The solution to the Lyapunov equation is characterized in Section III. A necessary and sufficient condition for our data informativity is provided in Section IV. Section V presents a data-driven method for solving the Lyapunov equation based on the data informativity. This letter is concluded in Section VI.

Notation:

(i) *Sets*: Let $\mathbf{R}$ and $\mathbf{R}_+$ be the set of the real numbers and positive real numbers, respectively. For $n \in \{1, 2, \ldots\}$, $\mathbf{Stab}(n) \subset \mathbf{R}^{n\times n}$ represents the set of $n \times n$ Hurwitz matrices. The cardinality of a set $\mathbf{S}$ is denoted by $|\mathbf{S}|$.

(ii) *Matrices*: For $(n, m) \in \{1, 2, \ldots\}^2$, we use $I_n \in \mathbf{R}^{n\times n}$, $\mathbf{0}_{n\times m} \in \mathbf{R}^{n\times m}$, and $\mathbf{0}_n \in \mathbf{R}^n$ to denote the $n \times n$ identity matrix, the $n \times m$ zero matrix, and the $n \times 1$ zero vector. The image of a matrix $A \in \mathbf{R}^{n\times m}$ is written as $\mathrm{Im}(A)$.

(iii) *Vector spaces*: The dimension of a vector space $\mathbf{V}$ is written as $\dim(\mathbf{V})$. For a set $\mathbf{S} \subseteq \mathbf{R}^n$, we denote the linear span of $\mathbf{S}$ by $\mathrm{span}(\mathbf{S})$ and the orthogonal complement of $\mathbf{S}$ by

$$\mathbf{S}^\perp := \{x \in \mathbf{R}^n \mid \forall y \in \mathbf{S} \;\; x^\top y = 0\}.$$

Note that both $\mathrm{span}(\mathbf{S})$ and $\mathbf{S}^\perp$ are vector subspaces of $\mathbf{R}^n$.

II. PROBLEM FORMULATIONS

Consider the linear system

$$\dot{x}(t) = Ax(t), \tag{1}$$

where $x(t) \in \mathbf{R}^n$ is the state and $A \in \mathbf{Stab}(n)$ is a Hurwitz matrix. Throughout this paper, the solution of (1) with $x(0) = x_0$ is written as $x(t, x_0)$.

Associated with the system, we focus on the Lyapunov equation

$$AP + PA^\top = -Q, \tag{2}$$

where $Q \in \mathbf{R}^{n\times n}$ is a constant matrix (not necessarily symmetric) and $P \in \mathbf{R}^{n\times n}$ is an unknown matrix. As is well-known, (2) has the unique solution

$$\Phi(A, Q) := \int_0^\infty e^{At} Q e^{A^\top t}\, dt. \tag{3}$$

if $A \in \mathbf{Stab}(n)$ [14].

We are interested in solving the Lyapunov equation (2) without exact information about A, but with a state trajectory of (1) on a certain time interval. In this letter, a dataset is given as state trajectory data, i.e.,

$$\mathcal{D}_T := \bigcup_{t\in[0,T)} \{(t, x(t, x_0))\}, \tag{4}$$

where $T \in \mathbf{R}_+ \cup \{\infty\}$.

This letter deals with the following question: Can we uniquely determine the solution P to the Lyapunov equation (2) only from the dataset $\mathcal{D}_T$? To answer this question, we introduce the notion of the *data informativity*. Let

$$\mathbf{\Sigma}(\mathcal{D}_T) := \{\tilde{A} \in \mathbf{Stab}(n) \mid \forall t \in [0, T) \;\; \dot{x}(t, x_0) = \tilde{A}x(t, x_0)\}.$$

This set is a collection of system matrices that is Hurwitz and consistent to the given dataset $\mathcal{D}_T$. Notice that $\mathbf{\Sigma}(\mathcal{D}_T)$ is nonempty since we always have $A \in \mathbf{\Sigma}(\mathcal{D}_T)$. Based on this notation, the data informativity for the Lyapunov equation is specified.

Definition 1: For the system (1), suppose that a dataset $\mathcal{D}_T$ in (4) is given. The dataset $\mathcal{D}_T$ is said to be *informative for the Lyapunov equation (2)* if

$$\Phi(A_1, Q) = \Phi(A_2, Q) \tag{5}$$

holds for any $(A_1, A_2) \in \mathbf{\Sigma}(\mathcal{D}_T) \times \mathbf{\Sigma}(\mathcal{D}_T)$. ■

This notion represents whether the dataset $\mathcal{D}_T$ has enough information to uniquely determine the solution to the Lyapunov equation (2). We emphasize that Definition 1 is distinguished from the situation where the dataset has enough information to identify the system. In fact, the requirement of Definition 1 could hold even if $|\mathbf{\Sigma}(\mathcal{D}_T)| > 1$.

