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A data-driven approach to system invertibility and input reconstruction

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***Abstract*— We consider the problems of system invertibility and input reconstruction for linear time-invariant (LTI) systems using only measured data. The two problems are connected in the sense that input reconstruction is possible provided that the system is left invertible. To verify the latter property without model knowledge, we leverage behavioral systems theory and develop two data-driven algorithms: one based on input/state/output data and the other based only on input/output data. We then consider the problem of input reconstruction for both noise-free and noisy data settings. In the case of noisy data, a statistical approach is leveraged to formulate the problem as a maximum likelihood estimation (MLE) problem. The proposed approaches are finally illustrated with numerical examples that show: exact input reconstruction in the noise-free setting; and the better performance of the MLE-based approach compared to the standard least-norm solution.**

***Index Terms*— System invertibility, input reconstruction, maximum likelihood estimation, behavioral systems theory.**

I. Introduction

In recent years there has been an increasing interest in developing methods that enable classic analysis and design control problems to be tackled without having access to a model of the system, but instead directly using data. This is particularly advantageous when dealing with complex systems for which models are not easy to obtain either from first principles or system identification. A powerful result in support of these so-called *direct data-driven* methods has been without any doubt the *fundamental lemma* developed by Willems and co-workers [1] in the context of behavioral systems theory, see [2] for a recent survey. Building on this framework, this paper considers the problems of system invertibility and input reconstruction for discrete-time linear time-invariant (LTI) systems based on measured data, for both noise-free and noisy cases.

In the model-based setting, the problem of system invertibility and input reconstruction may be traced back to the seminal contributions by Sain and Massey [3], and Silverman [4]. It has been since then an active area of research, which has found several applications, see, for example, [5] and references therein. Recent works considered the problem of initial state and input reconstruction [6] and simultaneous input reconstruction and state estimation [7]. Strong invertibility of linear systems has been studied in [8], whereas the problem of input reconstruction for switched systems is studied in [9].

Differently from the previously mentioned papers, all based on model knowledge, we are interested in developing efficient algorithms for these tasks which only make use of measured data. That is, the main goal is that of reconstructing the input trajectory for a given output trajectory and initial conditions based on past measured input/output data. The key observation we leverage is that the data-driven input reconstruction problem can be seen as an *inverse* of the problem of data-driven simulation [10], which consists of computing the output trajectory for a given input trajectory and initial conditions based on past measured input/output data. It is known that input reconstruction can be achieved only if the system is left invertible. Therefore, we also develop data-driven criteria to verify the left invertibility of the system. If this system is positive, we show that input reconstruction can be achieved with a minimum delay only, which is known as the inherent delay of the system [3].

Prior work has considered partially related problems in the data-based setting. Data-driven state observers have been considered in [11], [12] and the problem of data-driven input reconstruction has been studied in [13], [14]. However, papers [13], [14] assume that the system is left invertible and do not provide theoretical guarantees on how to verify such property based on measured data. In fact, it will be shown here that [13, Condition (7)], which is assumed for input reconstruction, can be guaranteed by the left invertibility of the system (see Remark 5). Furthermore, this problem has not been studied yet when working with noisy data.

The work has two main contributions. First, for the case of noise-free data, we prove novel data-based tests and algorithms based on behavioral systems theory to verify the left invertibility of an LTI system and to reconstruct the input trajectory from a given output trajectory. Namely, Algorithm 1 and Algorithm 2 deal with the problem of left invertibility when input/state/output and input/output data are available, respectively. Algorithm 3 instead solves exactly the input reconstruction problem. Second, we develop a statistical framework inspired by the signal matrix model (SMM) proposed in [15] to handle the case where the output is corrupted by additive noise. The resulting method provides a maximum likelihood input reconstruction estimate. Numerical tests show a better performance in average accuracy and dispersion of the estimate when compared to standard heuristics that do not leverage any distributional information of the noise.

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Notation and Preliminaries. The set of real $k \times m$ matrices is denoted by $\mathbb{R}^{k\times m}$. The transpose and the Moore-Penrose pseudo-inverse of $A \in \mathbb{R}^{k\times m}$ are denoted by $A^\top$ and $A^\dagger$, respectively. The (i,j)-th entry of any matrix A is denoted by $(A)_{i,j}$. The identity matrix of dimension k/appropriate is denoted by I_k/I and the zero matrix of appropriate dimensions is denoted by 0. The normal-rank of a polynomial matrix or a matrix transfer function in z is defined as the maximal rank over all possible values of z. The symbol $\Lambda(A,B) := \{\lambda \in \mathbb{C} \mid \det(A - \lambda B) = 0\}$ denotes the set of generalized eigenvalues (GEVs) of a pair of square matrices (A, B). We define

$$\mathrm{col}(A_1, A_2, \ldots, A_r) := \begin{bmatrix} A_1^\top & A_2^\top & \ldots & A_r^\top \end{bmatrix}^\top.$$

with the tacit assumption that the matrices have the same number of columns. With z_d we denote an offline (*data*) trajectory of length T defined as

$$z_\mathrm{d} := \big(z_\mathrm{d}(k), z_\mathrm{d}(k+1), \ldots, z_\mathrm{d}(k+T-1)\big) \in (\mathbb{R}^q)^T.$$

