IEEE Control Systems Letters paper presented at
2023 62nd IEEE Conference on Decision and Control (CDC)
December 13-15, 2023. Marina Bay Sands, Singapore

Influencing Opinions in a Nonlinear Pinning Control Model

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***Abstract*— This letter studies how opinions and subsequent actions of groups of individuals are shaped by opinion leaders, nowadays denoted influencers. We model an influencer as a pinner that exerts a control input on a small subset of individuals, and leverages the interaction network to affect the action of a large fraction of individuals. We provide sufficient conditions so that a given agent takes the same action as the pinner. Based on these conditions, we design a heuristic for the pinned node selection that maximizes the number of nodes taking the action elected by the pinner. The performance of the heuristic is then numerically tested against standard pinning strategies.**

I. Introduction

From understanding how to protect democracy against foreign cyber interference [1] to the design of awareness campaigns enhancing health literacy and trust in vaccinations [2], several pressing societal challenges require a deeper understanding of how opinions can be shaped and manipulated. While opinion dynamics models have shed light on some essential mechanisms for the emergence of consensus, the role of external influences on the opinion shaping process has only been partially unravelled [3], [4].

Control theory has recently attempted to contribute in this area, as the actions of an external entity in a social group can be viewed as a control signal acting on select nodes of the network that describes the interactions within the group [5]–[7]. A glaring analogy has been established with pinning control [8], [9], whereby an opinion leader (or *influencer*) is identified as the pinner and affects the state (opinion) of the other nodes without being affected by them [10]–[12].

The analogy between the action of a pinner in a network system and that of an influencer has been explored in classic opinion dynamics models, based on averaging updating rules that imply that the more divergent two agents' opinions are, the more they tend to get closer [13]–[15]. To overcome this paradox, a discontinuous model, called bounded confidence, considered that interactions only take place when the opinion difference between neighboring nodes is below a given threshold [16], [17]. An alternative model has been recently presented in [18], [19], where the influence an agent has on the opinion of the others is capped by a saturation.

Here, we introduce the pinning control formalism in the modeling framework proposed in [18], [19] to describe a two-options problem. We complement the model with an output function describing the action (choice between the alternative options) associated to the opinion, as typically done in continuous opinion and discrete actions (CODA) models [20]. Then, we add a virtual node, the pinner, which corresponds to (one or more) influencers trying to steer the action of the group towards one of the two options. The pinner will have to decide the pinned nodes, that is, the nodes it will directly attempt to influence. The rest of the network will be instead indirectly affected by the pinner through the interaction topology.

Decomposing the network in layers, we are able to provide sufficient conditions on the interaction topology, the set of pinned nodes, and the pinning control gain guaranteeing that a given subset of agents in the network will take the same action as the pinner. On the basis of our theoretical findings, we design a heuristic control strategy that aims to select the set of pinned nodes maximizing the number of nodes that will agree with the pinner on the action to take. We observe that the proposed heuristic outperforms classic pinning strategies based on centrality metrics [21], [22].

Graph notation. A weighted directed graph is the triplet $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{W}\}$, where $\mathcal{V}$ is the set of nodes, $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$ is the set of edges, and $\mathcal{W}$ is the set weight function that associates to every edge $(i,j) \in \mathcal{E}$ a positive weight w_{ji}. Following the notation used e.g. in [23], the ij-th element of the adjacency matrix A associated to $\mathcal{G}$ is defined as $a_{ij} = w_{ij}$, if $(j,i) \in \mathcal{E}$, whereas it is zero otherwise. Given a node i, its weighted in-degree is the sum of the weights of its incoming edges, that is, $\delta_i^{\text{in}} = \sum_{j:(j,i)\in\mathcal{E}} a_{ij} = \sum_{j=1}^{N} = a_{ij}$. An illustration of the notation used in this manuscript is given in Figure 1.

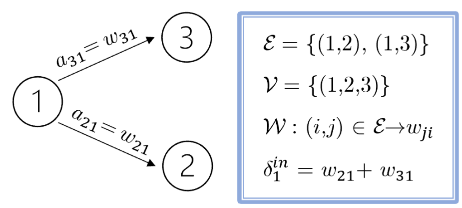

Fig. 1. Illustration of the graph notation used in the manuscript on a sample 3-node graph $\mathcal{G}$.