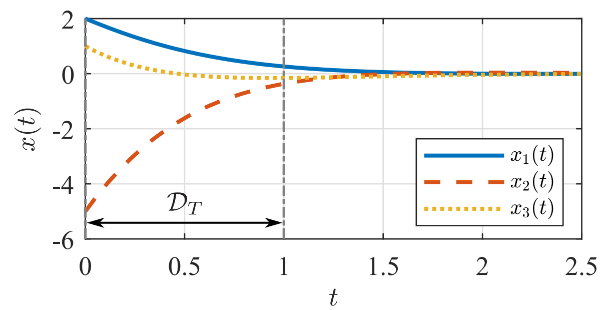

Fig. 1: The dataset $\mathcal{D}_T$ in Example 1.

Example 1: Consider the system (1) with

$$A = \begin{bmatrix} -2 & 0 & 1 \\ 2 & -1.5 & -1.5 \\ 0 & 0.5 & -1.5 \end{bmatrix} \in \mathbf{Stab}(3).$$

For this system, the dataset $\mathcal{D}_T$ is given by $x_0 = [2 \;\; -5 \;\; 1]^\top$, $T = 1$, and (4), as shown in Fig. 1.

Then, $|\mathbf{\Sigma}(\mathcal{D}_T)| > 1$ holds but $\mathcal{D}_T$ is informative for the Lyapunov equation (2) with

$$Q = \begin{bmatrix} 1 & -1 & -1 \\ -4 & 7 & 1 \\ 2 & -5 & -1 \end{bmatrix}. \tag{6}$$

The fact $|\mathbf{\Sigma}(\mathcal{D}_T)| > 1$ can be verified from

$$\tilde{A} = \begin{bmatrix} -3 & -0.5 & 0.5 \\ 1 & -2 & -2 \\ -1 & 0 & -2 \end{bmatrix} \in \mathbf{\Sigma}(\mathcal{D}_T) \tag{7}$$

and $A \neq \tilde{A}$. Moreover, one can prove that (5) holds for any $(A_1, A_2) \in \mathbf{\Sigma}(\mathcal{D}_T) \times \mathbf{\Sigma}(\mathcal{D}_T)$. For example, we have $\Phi(A, Q) = \Phi(\tilde{A}, Q)$, where $(A, \tilde{A}) \in \mathbf{\Sigma}(\mathcal{D}_T) \times \mathbf{\Sigma}(\mathcal{D}_T)$. ■

Example 2: Consider the same system and dataset in Example 1 and $Q = I_3$. Then, $\mathcal{D}_T$ is not informative for the Lyapunov equation (2), which is shown as follows. Since

$$\Phi(A, I_3) = \begin{bmatrix} 0.3062 & 0.1250 & 0.1125 \\ 0.1250 & 0.5062 & -0.0063 \\ 0.1125 & -0.0063 & 0.3313 \end{bmatrix},$$

$$\Phi(\tilde{A}, I_3) = \begin{bmatrix} 0.1697 & -0.0154 & 0.0029 \\ -0.0154 & 0.3620 & -0.1197 \\ 0.0029 & -0.1197 & 0.2486 \end{bmatrix},$$

we have $\Phi(A, I_3) \neq \Phi(\tilde{A}, I_3)$. This implies that there exists $(A_1, A_2) \in \mathbf{\Sigma}(\mathcal{D}_T) \times \mathbf{\Sigma}(\mathcal{D}_T)$ such that (5) does not hold. ■

In this letter, we are interested in an equivalent condition for the dataset that uniquely determines the solution to the Lyapunov equation (2). Therefore, we consider the following problem concerning Definition 1.

Problem 1: Consider the system (1) and the Lyapunov equation (2). Suppose that the dataset $\mathcal{D}_T$ in (4) is given. Determine whether $\mathcal{D}_T$ is informative for the Lyapunov equation (2). ■

III. CHARACTERIZATION OF SOLUTION TO LYAPUNOV EQUATION

There is a close relationship between a unique solution to the Lyapunov equation (2) and the controllable subspace of the system

$$\dot{x}(t) = Ax(t) + bu(t). \tag{8}$$

In this section, we address this relationship as a preliminary to Problem 1.

Consider the system (8), where $A \in \mathbf{Stab}(n)$ is a Hurwitz matrix and $b \in \mathbf{R}^n$ is a vector. For this system, we define $M_k(A, b) \in \mathbf{R}^{n\times k}$ by

$$M_k(A,b) := \begin{bmatrix} b & Ab & \cdots & A^{k-1}b \end{bmatrix}$$

for $k \in \{1, 2, \ldots\}$. The matrix $M_n(A, b)$ is known as the controllability matrix of the system (8). Then, let us introduce the set $\mathbf{\Sigma}_{\rm c}(A, b) \subseteq \mathbf{R}^{n\times n}$ by

$$\begin{aligned}&\mathbf{\Sigma}_{\rm c}(A,b)\\ &\quad = \{\tilde{A} \in \mathbf{Stab}(n) \mid M_{n+1}(\tilde{A}, b) = M_{n+1}(A, b)\}\end{aligned} \tag{9}$$

The set $\mathbf{\Sigma}_{\rm c}(A, b)$ provides an *equivalence class* of the Hurwitz matrices associated to the controllable subspaces of (8).