Associated with it, we also define two shifted trajectories

$$z_\mathrm{d}^+ := \big(z_\mathrm{d}(k+1), z_\mathrm{d}(k+2), \ldots, z_\mathrm{d}(k+T-1)\big),$$
$$z_\mathrm{d}^- := \big(z_\mathrm{d}(k), z_\mathrm{d}(k+1), \ldots, z_\mathrm{d}(k+T-2)\big).$$

Further, we denote a generic trajectory of length L as $z|_{[k,k+L-1]} := \mathrm{col}(z(k), z(k+1), \ldots, z(k+L-1))$.

Definition 1: A q-variate time series z_d is persistently exciting of order $L \in \mathbb{N}$ if the Hankel matrix with L-block rows defined as

$$\mathcal{H}_L(z_\mathrm{d}) = \begin{bmatrix} z_\mathrm{d}(k) & z_\mathrm{d}(k+1) & \cdots & z_\mathrm{d}(k+T-L) \\ z_\mathrm{d}(k+1) & z_\mathrm{d}(k+2) & \cdots & z_\mathrm{d}(k+T-L+1) \\ \vdots & \vdots & \ddots & \vdots \\ z_\mathrm{d}(k+L-1) & z_\mathrm{d}(k+L) & \cdots & z_\mathrm{d}(k+T-1) \end{bmatrix}$$

has full row rank, i.e., its rank is qL.

II. Background

A. Model-Based System Invertibility

Consider the discrete-time linear time-invariant system

$$x(k+1) = Ax(k) + Bu(k), \tag{1a}$$
$$y(k) = Cx(k) + Du(k). \tag{1b}$$

Here, at each time instant k, $x(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}^m$ is the control input vector, and $y(k) \in \mathbb{R}^p$ is the output vector. Matrices A, B, C, D are of appropriate dimensions. We assume that the system is minimal. The $p \times m$ matrix transfer function of the system is given by

$$G(z) = C(zI - A)^{-1}B + D.$$

Definition 2: [3, Definition 4] Let $\tau \geq 0$ be a finite integer. We say G is τ-delay left invertible if it has a τ-delay left inverse, i.e., there exists $\widehat{G}$ such that $\widehat{G}(z)G(z) = z^{-\tau} I_m$. Furthermore, we say G is left invertible if there exists a nonnegative integer κ such that G is κ-delay left invertible.

It can be seen that if G has a τ-delay left inverse, then it has a τ_1-delay left inverse for all $\tau_1 > \tau$. The smallest τ for which G has a left inverse is called the inherent delay of the left invertible system G [3], which we denote by τ_0. Note that $\tau_0 \leq n$ [3, Corollary 1]. Related to system (1), we define the following matrices

$$\mathcal{O}_\tau = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^\tau \end{bmatrix}, \quad \mathcal{T}_\tau = \begin{bmatrix} D & 0 & \cdots & 0 \\ CB & D & \cdots & 0 \\ CAB & CB & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{\tau-1}B & CA^{\tau-2}B & \cdots & D \end{bmatrix}.$$

Also, $\mathcal{T}_0 = D$ and $\mathcal{T}_{-1} = 0$. We recall the following result.

Theorem 3: The following statements are equivalent:

i) $\exists\ \tau \geq 0$ such that system (1) is τ-delay left invertible.

ii) normal-rank $\begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} = n + m.$

iii) normal-rank $G(z) = m$.

iv) $\exists\ \tau \geq 0$ such that rank $\mathcal{T}_\tau$ − rank $\mathcal{T}_{\tau-1} = m$.

v) $\exists$ a matrix Q and $\exists\ \tau \geq 0$ such that $Q\mathcal{T}_\tau = \begin{bmatrix} I_m & 0 \end{bmatrix}$.

Proof of the above theorem can be easily obtained by leveraging [3, Theorem 2], [6, Theorem 1], and [7, Proposition 2].

Remark 1: For left invertibility, $p \geq m$ is necessary.

B. Data-Driven Simulation

To begin with, we recall the fundamental lemma.