II. The opinion dynamics model

A. Uncontrolled dynamics and parameter setting

We consider an ensemble of N interconnected agents discussing a topic and assume that the opinion diffusion process occurs on a weighted directed graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{W}\}$, where nodes in $\mathcal{V}$ represent the N individuals, an edge $(i,j) \in \mathcal{E}$ implies that node i influences node j, whereas the function $\mathcal{W}$ associates to each edge a positive scalar modulating the strength of the interaction. Based on their opinion on the topic under discussion, each agent has to take a decision between two alternative actions $\mathcal{A}_{-1}$ and $\mathcal{A}_{+1}$.

Following [24], we identify the evolution over time of the opinion of agent i as its scalar state $x_i(t)$, and model the corresponding action $y_i(t)$ that agent i would take given its opinion at time t as a discrete variable. In the absence of external influences, our model is described by

$$\dot{x}_i(t) = -dx_i(t) + c\tanh\left(\alpha x_i(t) + \sum_{k=1}^{N} a_{ik}x_k(t)\right), \quad \text{(1a)}$$

$$y_i(t) = \operatorname{sgn}(x_i(t)), \quad \text{(1b)}$$

where a_{ij} is the ij-th entry of the adjacency matrix A associated to $\mathcal{G}$ ($a_{ij} \neq 0$ if j influences i), $d > 0$ captures the resistance each agent has to changing opinion, the attention parameter $c \geq 0$ weighs the opinion exchange term, and $\alpha > 0$ modulates how much agent i reinforces its own opinion;

$y_i(t) = -1$ ($y_i(t) = 1$) corresponds to agent i preferring $\mathcal{A}_{-1}(\mathcal{A}_{+1})$ at time t, whereas $y_i(t) = 0$ corresponds to agent i being undecided. We say that agent i has a stronger opinion than j at time t if $|x_i(t)| > |x_j(t)|$. Note that the strength of an opinion is measured by its distance from the undecided state 0, and thus it is possible to compare strengths of opinions corresponding to discordant actions.

In this study, we set the parameters c, d, and α in (1) so that $c > d/\alpha$. This ensures that the single-agent dynamics in the absence of interactions (i.e., when $a_{ik} = 0$ for all k), has an unstable fixed point at 0 and two stable fixed points in $\bar{x}$ and $-\bar{x}$, which are the two solutions of the implicit equation

$$\frac{x}{\tanh(\alpha x)} = \frac{c}{d}, \quad (2)$$

and, for any finite α, have magnitude smaller than c/d [18]. Note that the agent opinion may change also in the absence of interactions, whereby model (1) mimics the opinion formation process, where an agent may modulate the strength of its opinion based on collected information or critical thinking [24]. A reinforcement effect is observed when, as in our study, c is selected to be larger than d/α: the left panel of Figure 2 shows that, when the agents at time 0 would all take the same action $\mathcal{A}_{-1}$ (that is, $x_i(0) < 0$ for all i),they would asymptotically take the same action, but with a stronger opinion.

B. Pinning control to influence opinions

Different from [24], here we consider the case where (one or more) influencers, labeled with the greek letter ι, try to steer the decision towards one of the two options. To capture this scenario, we model the effect of the influencers through a virtual node ι with no incoming edges that, in agreement with the literature on consensus and synchronization in network systems, is called pinner and is unidirectionally coupled to a subset $\mathcal{D}_1 \subseteq \mathcal{V}$ of so-called *pinned nodes* [8]–[10], [25]–[27]. We assume that its opinion, independent of that of the other agents, is already formed (or formed on a much shorter timescale), so that $x_\iota(t) = \bar{x}_\iota$ for all t, with $\bar{x}_\iota$ corresponding to one of the two equilibria $\pm\bar{x}$ of the decoupled single-agent dynamics. The action associated to the constant opinion of the pinner is $\bar{y}_\iota(t) = \operatorname{sgn}(\bar{x}_\iota)$ for all t.