The following theorem addresses the relationship between $\Phi(A, Q)$ (i.e., the unique solution to the Lyapunov equation (2)) and the equivalence class $\mathbf{\Sigma}_{\rm c}(A, b)$.

Lemma 1: Consider the system (8) and the Lyapunov equation (2). Then, (5) holds for any $(A_1, A_2) \in \mathbf{\Sigma}_{\rm c}(A, b) \times \mathbf{\Sigma}_{\rm c}(A, b)$ if and only if there exists a matrix $W \in \mathbf{R}^{n\times n}$ such that

$$Q = M_n(A,b)WM_n^\top(A,b) \tag{10}$$

holds. ■

Proof: See Appendix I. ■

Lemma 1 indicates that $\Phi(A, Q) = \Phi(\tilde{A}, Q)$ holds for all $\tilde{A} \in \mathbf{\Sigma}_{\rm c}(A, b)$ if (10) holds for some W. In other words, a solution P to (2) with (10) is uniquely determined only by the knowledge of the controllable subspace of (8).

IV. INFORMATIVITY ANALYSIS

A. Solution to Data Informativity Problem

This section addresses Problem 1. In particular, we provide a necessary and sufficient condition of the data informativity for the Lyapunov equation (2), which is characterized by the possibility of the matrix decomposition $Q = X_0WX_0^\top$, called the *data-basis decomposition*.

A solution to Problem 1 is obtained by Lemma 1 and the following two facts:

(I) The set $\mathbf{\Sigma}(\mathcal{D}_T)$ is equal to $\mathbf{\Sigma}_{\rm c}(A, x_0)$.

(II) We can construct a matrix X_0 from $\mathcal{D}_T$ such that $\mathrm{Im}(M_n(A, x_0)) = \mathrm{Im}(X_0)$.

To formulate these facts, we first introduce several notions. Let

$$\mathbf{X}(\mathcal{D}_T) := \mathrm{span}\left(\bigcup_{t\in[0,T)} \{x(t,x_0)\}\right) \subseteq \mathbf{R}^n$$

be the minimum subspace containing the state trajectory $x(t, x_0)$ on $[0, T)$. By using this set, *the degree of* $\mathcal{D}_T$ and *a data-basis matrix of* $\mathcal{D}_T$ are defined as follows.

Definition 2: Let a dataset $\mathcal{D}_T$ in (4) be given. Then, $\deg(\mathcal{D}_T) := \dim(\mathbf{X}(\mathcal{D}_T))$ is called *the degree of* $\mathcal{D}_T$. ■

Definition 3: Assume that $\deg(\mathcal{D}_T) \neq 0$. Then, the matrix $X_0 \in \mathbf{R}^{n\times \deg(\mathcal{D}_T)}$ is called a *data-basis matrix of* $\mathcal{D}_T$ if

$$\mathrm{Im}(X_0) = \mathbf{X}(\mathcal{D}_T) \tag{11}$$

holds. ■

Example 3: Consider the same system and dataset in Example 1. Then, we have $\deg(\mathcal{D}_T) = 2$ because $\mathbf{X}(\mathcal{D}_T)$ is a 2-dimensional plane in $\mathbf{R}^3$. On the other hand, we can find that $\{x_0, x(0.5, x_0)\}$ is a basis of $\mathbf{X}(\mathcal{D}_T)$ and thus

$$X_0 = \begin{bmatrix} x_0 & x(0.5,x_0)\end{bmatrix} = \begin{bmatrix} 2 & 0.8221 \\ -5 & -1.6142 \\ 1 & -0.0299 \end{bmatrix} \tag{12}$$

is a data-basis matrix of $\mathcal{D}_T$. ■

By using these notations, (I) and (II) are formalized by the following theorem.

Lemma 2: Consider the system (1). Suppose that $\mathcal{D}_T$ in (4) is given. Assume that $\deg(\mathcal{D}_T) \neq 0$ and let $X_0 \in \mathbf{R}^{n\times \deg(\mathcal{D}_T)}$ be a data-basis matrix of $\mathcal{D}_T$. Then, the following relations hold:

(i) $\mathbf{\Sigma}(\mathcal{D}_T) = \mathbf{\Sigma}_{\rm c}(A, x_0)$.

(ii) $\mathrm{Im}(X_0) = \mathrm{Im}(M_n(A, x_0))$. ■

Example 4: Consider the same system and dataset in Example 1. Then, let us verify Lemma 2.