Lemma 4: [1, Theorem 1] Assume that system (1) is controllable and, given $L \in \mathbb{N}$ with $L > n$, the observed trajectory $\mathrm{col}(u_d, y_d)$ is such that u_d is persistently exciting of order $L + n$. Then $\mathrm{col}(u|_{[k,k+L-1]}, y|_{[k,k+L-1]})$ is a trajectory of system (1) if and only if there exists $g \in \mathbb{R}^{T-L+1}$ such that

$$\begin{bmatrix} \mathcal{H}_L(u_d) \\ \mathcal{H}_L(y_d) \end{bmatrix} g = \begin{bmatrix} u|_{[k,k+L-1]} \\ y|_{[k,k+L-1]} \end{bmatrix}.$$

The problem of data-driven simulation can then be stated as follows [10]. Given an observed input/output trajectory $\mathrm{col}(u_\mathrm{d}, y_\mathrm{d})$, initial condition $\mathrm{col}(u_\mathrm{ini}, y_\mathrm{ini})$ of length T_ini, and input trajectory $u|_{[k,k+L-1]} := \mathrm{col}(u(k), u(k+1), \ldots, u(k+L-1))$: compute the output trajectory $y|_{[k,k+L-1]}$. Note that, using Lemma 4, the concatenation $\mathrm{col}(u_\mathrm{ini}, y_\mathrm{ini}) \wedge \mathrm{col}(u, y)|_{[k,k+L-1]} =: \mathrm{col}(\bar{u}|_N, \bar{y}|_N)$ is a trajectory of the system if and only if there exists g such that

$$\begin{bmatrix} \mathcal{H}_N(u_\mathrm{d}) \\ \mathcal{H}_N(y_\mathrm{d}) \end{bmatrix} g = \begin{bmatrix} \bar{u}|_N \\ \bar{y}|_N \end{bmatrix}, \tag{2}$$

where $N = L + T_\mathrm{ini}$. Defining $\mathrm{col}(U_p, U_f) := \mathcal{H}_N(u_\mathrm{d})$ and $\mathrm{col}(Y_p, Y_f) := \mathcal{H}_N(y_\mathrm{d})$, (2) is equivalent to

$$\begin{bmatrix} U_p \\ Y_p \\ U_f \\ Y_f \end{bmatrix} g = \begin{bmatrix} u_\mathrm{ini} \\ y_\mathrm{ini} \\ u|_{[k,k+L-1]} \\ y|_{[k,k+L-1]} \end{bmatrix} \begin{matrix} mT_\mathrm{ini} \\ pT_\mathrm{ini} \\ mL \\ pL \end{matrix}. \tag{3}$$

To solve the problem of simulation, we compute g from the first three block equations of (3), and then from the fourth block equation, we compute the output trajectory as $y|_{[k,k+L-1]} = Y_f g$ [10, Algorithm 1]. In view of this, it would be tempting to say that we can compute the input trajectory as $u|_{[k,k+L-1]} = U_f g$, where g is computed by the other three block equations $\mathrm{col}(U_p, Y_p, Y_f)g = \mathrm{col}(u_\mathrm{ini}, y_\mathrm{ini}, y|_{[k,k+L-1]})$. However, we observe on passing here, deferring a proof of it to the technical part later, that this is incorrect. This is because (3) is just a feasibility condition for the realization of input/output data and cannot be seen as an inversion test. We can only reconstruct the inputs provided that the system is left invertible. Thus, we aim to develop data-driven criteria for the left invertibility of an LTI system (see Section III) and retrieve the inputs that have led to the outputs based on (noisy) measured data (see Section IV).

III. Data-Driven Criteria for Left Invertibility

In this section, we provide data-driven algorithms for the left invertibility of the system. First, we consider the case where we have access to the input/state/output data, and then the scenario where we have only input/output data.

A. Input/State/Output Data Case

Assumption 1: The matrix $\operatorname{col}(\mathcal{H}_1(x_\mathrm{d}^-), \mathcal{H}_1(u_\mathrm{d}^-))$ has full row rank.

The above assumption is equivalent to the identifiability of system (1) [16, Proposition 6]. Moreover, it can be enforced if the system is controllable and u_d is persistently exciting of order $n+1$ [1, Corollary 2 (ii)].