The pinner influences the decision process of the other agents by directly affecting the opinions of the pinned nodes,

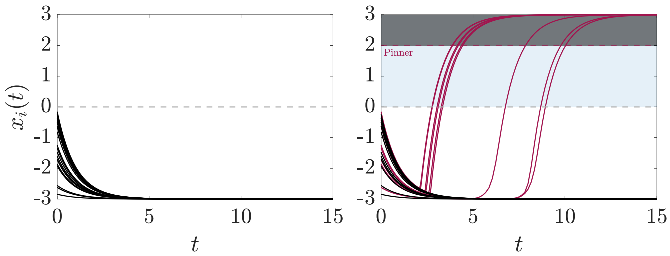

Fig. 2. The left panel displays the opinion dynamics of $N = 30$ agents coupled on an Erdös-Rényi graph with probability $p = 0.6$, and evolving according to (1) with $c = 3, d = \alpha = 1$, and initial conditions uniformly randomly selected in $[-3, 0]$. The right panel depicts the dynamics of the same network in the presence of a pinner (in dashed red) that is connected to 3 agents according to (3) and steers the other agents towards a positive opinion. When the agents enter the blue shaded area, they will always take the same action as the pinner, thus belonging to the set $\mathcal{Q}$ defined in (4). Also, their opinion will become stronger than that of the pinner: as they enter the gray shaded area, that they also belong to the set $\mathcal{Q}_{\rm str}$ defined in (6). In both panels, black and red lines identify the opinion dynamics of agents opting for $\mathcal{A}_{-1}$ and $\mathcal{A}_{+1}$ at the end of the simulation, respectively.

and exploiting the network structure to diffuse its effect on the rest of the network. The presence of the control action from the pinner to the pinned nodes modifies model (1) as

$$\dot{x}_i(t) = -dx_i(t) + c\tanh\big(\alpha x_i(t) + \sum_{k=1}^{N} a_{ik}x_k(t) + h_i\kappa_\iota\bar{x}_\iota\big),$$
$$y_i(t) = \operatorname{sgn}(x_i(t)), \quad (3)$$

for $i = 1, \ldots, N$, where the control gain $\kappa_\iota > 0$ modulates the influence that the pinner has on the dynamics of the pinned nodes; and, $\forall i = 1, \ldots, N,\ h_i = 1$ if i is pinned, i.e. $i \in \mathcal{D}_1$, whereas $h_i = 0$ otherwise. The right panel of Figure 2 shows the effects of a pinner on a group of interconnected agent that at time 0 would take the opposite action of the pinner, with the pinner able to convince a fraction of them to change their opinion and subsequent action.

C. Control objectives

The aim of the control input $h_i\kappa_\iota\bar{x}_\iota$ in (3) is to select the set of pinned nodes $\mathcal{D}_1$ that maximizes the number of individuals that, after a sufficient amount of time, will take the same action as the pinner. To formally define this control design problem, we introduce the set

$$\mathcal{Q} = \{i \in \mathcal{V} : \text{there exists } \bar{t} \text{ such that } y(t) = \bar{y}_\iota \text{ for all } t > \bar{t}\}, \quad (4)$$

whose composition will depend on the choice of κ_ι and $\mathcal{D}_1$. For any κ_ι, we can state the following optimization problem:

$$\max_{\mathcal{D}_1} |\mathcal{Q}(\kappa_\iota, \mathcal{D}_1)| \quad \text{(5a)}$$

subject to

$$|\mathcal{D}_1| = M, \quad \text{(5b)}$$

with $M < N$ as the number of nodes that are directly controlled may be limited by physical or economic constraints.

Depending on the context, one may be interested not only in convincing agents to take the same action as the pinner, but also to make their opinion at least as strong as that of the pinner. In such case, we can define the set $\mathcal{Q}_{\mathrm{str}} \subseteq \mathcal{Q}$ as

$$\begin{aligned}\mathcal{Q}_{\mathrm{str}} = \{i \in \mathcal{V} : & \text{ there exists } \bar{t} \text{ such that} \\ & y(t) = \bar{y}_\iota \text{ and } |x(t)| \geq |x_\iota| \text{ for all } t > \bar{t}\},\end{aligned} \tag{6}$$

and state the following problem:

$$\max_{\mathcal{D}_1} |\mathcal{Q}_{\mathrm{str}}(\kappa_\iota, \mathcal{D}_1)| \tag{7a}$$

subject to

$$|\mathcal{D}_1| = M. \tag{7b}$$

The right panel of Figure 2 reports an instance where, for a given choice of κ_ι and $\mathcal{D}_1$, $\mathcal{Q}_{\mathrm{str}}$ and $\mathcal{Q}$ coincide. If this happens for all possible choices of κ_ι and $\mathcal{D}_1$, then problems (5) and (7) would also coincide.