One can find that (i) holds for the dataset $\mathcal{D}_T$. For example, $\tilde{A} \in \mathbf{\Sigma}(\mathcal{D}_T)$ in (7) is an element of the equivalence class $\mathbf{\Sigma}_{\rm c}(A, x_0)$ because we have

$$\begin{aligned} M_{n+1}(\tilde{A}, x_0) &= M_{n+1}(A, x_0) \\ &= \begin{bmatrix} 2 & -3 & 2 & 7 \\ -5 & 10 & -15 & 10 \\ 1 & -4 & 11 & -24 \end{bmatrix}. \end{aligned} \tag{13}$$

On the other hand, (ii) is checked by (12), (13), and a simple calculation. ■

Proof of Lemma 2: Let us prove (i). From Proposition 3 in [15], we obtain

$$\mathbf{\Sigma}(\mathcal{D}_T) = \{\tilde{A} \in \mathbf{Stab}(n) \mid (A - \tilde{A})A^i x_0 = \mathbf{0}_n \quad (i = 0, 1, \ldots, n-1)\}.$$

Thus, we only show that

$$(A - \tilde{A})A^i x_0 = 0 \tag{14}$$

for each $i = 0, 1, \ldots, n-1$ is equivalent to $M_{n+1}(\tilde{A}, x_0) = M_{n+1}(A, x_0)$.

Let (14) holds for each $i = 0, 1, \ldots, n-1$. The equation (14) with $i = 0$ provides $Ax_0 = \tilde{A}x_0$. By substituting this relation into (14), we obtain $A^i x_0 = \tilde{A}^i x_0$ for all $i = 1, 2, \ldots, n$, which indicates $M_{n+1}(\tilde{A}, x_0) = M_{n+1}(A, x_0)$.

By using a similar way, the converse result is shown. Let $M_{n+1}(\tilde{A}, x_0) = M_{n+1}(A, x_0)$. This provides $Ax_0 = \tilde{A}x_0$.

This relation and $A^{i+1}x_0 = \tilde{A}^{i+1}x_0$ derive (14). Hence, (i) is proved.

Next, we give a proof of (ii). From (11), it is suffice to show $\mathbf{X}(\mathcal{D}_T) = \mathrm{Im}(M_n(A, x_0))$. By using Lemma 16.6.2 in [14] and simple calculation, we have

$$\mathbf{X}(\mathcal{D}_T) = \bigcup_{t\in[0,T)} \mathrm{Im}(e^{At}x_0) = \left(\bigcap_{t\in[0,T)} \ker(x_0^\top e^{A^\top t})\right)^\perp = \mathrm{Im}(M_n(A, x_0)).$$

Thus, (ii) is proved. ■

By using Lemma 1, Lemma 2, and the definition of data-basis matrices, we eventually obtain a solution to Problem 1 as follows.

Theorem 1: Consider Problem 1. Let $d := \deg(\mathcal{D}_T)$. Then, the following statements hold:

(i) Suppose $d \neq 0$ and let a data-basis matrix $X_0 \in \mathbf{R}^{n\times d}$ of $\mathcal{D}_T$ be given. Then, $\mathcal{D}_T$ is informative for the Lyapunov equation (2) if and only if there exists a matrix $W \in \mathbf{R}^{d\times d}$ such that $Q = X_0WX_0^\top$.

(ii) Suppose $d = 0$. Then, $\mathcal{D}_T$ is informative for the Lyapunov equation (2) if and only if $Q = \mathbf{0}_{n\times n}$. ■

Theorem 1 indicates that the data informativity is characterized by the possibility of the matrix decomposition in the form of $Q = X_0WX_0^\top$. This condition can be easily checked because the decomposition can be regarded as a linear matrix equation with the unknown $W \in \mathbf{R}^{d\times d}$. We call this decomposition *data-basis decomposition*.

B. Examples

Consider the same system and dataset in Example 1 and $Q \in \mathbf{R}^{n\times n}$ in (6). Then, let us verify that $\mathcal{D}_T$ is informative for the Lyapunov equation (2) by using Theorem 1.

In this case, we can apply the data-basis decomposition to Q. In fact, we can find that $Q = X_0WX_0^\top$ holds for $X_0 \in \mathbf{R}^{3\times 2}$ in (12) and

$$W = \begin{bmatrix} 0.9017 & 0.1153 \\ -3.2866 & 3.8577 \end{bmatrix}. \tag{15}$$

Hence, the dataset $\mathcal{D}_T$ is informative for the Lyapunov equation (2).

Meanwhile, we can check the data informativity for other cases. For example, $\mathcal{D}_T$ is not informative for the Lyapunov equation (2) with $Q = I_3$. This is because there is no feasible solution $W \in \mathbf{R}^{2\times 2}$ to the linear matrix equation $I_3 = X_0WX_0^\top$ for X_0 in (12).

V. DATA-DRIVEN COMPUTATION

A. Solution to Data-driven Computation Problem

This section provides a data-driven method for solving (2) based on our data informativity and the data-basis decomposition of Q. A key idea of the proposed method is constructing a generalized discrete-time Lyapunov equation equivalent to the original Lyapunov equation (2) by using the dataset $\mathcal{D}_T$.