Theorem 5: Let Assumption 1 hold. Then, system (1) is left invertible if and only if

$$\text{normal-rank} \begin{bmatrix} \mathcal{H}_1(x_\mathrm{d}^+) - \lambda \mathcal{H}_1(x_\mathrm{d}^-) \\ \mathcal{H}_1(y_\mathrm{d}^-) \end{bmatrix} = n + m. \qquad (4)$$

Proof: Note that

$$\begin{aligned} &\begin{bmatrix} \mathcal{H}_1(x_\mathrm{d}^+) - \lambda \mathcal{H}_1(x_\mathrm{d}^-) \\ \mathcal{H}_1(y_\mathrm{d}^-) \end{bmatrix} \\ &= \begin{bmatrix} A\mathcal{H}_1(x_\mathrm{d}^-) + B\mathcal{H}_1(u_\mathrm{d}^-) - \lambda \mathcal{H}_1(x_\mathrm{d}^-) \\ C\mathcal{H}_1(x_\mathrm{d}^-) + D\mathcal{H}_1(x_\mathrm{d}^-) \end{bmatrix} \\ &= \begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} \begin{bmatrix} \mathcal{H}_1(x_\mathrm{d}^-) \\ \mathcal{H}_1(u_\mathrm{d}^-) \end{bmatrix}. \end{aligned}$$

Since the matrix $\operatorname{col}(\mathcal{H}_1(x_\mathrm{d}^-), \mathcal{H}_1(u_\mathrm{d}^-))$ has full row rank, (4) holds if and only if Theorem 3 ii) holds. ■

Note that if Assumption 1 does not hold, condition (4) provides only a sufficient test for the left invertibility. Also, because $\mathcal{H}_1(x_\mathrm{d}^+) - \lambda \mathcal{H}_1(x_\mathrm{d}^-)$ is not a square matrix, it is not straightforward how to verify (4) for left invertibility of the system. In fact, if $\mathcal{H}_1(x_\mathrm{d}^+) - \lambda \mathcal{H}_1(x_\mathrm{d}^-)$ were a square matrix, it was sufficient to pick a λ that was not the generalized eigenvalue of the matrix pair $\left(\mathcal{H}_1(x_\mathrm{d}^+), \mathcal{H}_1(x_\mathrm{d}^-)\right)$. Nevertheless, similar to [17, Algorithm 1], the following algorithm is proposed to verify the left invertibility (4) for system (1).

Algorithm 1: Data-driven left invertibility test.

Input: Observed data u_d, x_d, and y_d.

Output: The system is left invertible/not left invertible.

1: Perform the SVD: $U^\top \mathcal{H}_1(x_\mathrm{d}^-) V = \left[\begin{smallmatrix} S & 0 \\ 0 & 0 \end{smallmatrix}\right]$, and let $r = \operatorname{rank} \mathcal{H}_1(x_\mathrm{d}^-)$.

2: Partition the matrix $\mathcal{H}_1(x_\mathrm{d}^+)$ as $U^\top \mathcal{H}_1(x_\mathrm{d}^+) V = \left[\begin{smallmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{smallmatrix}\right]$, where $H_{11} \in \mathbb{R}^{r \times r}$ and partition $\mathcal{H}_1(y_\mathrm{d}^-)$ conformably as $\mathcal{H}_1(y_\mathrm{d}^-) V = \begin{bmatrix} Y_{11} & Y_{12} \end{bmatrix}$.

3: Compute the GEVs of the matrix pair (H_{11}, S).

4: Compute the rank of $H_{\lambda_1} = \left[\begin{smallmatrix} H_{11} - \lambda_1 S & H_{12} \\ H_{21} & H_{22} \\ Y_{11} & Y_{12} \end{smallmatrix}\right]$ for some $\lambda_1 \notin \Lambda(H_{11}, S)$.

5: If $\operatorname{rank} H_{\lambda_1} = n + m$, then the system is left invertible. Otherwise, it is not left invertible.

B. Input/Output Data Case

Assumption 2: $\operatorname{rank} \mathcal{H}_{L_1}(\operatorname{col}(u_\mathrm{d}, y_\mathrm{d})) = n + mL_1$, for $L_1 > n$.

This assumption is conceptually similar to Assumption 1, but gives an identifiability condition based on input/output data [18, Theorem 4]. It is interesting to note that if the system is controllable and the inputs u_d are persistently exciting of order $n + L_1$, then Assumption 2 holds [19, Theorem 2].

Following Theorem 3 iv), we will now provide an algorithm to verify the τ-delay left invertibility of the system based on input/output measurements. The basic idea is to compute the Toeplitz matrix $\mathcal{T}_\tau$ based on input/output data and then use Theorem 3 iv). Note that the entries of $\mathcal{T}_\tau$ are the matrix impulse responses of the system. These matrix impulse responses can be computed iteratively based on input/output measurements, following [20, Algorithm 5]. Note that this algorithm works under the assumptions of Lemma 4. Recently, it has been shown that Lemma 4 holds if and only if Assumption 2 is satisfied [18, Theorem 4]. The preceding discussion can be put together in the following algorithm.

Algorithm 2: τ-delay left invertibility test from input/output data.

Input: Observed data u_d, y_d, and delay τ.

Output: The system is τ-delay left invertible/not τ-delay left invertible.

1: Build the Toeplitz matrix $\mathcal{T}_\tau$ by computing the matrix impulse responses of the system iteratively based on only observed data u_d, y_d by using [20, Algorithm 5].