III. Main results

We first show that the proposed opinion dynamics model is well-posed, as the controlled dynamics (3) are bounded. Then, we provide sufficient conditions so that an agent belongs to Q_{str} (and therefore to $\mathcal{Q}$ as well).

Lemma 1. *Under the dynamics* (3)*, the absolute value of the agents' opinion is asymptotically bounded by c/d, that is, for all $i = 1, \ldots, N$,*

$$\limsup_{t \to +\infty} |x_i(t)| \leq \frac{c}{d}. \tag{8}$$

If, additionally, $|x_i(0)| \leq c/d$, then, for all $t \in \mathbb{R}_{\geq 0}$,

$$|x_i(t)| \leq \frac{c}{d}. \tag{9}$$

Proof. Let us define two auxiliary dynamical systems

$$\dot{\bar{x}}_i = -d\bar{x}_i + c, \qquad \bar{x}_i(0) = x_i(0). \tag{10a}$$

$$\dot{\underline{x}}_i = -d\underline{x}_i - c, \qquad \underline{x}_i(0) = x_i(0). \tag{10b}$$

As $\tanh(\cdot) \in [-1, 1]$, from (3) and the Comparison Theorem for ordinary differential equations [28], we have that

$$\underline{x}_i(t) \leq x_i(t) \leq \bar{x}_i(t), \quad \forall t \geq 0. \tag{11}$$

As $\lim_{t \to +\infty} \bar{x}_i(t) = c/d$ and $\lim_{t \to +\infty} \underline{x}_i(t) = -c/d$, inequality (8) follows. Next, note that $\underline{x}_i(t) = (x_i(0) + c/d)\exp(-dt) - c/d$ and $\bar{x}_i(t) = (x_i(0) - c/d)\exp(-dt) + c/d$. As $|x_i(0)| \leq c/d$, from (11), inequality (9) follows. □

Remark 1. *We consider opinions that are bounded in a set centered at the undecided opinion $x_i = 0$. Considering Lemma 1, from now on we will assume $|x_i(0)| \leq c/d$, so that c/d will represent the maximum strength an opinion can have at any time instant.*

We define $\lambda_i(t) = \sum_{k=1}^{N} a_{ik} x_k(t) + h_i \kappa_\iota \bar{x}_\iota$ as the interaction term in (3), which can be rewritten as

$$\dot{x}_i(t) = -dx_i(t) + c\tanh\left(\alpha x_i(t) + \lambda_i(t)\right), \ |x_i(0)| \leq c/d, \tag{12a}$$

$$y_i(t) = \mathrm{sgn}(x_i(t)). \tag{12b}$$

In what follows, we first provide a condition on the absolute value and sign of $\lambda_i(t)$ so that agent i belongs to $\mathcal{Q}$, that is, in finite time, agent i will take the same action of the pinner, and its opinion will be at least as strong as that of the pinner, so that i also belongs to $\mathcal{Q}_{\mathrm{str}}$. Then, we provide conditions on the control gain κ_ι and on the network topology such that the sufficient condition on $\lambda_i(t)$ is fulfilled.

Let us define $t_{1,i}$ as the first instant such that $y_i(t) \in \{\bar{y}_\iota, 0\}$, with $t_{1,i} = +\infty$ if such an instant does not exist, and

$$\tilde{\lambda} := -\tanh^{-1}\left(\sqrt{1 - d/c\alpha}\right) + \frac{c\alpha}{d}\sqrt{1 - d/c\alpha}. \tag{13}$$

Next, we define the set $\mathcal{T}_i := \{\tau : \forall t > \tau, \ y_i(t) = \bar{y}_\iota \text{ and } |x_i(t)| \geq |\bar{x}_\iota|\}$ and the scalar

$$t_{2,i} := \begin{cases} \min \mathcal{T}_i, & \text{if } \mathcal{T}_i \neq \varnothing, \\ +\infty, & \text{otherwise.} \end{cases} \tag{14}$$

In simple words, $t_{2,i}$, when finite, is the smallest time instant such that node i takes the same action as the pinner with an at least as strong opinion, thereby guaranteeing that $i \in \mathcal{Q}_{\mathrm{str}}$.