Consider the following problem about a data-driven computation of the solution to the Lyapunov equation (2).

Problem 2: Consider the situation in Problem 1. Assume that $A \in \mathbf{Stab}(n)$ is unknown but $\mathcal{D}_T$ is informative for the Lyapunov equation (2). Then, calculate $\Phi(A, Q)$. ■

A solution to Problem 2 is formulated as follows. For given $t_i \in [0, \infty)$ $(i = 1, 2, \dots, d)$, let

$$X(t) := \begin{bmatrix} x(t + t_1, x_0) & x(t + t_2, x_0) & \cdots & x(t + t_d, x_0) \end{bmatrix},$$

where $d = \deg(\mathcal{D}_T)$. Note that $X(t)$ $(t \in [0, h])$ can be constructed from the dataset $\mathcal{D}_T$ if

$$h + t_i < T \quad (i = 1, 2, \dots, d) \tag{16}$$

holds. Then, the following theorem is a solution to Problem 2.

Theorem 2: Consider Problem 2. Let $d := \deg(\mathcal{D}_T)$. Then, the following two statements hold.

(i) Suppose $d \neq 0$. Assume that $X(0)$ is a data-basis matrix of $\mathcal{D}_T$ and $W \in \mathbf{R}^{d\times d}$ is a matrix satisfying $Q = X(0)WX^\top(0)$. Let $h \in \mathbf{R}_+$ be a positive number satisfying (16). Then, the *generalized discrete-time Lyapunov equation*

$$X(h)VX^\top(h) - X(0)VX^\top(0) = -\int_0^h X(t)WX^\top(t)\, dt \tag{17}$$

has a unique solution $V \in \mathbf{R}^{d\times d}$ and $\Phi(A, Q) = X(0)VX^\top(0)$ holds.

(ii) Suppose $d = 0$. Then, $\Phi(A, Q) = \mathbf{0}_{n\times n}$ holds. ■

Proof: The statement (ii) is directly derived from Theorem 1 (ii) and (3). On the other hand, (i) is the consequence of the following three facts.

(a) There exists a matrix $\tilde{V} \in \mathbf{R}^{d\times d}$ satisfying

$$\Phi(A, Q) = X(0)\tilde{V}X^\top(0). \tag{18}$$

(b) The matrix $\tilde{V}$ in (a) is a solution to (17).

(c) The solution $V \in \mathbf{R}^{d\times d}$ to (17) is unique if it has at least one solution.

The fact (a) is directly derived by Lemma 3 in Appendix I-A. In the following, we show (b) and (c).

The fact (b) is proved as follows. Since A is Hurwitz, (2) is equivalent to the discrete-time Lyapunov equation

$$e^{Ah}P(e^{Ah})^\top - P = -\int_0^h e^{At}Q(e^{At})^\top\, dt.$$

In this equation, we can replace $\int_0^h e^{At}Q(e^{At})^\top dt$ with $\int_0^h X(t)WX^\top(t)dt$ because

$$e^{At}Q(e^{At})^\top = e^{At}X(0)WX^\top(0)(e^{At})^\top = X(t)WX^\top(t),$$

where we use

$$e^{At}X(0) = X(t). \tag{19}$$

Therefore, we obtain the following relation:

$$e^{Ah}\Phi(A, Q)(e^{Ah})^\top - \Phi(A, Q) = -\int_0^h e^{At}Q(e^{At})^\top\, dt. \tag{20}$$

Moreover, we have

$$e^{Ah}\Phi(A,Q)(e^{Ah})^\top = e^{Ah}X(0)\tilde{V}X^\top(0)(e^{Ah})^\top = X(h)\tilde{V}X^\top(h) \quad (21)$$

from (18) and (19). By substituting (18) and (21) into (20), we eventually obtain the relation (17) with $V = \tilde{V}$, which proves (b).

Next, we show (c). Let $V_1, V_2 \in \mathbf{R}^{d\times d}$ be solutions to (17). Then, we have

$$e^{Ah}X(0)(V_1 - V_2)X^\top(0)(e^{Ah})^\top - X(0)(V_1 - V_2)X^\top(0) = \mathbf{0}_{n\times n}. \quad (22)$$

from (17) and (19). Since (22) is a variant of a discrete-time Lyapunov equation and e^{Ah} is Schur, (22) implies $X(0)(V_1 - V_2)X^\top(0) = \mathbf{0}_{n\times n}$. This relation and the column full rank property of $X(0)$ provide $V_1 = V_2$. This proves (c).

These facts complete the proof. ■

Theorem 2 claims that the solution P to the Lyapunov equation (2) is obtained by solving the generalized discrete-time Lyapunov equation (17) constructed by the dataset $\mathcal{D}_T$. Notice that Theorem 2 only assumes that the dataset $\mathcal{D}_T$ is informative for the Lyapunov equation (2). This indicates that Theorem 2 might be applicable even when the previous results in [5], [6] cannot be applied.