2: Compute $\operatorname{rank} \mathcal{T}_\tau - \operatorname{rank} \mathcal{T}_{\tau-1} =: \gamma$.

3: If $\gamma = m$, the system is τ-delay left invertible. Otherwise, it is not τ-delay left invertible.

Remark 2: The delay τ is usually unknown in practice. Furthermore, to reduce the computations, it is desirable to have a minimal bound for the delay. The smallest possible delay is thus defined as [5, Page 30]

$$\tau^* := \min\{\tau \geq 0 \mid \operatorname{rank} \mathcal{T}_\tau - \operatorname{rank} \mathcal{T}_{\tau-1} = m\}.$$

Conditions under which τ^* is finite are given in [5, Proposition 6]. Note that τ^* is always less than or equal to the system order n.

Remark 3: Tests to verify the right invertibility of LTI systems (see, for example, [3, Section V]) based on measured data can be developed analogously. This represents novel results as to the best of the authors' knowledge there are no data-driven tests to verify this property, which is required to solve, for example, the output matching problem [10, Section 5], and is typically only assumed.

IV. Data-Driven Input Reconstruction

A. Noise-free data

It can be seen, from system (1), that

$$y|_{[k,k+\tau]} = \mathcal{O}_\tau x(k) + \mathcal{T}_\tau u|_{[k,k+\tau]}. \qquad (5)$$

If system (1) is τ-delay left invertible, there exists a matrix Q such that $Q\mathcal{T}_\tau = \begin{bmatrix} I_m & 0 \end{bmatrix}$ (see Theorem 3 v)). Pre-multiplying both sides of (5) by Q, we have

$$u(k) = -Q\mathcal{O}_\tau x(k) + Qy|_{[k,k+\tau]}. \qquad (6)$$

From (6), it can be seen that to compute the input at timestamp k, we not only need the output at timestamp k but also at timestamps $k+1, k+2, \dots, k+\tau$ together with the state at timestamp k. Here, the state sequence can be obtained iteratively as follows

$$x(k+1) = (A - BQ\mathcal{O}_\tau)x(k) + BQy|_{[k,k+\tau]}. \qquad (7)$$

System (7)-(6) can be seen as the state-space model of the τ-delay left-inverse of system (1). Thus, if we know the system matrices A, B, C, D, initial conditions, and measured outputs, we can reconstruct the control inputs. However, here we are interested in reconstructing the inputs for given outputs based on measured past input/output data and initial conditions. In the following, we formally state the problem of data-driven input reconstruction.

Problem 1: Given an observed input/output trajectory $\text{col}(u_\text{d}, y_\text{d})$, initial condition $\text{col}(u_\text{ini}, y_\text{ini})$ of length T_ini, and output trajectory $y|_{[k,k+L-1+\tau]}$; reconstruct the unique input trajectory $u|_{[k,k+L-1]}$.

We now provide necessary and sufficient conditions under which the above problem has a solution.

Theorem 6: Let Assumption 2 hold for $L_1 = L + T_\text{ini} + \tau$. Then, Problem 1 has a solution if and only if system (1) is τ-delay left invertible.

Proof: Let system (1) be τ-delay left invertible. From Assumption 2, the concatenation $\text{col}(u_\text{ini}, y_\text{ini}) \wedge \text{col}(u, y)|_{[k,k+L-1+\tau]} =: \text{col}(\bar{u}|_{L_1}, \bar{y}|_{L_1})$ is a trajectory of the system if and only if there exists g such that

$$\begin{bmatrix} \mathcal{H}_{L_1}(u_\text{d}) \\ \mathcal{H}_{L_1}(y_\text{d}) \end{bmatrix} g = \begin{bmatrix} \bar{u}|_{[k,k+L-1+\tau]} \\ \bar{y}|_{[k,k+L-1+\tau]} \end{bmatrix}, \qquad (8)$$

with $L_1 = L + T_\text{ini} + \tau$. Define $\text{col}(U_p, U_{f_\tau}) := \mathcal{H}_{L_1}(u_\text{d})$ and $\text{col}(Y_p, Y_{f_\tau}) := \mathcal{H}_{L_1}(y_\text{d})$, (8) becomes

$$\begin{bmatrix} U_p \\ Y_p \\ Y_{f_\tau} \\ U_{f_\tau} \end{bmatrix} g = \begin{bmatrix} u_\text{ini} \\ y_\text{ini} \\ y|_{[k,k+L-1+\tau]} \\ u|_{[k,k+L-1+\tau]} \end{bmatrix} \begin{matrix} mT_\text{ini} \\ pT_\text{ini} \\ p(L+\tau) \\ m(L+\tau) \end{matrix}. \qquad (9)$$