Theorem 1. *Under the dynamics described by Eqs.* (12)*, if*

$$\exists \epsilon > 0 : |\lambda_i(t)| \geq |\tilde{\lambda}| + \epsilon, \tag{15}$$

$$\mathrm{sgn}(\lambda_i(t)) = \mathrm{sgn}(\bar{x}_\iota), \tag{16}$$

for all $t \geq 0$, then

$$\exists t_{1,i} < +\infty : y_i(t) = \bar{y}_\iota, \forall t > t_{1,i}, \tag{17a}$$

$$\exists t_{2,i} \in [t_{1,i}, +\infty[\ : |x_i(t)| \geq |\bar{x}_\iota|, \forall t > t_{2,i}. \tag{17b}$$

Proof. For clarity, in the proof we consider $\bar{y}_\iota = 1$, but the derivations hold *ceteris paribus* for $\bar{y}_\iota = -1$.

Existence of a finite $t_{1,i}$: Let us start by showing that if $x_i(0) < 0$ then there exists a time instant $\tilde{t}$ such that $x_i(\tilde{t}) = 0$. Note that, as the hyperbolic tangent is strictly monotone increasing, assumptions (15)-(16) imply that

$$\dot{x}_i(t) \geq f(x_i, \epsilon) := -dx_i + c\tanh(\alpha x_i + \tilde{\lambda} + \epsilon), \tag{18}$$

for all $x_i \in [-c/d, 0]$. Function f has two stationary points, whereby setting $\partial f(x_i, \epsilon)/\partial x_i = 0$, one obtains

$$x^\star_{i,1} = \frac{-c\alpha/d\sqrt{1 - d/c\alpha} - \epsilon}{\alpha},$$

$$x^\star_{i,2} = \frac{2\tanh^{-1}(\sqrt{1 - d/c\alpha}) - c\alpha/d\sqrt{1 - d/c\alpha} - \epsilon}{\alpha}.$$

Evaluating the function at $x^\star_{i,1}$ and $x^\star_{i,2}$, respectively, yields

$$f(x^\star_{i,1}, \epsilon) = \epsilon d/\alpha > 0, \tag{19}$$

$$f(x^\star_{i,2}, \epsilon) = 2\phi(c) + \epsilon d/\alpha > \epsilon d/\alpha > 0, \tag{20}$$

where we used that, for all $c > d/\alpha$, $\phi(c) = (c\sqrt{1 - d/c\alpha} - d/\alpha \tanh^{-1}(\sqrt{1 - d/c\alpha}) > 0$. Noting that

1) f is continuous and differentiable;
2) f is positive at the extrema of the interval $[-c/d, 0]$, whereby $f(0, \epsilon) = c\tanh(\tilde{\lambda} + \epsilon) > 0$, and $f(-\frac{c}{d}, \epsilon) = c + c\tanh(-\alpha\frac{c}{d} + \tilde{\lambda} + \epsilon) > 0$ as $\tanh(\cdot) > -1$;
3) f is positive and lower bounded by $\epsilon d/\alpha$ at both its stationary points;

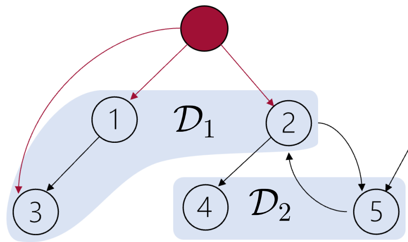

Fig. 3. Decomposition in layers of a sample graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$: the pinner (in red) has 3 outgoing edges that point to the nodes in $\mathcal{D}_1 = \{1, 2, 3\}$, whereas the set $\mathcal{D}_2$ is composed by nodes 4 and 5 that are at two steps away from the pinner. Note that $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$ does not encompass node 6, which is not influenced by the pinner, and therefore in this case $\mathcal{D} \subset \mathcal{V}$.

we obtain $f(x_i, \epsilon) \geq \varepsilon$, for all $x_i \in [-c/d, 0]$, with

$$\varepsilon = \min\{c \tanh(\tilde{\lambda}+\epsilon), c+c\tanh(-\alpha c/d+\tilde{\lambda}+\epsilon), \epsilon d/\alpha\} > 0.$$

Therefore, from (18) we then have

$$\dot{x}_i(t) \geq f(x_i, \epsilon) \geq \varepsilon > 0,\ \forall t : x_i(t) \in [-c/d, 0]. \tag{21}$$

In turn, this implies that $x_i(t) > x_i(0) + t\varepsilon$ for all t such that $x_i(t) < 0$. As $x_i(0) \geq -c/d$, we can then conclude that $\tilde{t} \leq c/d\varepsilon$. Then, from the continuity of $f(\cdot)$ and as $f(0, \epsilon) \geq \varepsilon > 0$ we have $t_{1,i} = \tilde{t}$ (17a). Note that this also proves the existence of $t_{1,i}$ when $x_i(0) = 0$.