B. Examples

Consider the same system and dataset in Example 1 and $Q \in \mathbf{R}^{n\times n}$ in (6). Then, let us compute $\Phi(A,Q)$ by using Theorem 2. Note that $\mathcal{D}_T$ is informative for the Lyapunov equation (2) from Section IV-B. The true value of $\Phi(A,Q)$ is given by

$$\Phi(A,Q) = \begin{bmatrix} 0.3 & -0.325 & -0.375 \\ -1.075 & 1.8 & 0.35 \\ 0.475 & -1.15 & 0.2 \end{bmatrix}. \quad (23)$$

By picking $h = 0.4$ and $X(t) = [x(t, x_0) \;\; x(t+0.5, x_0)]$, we have $X(0)$ in the right-hand side of (12) and

$$X(h) = \begin{bmatrix} 1.0027 & 0.3350 \\ -2.0693 & -0.5138 \\ 0.0639 & -0.1562 \end{bmatrix}.$$

Moreover, we obtain W satisfying (15) and

$$\int_0^h X(t)WX^\top(t)\,dt = \begin{bmatrix} 0.2280 & -0.2499 & -0.2061 \\ -0.8485 & 1.4558 & 0.2412 \\ 0.3925 & -0.9560 & 0.1710 \end{bmatrix}.$$

Therefore, solving (17) yields

$$V = \begin{bmatrix} 0.1805 & 0.1120 \\ -0.7385 & 0.9001 \end{bmatrix}$$

and we eventually obtain $\Phi(A,Q)$ by

$$X(0)VX^\top(0) = \begin{bmatrix} 0.3000 & -0.3250 & -0.2750 \\ -1.0750 & 1.8000 & 0.3500 \\ 0.4750 & -1.1500 & 0.2000 \end{bmatrix}.$$

This matrix is equal to the true value in (23). Hence, the solution to the Lyapunov equation (2) is successfully computed from the dataset $\mathcal{D}_T$. It is worth noting that the proposed method is applicable even if $\mathcal{D}_T$ is not informative for system identification (see Example 1).

VI. CONCLUSIONS

This letter addresses the data informativity for the Lyapunov equation (2). This notion represents a property whether the solution to the Lyapunov equation is uniquely determined by given data. First, we clarify the relationship between the solution to the Lyapunov equation and the controllable subspace of the system (8). Second, we show that the data informativity for the Lyapunov equation (2) is equivalent to the possibility of the data-basis decomposition of Q. Finally, based on this result, we present a data-driven method for solving the Lyapunov equation. This method has a potential to compute the solution from given data even if the data do not contain sufficient information to identify the system.

In future work, we plan to consider data informativity problems for other classes of Lyapunov equations, e.g., $A^\top P + PA = -Q$.

APPENDIX I
PROOF OF LEMMA 1

A. Proof of If Part

As a preliminary to the proof, we introduce the following result.

Lemma 3: Let $A \in \mathbf{Stab}(n)$, $b \in \mathbf{R}^n$, and $Q \in \mathbf{R}^{n\times n}$ be given in a similar way to Lemma 1. Suppose $d = \operatorname{rank}(M_n(A,b))$ and let $\bar{M} := M_d(A,b)$ If $Q = \bar{M}\bar{W}\bar{M}^\top$, there exists a matrix $V(\bar{W},\bar{M}) \in \mathbf{R}^{d\times d}$ satisfying $\Phi(A,Q) = \bar{M}V(\bar{W},\bar{M})\bar{M}^\top$. ■

Proof: This lemma is a variant of Theorem 1 in [16] and is proved in a similar way to the original proof and the following fact: There exists a coprime monic polynomial α which has the degree d satisfying $\alpha(A)b = \mathbf{0}_n$. This fact is derived from the linear dependence of the vectors A^ib ($i = 0, 1, \ldots, d$). ■

By using Lemma 3, let us prove the *if part* of Lemma 1, i.e., (5) holds for any $(A_1, A_2) \in \mathbf{\Sigma}_c(A,b)\times\mathbf{\Sigma}_c(A,b)$. From (10) and $\operatorname{Im}(M_n(A,b)) = \operatorname{Im}(\bar{M})$, there exists $\bar{W} \in \mathbf{R}^{d\times d}$ satisfying we have $Q = \bar{M}\bar{W}\bar{M}^\top$. Therefore, Lemma 3 and (9) provide (5) for all $(A_1, A_2) \in \mathbf{\Sigma}_c(A,b)\times\mathbf{\Sigma}_c(A,b)$. This completes the proof.

B. Proof of Only-if Part

Let $U \in \mathbf{R}^{n\times n}$ be a unitary matrix such that $U = [B_1 \;\; B_2]$, $B_1 \in \mathbf{R}^{n\times d}$, $B_2 \in \mathbf{R}^{n\times(n-d)}$, and $\operatorname{Im}(B_1) = \operatorname{Im}(M_{n+1}(A,b))$. Then, the following facts prove the *only if part* of Lemma 1.