In view of (6), because we can compute inputs of length L for a given output of length $L+\tau$, the following relation can be derived from (9) by deleting the last $m\tau$ rows,

$$\begin{bmatrix} U_p \\ Y_p \\ Y_{f_\tau} \\ U_f \end{bmatrix} g = \begin{bmatrix} u_\text{ini} \\ y_\text{ini} \\ y|_{[k,k+L-1+\tau]} \\ u|_{[k,k+L-1]} \end{bmatrix} \begin{matrix} mT_\text{ini} \\ pT_\text{ini} \\ p(L+\tau) \\ mL \end{matrix}. \qquad (10)$$

We now compute a solution for g from the first three block equations and use that g to compute the input trajectory from the fourth block equation $u|_{[k,k+L-1]} = U_f g$.

Note that any solution g is of the form $g = \mathcal{V}^\dagger \zeta + \ker(\mathcal{V})$, where $\mathcal{V} := \text{col}(U_p, Y_p, Y_{f_\tau})$ and $\zeta := \text{col}(u_\text{ini}, y_\text{ini}, y|_{[k,k+L-1+\tau]})$. Thus, to prove the uniqueness of the input trajectory $u|_{[k,k+L-1]}$, we need to show that $\ker(\mathcal{V}) \subseteq \ker(U_f)$. From the invertibility assumption, we have $u(k) = Qy|_{[k,k+\tau]} - Q\mathcal{O}_\tau x(k)$, where $x(k)$ is uniquely determined by past input/output data, as the system is observable [10, Lemma 1]. Thus, row-sp$(U_f) \subseteq$ row-sp$(\mathcal{V})$. Hence, $\ker(\mathcal{V}) \subseteq \ker(U_f)$.

Conversely, suppose that Problem 1 has a solution. Then, $\ker(\mathcal{V}) \subseteq \ker(U_f)$, which implies row-sp$(U_f) \subseteq$ row-sp$(\mathcal{V})$. Hence, system (1) is τ-delay left-invertible. ■

Remark 4: It is assumed in [13, Condition (7)] that $\ker(\mathcal{V}) \subseteq \ker(U_f)$ to devise the input reconstruction method from outputs (IRO). It can be seen from the above proof that this condition holds under the left invertibility of the system. See also [14, Lemma 3].

Based on Theorem 6, we now provide an algorithm for input reconstruction.

Algorithm 3: Data-driven input reconstruction.

Input: Observed data u_d, y_d, initial conditions u_ini, y_ini, output trajectory $y|_{[k,k+L-1+\tau]}$.

Output: Reconstructed control input trajectory $u|_{[k,k+L-1]}$.

1: Compute a solution for g from

$$\begin{bmatrix} U_p \\ Y_p \\ Y_{f_\tau} \end{bmatrix} g = \begin{bmatrix} u_\text{ini} \\ y_\text{ini} \\ y|_{[k,k+L-1+\tau]} \end{bmatrix}. \qquad (11)$$

2: Compute

$$u|_{[k,k+L-1]} = U_f g.$$

Remark 5: Similar to [20, Algorithm 5], Algorithm 3 can be modified to compute the control input iteratively. Moreover, if the system is both left and right invertible and matrix D has full row rank, then $\tau = 0$ and Algorithm 3 reduces to the output matching algorithm [10, Algorithm 5].

B. Noisy data

In the previous section, we assumed that the measured signals were exact (noise-free). In this section, we relax this assumption and we consider the case where all the output trajectories (data, initial conditions, and future) are subject to additive Gaussian noise, that is,

$$\begin{aligned} &\widetilde{y}_\text{d} = y_\text{d} + \nu_\text{d}, \quad \widetilde{y}_\text{ini} = y_\text{ini} + \nu_\text{ini}, \quad \widetilde{y}_\text{s} = y_\text{s} + \nu_\text{s}, \\ &\nu_\text{d} \sim \mathcal{N}(0, \sigma^2 I), \quad \nu_\text{ini}, \nu_\text{s} \sim \mathcal{N}(0, \sigma_p^2 I), \end{aligned} \qquad (12)$$

where $y_\text{s} := y|_{[k,k+L-1+\tau]}$. In this case, (6) is not satisfied, Theorem 6 does not hold, and Algorithm 3 does not return a unique control input trajectory $u|_{[k,k+L-1]} =: u_\text{s}$.