Finally, if $x_i(0) > 0$, then $t_{1,i} = 0$ follows from the continuity of $f(\cdot)$. Indeed, as $f(0, \epsilon) \geq \varepsilon > 0$, then there exists a finite $\tilde{x}_i$ such that $0 < \tilde{x}_i \leq x_i(0)$ and $f(\tilde{x}_i, \epsilon) > 0$.

Existence of a finite $t_{2,i}$: Now, let us study the dynamics (12) for $t > t_{1,i}$. As $x_i(t) > 0$ for all $t > t_{1,i}$, to prove (17b), consider that

$$\begin{aligned} \dot{x}_i(t) &> -dx_i + c\tanh(\alpha x_i + \tilde{\lambda}) \\ &> g(x_i) := -dx_i + c\tanh(\alpha x_i). \end{aligned} \tag{22}$$

As $\dot{x}_i = g(x_i)$ is a bistable dynamical system with the stable fixed points at $\pm\bar{x}_\iota$, and starting at $x_i(\tilde{t}) > 0$, the dynamics governed by $g(x_i)$ monotonically converge to $\bar{x}_\iota$, thus from (22) the existence of a finite $t_{2,i}$ follows. □

Next, we leverage the result of Theorem 1 to guarantee that a given agent will belong to $\mathcal{Q}_{\text{str}}$. Namely, we define the extended graph $\widetilde{\mathcal{G}}$ obtained by adding the pinner and its ingoing edges to $\mathcal{G}$. Let $\mathcal{D} \subseteq \mathcal{V}$ be the set that includes all nodes that are destination of a directed path originating from s in $\widetilde{\mathcal{G}}$, and let $q \leq N$ be the maximum length of the shortest path from s to a node in $\mathcal{D}$. We focus on $\mathcal{D}$ as the opinion dynamics of the nodes in $\mathcal{V} \setminus \mathcal{D}$ cannot be affected either directly or indirectly by the pinner.

Next, we relabel the nodes in $\mathcal{V}$ so that the nodes belonging to $\mathcal{D}$ are the first $|\mathcal{D}|$, and partition $\mathcal{D}$ in q disjoint subsets (layers) $\mathcal{D}_1, \ldots, \mathcal{D}_q$, so that $i \in \mathcal{D}_l$ if the shortest path in $\widetilde{\mathcal{G}}$ that connects s to i has length l, for $l = 1, \ldots, q$ (the first layer $\mathcal{D}_1$ coincides with the set of pinned nodes). Finally, we define the set $\mathcal{B}_l := \{j \in \cup_{k=1}^{l} \mathcal{D}_k : j \in \mathcal{Q}_{\text{str}}\} \subseteq \cup_{k=1}^{l} \mathcal{D}_k$ of nodes in the layers $1, \ldots, l$ that in finite time will take the same action as the pinner with an at least as strong opinion.

Now, let us study the behavior of the nodes belonging to $\mathcal{D}_1$, that is, the pinned nodes. Denoting $\delta_i^{\text{in}} = \sum_{k=1}^{N} a_{ik}$ the weighted in-degree of node i, we can give

Corollary 1. *For any $i \in \mathcal{D}_1$, if*

$$\kappa_\iota > \frac{|\tilde{\lambda}| + \epsilon + \frac{c}{d}\delta_i^{\text{in}}}{|\bar{x}_\iota|}, \tag{23}$$

then $i \in \mathcal{B}_1$.