(a) There exist $F_{11} \in \mathbf{Stab}(d)$ and $F_{22} \in \mathbf{Stab}(n-d)$ satisfying $(\tilde{A}_1, \tilde{A}_2) \in \mathbf{\Sigma}_c(A,b)\times\mathbf{\Sigma}_c(A,b)$, where

$$\tilde{A}_1 = U\begin{bmatrix} F_{11} & \mathbf{0}_{d\times(n-d)} \\ \mathbf{0}_{(n-d)\times d} & F_{22} \end{bmatrix}U^\top, \quad (24)$$
$$\tilde{A}_2 = U\begin{bmatrix} F_{11} & \mathbf{0}_{d\times(n-d)} \\ \mathbf{0}_{(n-d)\times d} & F_{22} - I_{n-d} \end{bmatrix}U^\top.$$

(b) If $\Phi(\tilde{A}_1, Q) = \Phi(\tilde{A}_2, Q)$ holds, there exists $\tilde{W}_{11} \in \mathbf{R}^{d\times d}$ satisfying $Q = B_1 \tilde{W}_{11} B_1^\top$.
(c) The consequent of (b) is equivalent to the existence of $W \in \mathbf{R}^{d\times d}$ satisfying $Q = BWB^\top$.

The fact (c) is trivial. Therefore, we prove (a) and (b).

We first show (a). Let F_{ij} $((i,j) \in \{1,2\}^2)$ be matrices satisfying

$$A = U \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} U^\top. \tag{25}$$

In (25), we have

$$F_{21} = \mathbf{0}_{(n-d)\times n}, \tag{26}$$

which is derived as follows. Since $\mathrm{Im}(M_{n+1}(A, b))$ is an A-invariant subspace, $B_2^\top A B_1 = \mathbf{0}_{(n-d)\times d}$ holds. By substituting (25) into this relation, we obtain

$$B_2^\top \begin{bmatrix} B_1 & B_2 \end{bmatrix} \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} B_1^\top \\ B_2^\top \end{bmatrix} B_1 = F_{21} = \mathbf{0}_{(n-d)\times n},$$

where we use $B_1^\top B_1 = I_d$, $B_2^\top B_2 = I_{n-d}$, and $B_2^\top B_1 = \mathbf{0}_{(n-d)\times d}$. Moreover, the Hurwitz properties of F_{11} and F_{22} are inherited from that of A because of (25) and (26).

From (25) and (26), we can find that AB_1, $\tilde{A}_1 B_1$, and $\tilde{A}_2 B_1$ are all equal to $B_1 F_{11}$, which implies $M_{n+1}(A, b) = M_{n+1}(\tilde{A}_1, b) = M_{n+1}(\tilde{A}_2, b)$. This relation and $(\tilde{A}_1, \tilde{A}_2) \in \mathbf{Stab}(n) \times \mathbf{Stab}(n)$ prove (a).

Next, we prove (b). In the following, let

$$Q = U \begin{bmatrix} \tilde{W}_{11} & \tilde{W}_{12} \\ \tilde{W}_{21} & \tilde{W}_{22} \end{bmatrix} U^\top, \tag{27}$$

and we show that $\tilde{W}_{12}$, $\tilde{W}_{21}$, and $\tilde{W}_{22}$ are all equal to zero matrices. From (24) and (27), $\Phi(\tilde{A}_1, Q)$ is expressed as

$$\begin{aligned} \Phi(\tilde{A}_1, Q) &= \int_0^\infty e^{\tilde{A}_1 t} Q e^{\tilde{A}_1^\top t}\, dt \\ &= U \int_0^\infty \begin{bmatrix} e^{F_{11}t}\tilde{W}_{11}e^{F_{11}^\top t} & e^{F_{11}t}\tilde{W}_{12}e^{F_{22}^\top t} \\ e^{F_{22}t}\tilde{W}_{21}e^{F_{11}^\top t} & e^{F_{22}t}\tilde{W}_{22}e^{F_{22}^\top t} \end{bmatrix} dt\, U^\top \\ &= U \begin{bmatrix} \Phi(F_{11}, \tilde{W}_{11}) & \Psi(F_{11}, F_{22}, \tilde{W}_{12}) \\ \Psi(F_{22}, F_{11}, \tilde{W}_{21}) & \Phi(F_{22}, \tilde{W}_{22}) \end{bmatrix} U^\top, \end{aligned}$$

where

$$\Psi(F_{11}, F_{22}, \tilde{W}_{12}) := \int_0^\infty e^{F_{11}t}\tilde{W}_{12}e^{F_{22}^\top t}\, dt. \tag{28}$$