To address this, we build here on the recently proposed statistical framework for data-driven simulation and control known as signal matrix model (SMM) [15]. In the setting of Section II-B, SMM tackles the issue that with noisy data (3) is no longer exactly satisfied by modeling g as a hyperparameter of the future output trajectory estimation problem. That is, the vector g is chosen such that $Y_f g$ maximizes the conditional probability of observing the output $y|_{[k,k+L-1]}$ given the available data. We extend here SMM to the input reconstruction problem, by formulating a maximum likelihood estimation (MLE) problem whereby

g maximizes the conditional probability that the input $U_f g$ has generated the true output y_s. The framework is modified as follows. Given the data trajectory u_d, $\widetilde{y}_d$, initial conditions u_{ini}, $\widetilde{y}_{ini}$, output trajectory $\widetilde{y}_s$, the maximum likelihood input reconstruction problem can be cast, by using known formulas for Gaussian distribution [15, Eqs. (27)–(30)], as

$$\min_{g\in\mathcal{G}} \log\det(\Sigma_y) + \begin{bmatrix} Y_p g - \widetilde{y}_{ini} \\ Y_{f_\tau} g - \widetilde{y}_s \end{bmatrix}^\top \Sigma_y^{-1} \begin{bmatrix} Y_p g - \widetilde{y}_{ini} \\ Y_{f_\tau} g - \widetilde{y}_s \end{bmatrix}, \quad (13)$$

where $\mathcal{G} = \left\{ g \in \mathbb{R}^{T-L_1+1} \mid U_p g = u_{ini} \right\}$,

$$(\Sigma_y)_{i,j} = \sigma^2 \sum_{k=1}^{T-L_1+1-|i-j|} g_k g_{k+|i-j|} + \begin{cases} \sigma_p^2, & i=j \\ 0, & \text{otherwise} \end{cases}.$$

Note that, differently than in the noise-free case (cf. Algorithm 3), here the identity (11) is relaxed as the constraint $g \in \mathcal{G}$ only involves noise-free variables. Compared to [15, Eq. (30)], the key difference is the presence of $\left[Y_{f_\tau} g - \widetilde{y}_s\right] \neq 0$, which comes from the noisy output. By neglecting the off-diagonal terms in Σ_y (these terms are zero when Page matrices are used in place of Hankel [21]), (13) becomes

$$\min_{g\in\mathcal{G}} L_1 \log\left(\sigma^2\|g\|_2^2 + \sigma_p^2\right) + \frac{1}{\sigma^2\|g\|_2^2+\sigma_p^2}\|Y_p g - \widetilde{y}_{ini}\|_2^2 + \frac{1}{\sigma^2\|g\|_2^2+\sigma_p^2}\|Y_{f_\tau} g - \widetilde{y}_s\|_2^2, \quad (14)$$

whose solution is used to reconstruct the input as $u_s = U_f g$. Alternatively, one can employ a convex, possibly suboptimal, relaxation. An option inspired by [15, Eq. (33)] is the following quadratic program obtained by linearizing (in $\|g\|_2^2$) the first term around a value of g (arbitrary) and fixing the denominator in the last two terms at the same value of g

$$\min_{g\in\mathcal{G}} L_1\sigma^2\|g\|_2^2 + \|Y_p g - \widetilde{y}_{ini}\|_2^2 + \|Y_{f_\tau} g - \widetilde{y}_s\|_2^2. \quad (15)$$

The signal u_s has (modulo the numerical approximations discussed above and similarly employed in [15]) the statistical interpretation of being the input which has the highest (conditional) probability to have generated the response y_s.

V. Numerical Experiments

We consider the following benchmark system [22]

$$G(z) = \frac{0.1159(z^3+0.5z)}{z^4 - 2.2z^3 + 2.42z^2 - 1.87z + 0.7225}. \quad (16)$$

We consider Gaussian distributed random inputs with zero mean and unit variance of length $T = 100$ to excite the system and generate the required data, previously denoted by $\text{col}(u_d, y_d)$. Based on this, we verify, using Algorithm 2, that the system is left invertible with $\tau = 1$. The next step is then to compute the input trajectory for a given output trajectory y_s. As an example, we want to reconstruct the input trajectory of length $L = 30$ for a given output trajectory of length $L+\tau = 31$. We thus generate an output trajectory y_s of length 31 by considering again Gaussian distributed random input trajectory u_s of length 31 with zero mean and unit variance. In the noise-free case, we apply Algorithm 3 and verify that the reconstructed input trajectory exactly matches the true input trajectory, as expected from Theorem 6.

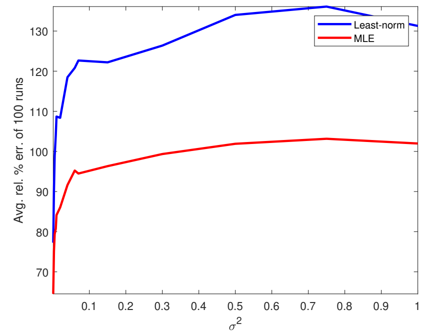

Fig. 1. Average relative percentage errors of LN and MLE.