Thus, $\Phi(\tilde{A}_1, Q) = \Phi(\tilde{A}_2, Q)$ implies the following three equations:

$$\Psi(F_{11}, F_{22}, \tilde{W}_{12}) = \Psi(F_{11}, F_{22} - I_{n-d}, \tilde{W}_{12}), \tag{29}$$
$$\Psi(F_{22}, F_{11}, \tilde{W}_{21}) = \Psi(F_{22} - I_{n-d}, F_{11}, \tilde{W}_{21}), \tag{30}$$
$$\Phi(F_{22}, \tilde{W}_{22}) = \Phi(F_{22} - I_{n-d}, \tilde{W}_{22}). \tag{31}$$

From (28) and (29), $\Psi(F_{11}, F_{22}, \tilde{W}_{12})$ is a solution to the two Sylvester equations

$$F_{11}P + PF_{22}^\top = -\tilde{W}_{12},$$
$$F_{11}P + P(F_{22} - I_{n-d})^\top = -\tilde{W}_{12}.$$

By subtracting these two equations, we obtain the relation $\Psi(F_{11}, F_{22}, \tilde{W}_{12}) = \mathbf{0}_{d\times(n-d)}$, which implies $\tilde{W}_{12} = \mathbf{0}_{d\times(n-d)}$ from (28). In addition, $\tilde{W}_{21} = \mathbf{0}_{(n-d)\times d}$ and $\tilde{W}_{22} = \mathbf{0}_{(n-d)\times(n-d)}$ are derived by applying a similar procedure to (30) and (31). These facts complete the proof.

References

[1] H.J. van Waarde, J. Eising, H.L. Trentelman, and M.K. Camlibel, Data informativity: a new perspective on data-driven analysis and control, IEEE Transactions on Automatic Control, vol. 65, no. 11, pp. 4753–4768, 2020.
[2] J. Eising, H. L. Trentelman, and M. K. Camlibel, Data informativity for observability: an invariance-based approach, 2020 European Control Conference (ECC), pp. 1057–1059, 2020.
[3] H.J. van Waarde and M. Mesbahi, Data-driven parameterizations of suboptimal LQR and H_2 controllers, IFAC-PapersOnLine, vol. 53, no. 2, pp. 4234–4239, 2020.
[4] H.J. van Waarde, J. Eising, M.K. Camlibel, and H.L. Trentelman, A tutorial on the informativity framework for data-driven control, 2022 IEEE 61st Conference on Decision and Control (CDC), pp. 1085–1090, 2022.
[5] I. Banno, S. Azuma, R. Ariizumi, T. Asai, and J. Imura, Data-driven estimation and maximization of controllability Gramians, 60th IEEE Conference on Decision and Control (CDC), pp. 5053–5058, 2021.
[6] D. Vrabie, O. Pastravanu, M. Abu-Khalaf, and F.L. Lewis, Adaptive optimal control for continuous-time linear systems based on policy iteration, Automatica, vol. 45, no. 2, pp. 477–484, 2009.
[7] Y. Jiang and Z.P. Jiang, Computational adaptive optimal control for continuous-time linear systems with completely unknown dynamics, Automatica, vol. 48, no. 10, pp. 2699–2704, 2012.
[8] Z. Wang and D. Liu, Data-based controllability and observability analysis of linear discrete-time systems, IEEE Transactions on Neural Networks, vol. 22, no. 12, pp. 2388–2392, 2011.
[9] H.R. Shaker and S. Lazarova-Molnar, A new data-driven controllability measure with application in intelligent buildings, Energy and Buildings, vol. 138, pp. 526–529, 2017.
[10] V.K. Mishra, I. Markovsky, and B. Grossmann, Data-driven tests for controllability, IEEE Control Systems Letters, vol. 5, no. 2, pp. 517–522, 2021.
[11] M. Tanemura and S. Azuma, Efficient data-driven estimation of passivity properties, IEEE Control Systems Letters, vol. 3, no. 2, pp. 398–403, 2019.
[12] Z.S. Hou and Z. Wang, From model-based control to data-driven control: Survey, classification and perspective, Information Sciences, vol. 235, no. 20, pp. 3–35, 2013.
[13] L. Ljung, *System identification: Theory for the user*, New Jersey, USA: Prentice-Hall, 1987.
[14] D.S. Bernstein, *Scalar, Vector, and Matrix Mathematics: Theory, Facts, and Formulas Revised and Expanded Edition*, New Jersey, USA: Princeton University Press, 2018.
[15] H.J. van Waarde, P. Tesi, and M. K. Camlibel, Topology reconstruction of dynamical networks via constrained Lyapunov equations, IEEE Transactions on Automatic Control, vol. 64, no. 10, pp. 4300–4306, 2019.
[16] E. de Souza and S.P. Bhattacharyya, Controllability, observability and the solution of $AX - XB = C$, Linear Algebra and its Applications, vol. 39, pp. 167–188, 1981.
[17] R.A. Smith, Matrix calculations for Liapunov quadratic forms, Journal of Differential Equations, vol. 2, no. 2, pp. 208–217, 1966.