We then consider, for the same simulation setup, the noisy scenario and we compare the proposed statistical approach (MLE) with the least-norm (LN) solution, which is an empirical way used in the literature to address data-driven simulation problems with noisy data [23]. This approach consists of reconstructing the input as $u_s = U_f g$, where g is obtained here by solving

$$\min_g \|g\|_2^2 \quad \text{subject to (11)}. \quad (17)$$

We consider for the noise model (12) different values of variance $\sigma_p^2 = \sigma^2$. For each noise level, we perform $M = 100$ Monte Carlo runs and for each run, we compute the relative (rel.) error defined as $\frac{\|u_s - \widehat{u}_s\|_2}{\|u_s\|_2}$, where $\widehat{u}_s$ denote the estimated input trajectory. The results are shown through average relative percentage error (avg. rel. % err.) plot in Fig. 1 and the box plots in Fig. 2, showing the error metric for different noise levels. A marked performance improvement is obtained with SMM both in terms of average accuracy and dispersion.

Moreover, the comparison with the input reconstruction obtained with the LN approach is reported in Table I in terms of percentage mean and variance of the error metric and shows again a clear improvement achieved using the maximum likelihood approach. To empirically test the sensitivity

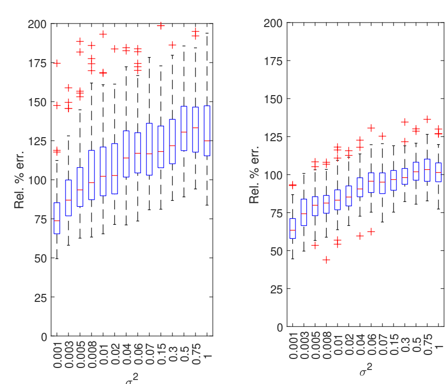

Fig. 2. Box plots (left: LN, right: MLE) for error vectors of 100 Monte Carlo runs.

TABLE I

MEAN (M) AND VARIANCE (V) OF THE INPUT RECONSTRUCTION ERROR

σ^2	LN-m [%]	MLE-m [%]	LN-v	MLE-v
0.005	98.8	79.4	0.0567	0.0112
0.01	108.6	84.1	0.0800	0.0139
0.02	108.3	86.0	0.0476	0.0102
0.04	118.4	91.6	0.0543	0.0105
0.06	120.8	95.2	0.0547	0.0097

of the statistical approach to the right assumption on the noise distribution, we also perform numerical experiments, where the noise contaminating the case is sampled according to a different distribution, e.g., uniform instead of Gaussian (but with the same first and second order statistics). We found that the previous results are only marginally altered.

To further investigate the performance of the MLE-based input reconstruction, we analyze the discretized version of the mass-spring-damper system from the literature [5, Example 3.5.2] using a sampling time of 1 s

$$A = \begin{bmatrix} 0.1250 & 0.0229 & 0.0694 & 0.0954 \\ 0.0229 & 0.1478 & 0.0954 & 0.1647 \\ -0.4336 & -0.2600 & -0.0919 & -0.1071 \\ -0.2600 & -0.6936 & -0.1071 & -0.1990 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.0852 \\ 0.0829 \\ 0.0694 \\ 0.0954 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \text{and} \quad D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The delay $\tau = 1$ is obtained by using Algorithm 2. The results, obtained using the same data-generating setup, are shown through avg. rel. percentage err. plot in Fig. 3. They confirmed the observations made in relation to system (16).

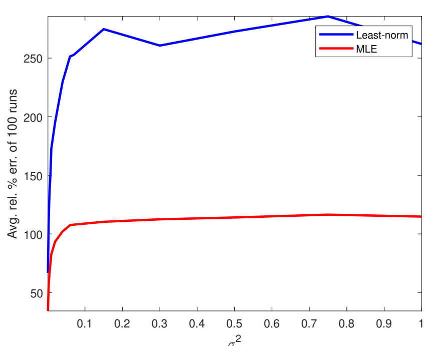

Fig. 3. Average relative percentage errors of LN and MLE.

VI. CONCLUSIONS

The work is concerned with the development of a data-driven approach based on behavioral system theory to infer the invertibility of an LTI system and to reconstruct the inputs for given outputs. Precisely, novel data-based tests are proposed to verify the system invertibility property and, if this holds, exactly reconstruct the input for given noise-free measurements of outputs. The more realistic scenario, where data are contaminated by noise is also tackled using a maximum likelihood estimation approach that consistently shows in simulation better performance compared to a least-norm approach. Interesting future directions include: leveraging these results to design an online fault detection and isolation scheme based on data; determining confidence regions on the reconstructed input in the case of noisy data by leveraging the statistical approach pursued here; and extending these results to nonlinear systems.

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