Proof. Let us recall that $\lambda_i(t) = \sum_{k=1}^{N} a_{ik}x_k + \kappa_\iota \bar{x}_\iota$. Then, as $|x_k| \leq c/d$ from Lemma 1 and Remark 1, and as $\sum_{k=1}^{N} a_{ik} = \delta_i^{\text{in}}$ by definition, then from (23) we have that (16) holds. Furthermore, the same arguments imply that

$$|\lambda_i(t)| \geq \kappa_\iota |\bar{x}_\iota| - \sum_{k=1}^{N} a_{ik}|x_k| \geq \kappa_\iota |\bar{x}_\iota| - \frac{c}{d}\delta_i^{\text{in}}.$$

Hence, also (15) holds, and therefore from Theorem 1 the thesis follows. □

We can now study the dynamics of a generic node $i \in \mathcal{D}_l$, $l \geq 2$. Let us define b_i as the sum of the weights of the edges entering node i from every node $j \in \mathcal{B}_{l-1}$, that is, $b_i := \sum_{j \in \mathcal{B}_{l-1}} a_{ij}$, which implies that $0 \leq b_i \leq \delta_i^{\text{in}}$. Then, we can give the following sufficient condition for a node i to belong to the set $\mathcal{B}_q \subseteq \mathcal{Q}_{\text{str}}$ of nodes in $\mathcal{D}$ that take the same action with an at least as strong opinion as the pinner.

Corollary 2. *For any $i \in \mathcal{D}_l$, and $l \geq 2$, if $\mathcal{B}_{l-1} \neq \varnothing$, and there exists $\epsilon > 0$ such that*

$$b_i > \frac{|\tilde{\lambda}| + \epsilon + \frac{c}{d}(\delta_i^{\text{in}} - b_i)}{|\bar{x}_\iota|}, \tag{24}$$

then $i \in \mathcal{B}_q$.

Proof. Let us note that

$$\lambda_i(t) = \sum_{j \in \mathcal{B}_{l-1}} a_{ij}x_j(t) + \sum_{j \notin \mathcal{B}_{l-1}} a_{ij}x_j(t). \tag{25}$$

Moreover, consider that, from Lemma 1, we have

$$\left| \sum_{j \notin \mathcal{B}_{l-1}} a_{ij}x_j(t) \right| \leq \frac{c}{d}(\delta_i^{\text{in}} - b_i) \tag{26}$$

and from Theorem 1 and the definition of $\mathcal{B}_{l-1}$ we have that

$$\left| \sum_{j \in \mathcal{B}_{l-1}} a_{ij}x_j(t) \right| \geq |\bar{x}_\iota| b_i, \tag{27}$$

for all $t \geq t_l := \max_{j \in \mathcal{B}_{l-1}} t_{2,j}$. Hence, combining Eqs. (26)-(27), and from (24), $\lambda_i(\tau)$, with $\tau = t - t_l$, satisfies (15)-(16), and thus the thesis follows from Theorem 1. □

Note that in (24) the lower bound of b_i is given in implicit form to underline the analogy with (23).

Corollaries 1 and 2 hold for any initial opinion in the set $[-c/d, c/d]$. Hence, they allow exploring layer by layer the part of the network whose dynamics is affected by the control signals and determine which agent we can guarantee

TABLE I

COMPARISON OF OUR STRATEGY (HEUR) AGAINST A RANDOM SELECTION, THE MAXIMIZATION OR MINIMIZATION OF THE OUT-DEGREE δ_{out}, IN-DEGREE δ_{in}, OR THE BETWEENNESS CENTRALITY bc FOR TWO ALTERNATIVE CHOICES OF INITIAL CONDITIONS.

Initial conditions	**Metrics**	**Strategy** heur	rand	$\delta_{\mathrm{out}}^{\max}$	$\delta_{\mathrm{out}}^{\min}$	$\delta_{\mathrm{in}}^{\max}$	$\delta_{\mathrm{in}}^{\min}$	$\mathrm{bc}^{\max}$	$\mathrm{bc}^{\min}$
$x_i(0) \sim U(-\frac{c}{d}, \frac{c}{d})$	m_1	0.92	0.65	0.90	0.54	0.90	0.58	0.90	0.57
	m_2	0.92	0.63	0.90	0.53	0.89	0.57	0.90	0.56
	m_3	0.81	0.61	0.80	0.53	0.79	0.57	0.80	0.56
	m_4	0.7	0.5	0.68	0.41	0.68	0.47	0.68	0.45
$x_i(0) = -\frac{c}{d}, \forall i$	m_1	0.30	0.081	0.27	0.070	0.14	0.070	0.22	0.050
	m_2	0.30	0.081	0.27	0.070	0.14	0.070	0.22	0.050
	m_3	0.22	0.061	0.20	0.050	0.11	0.050	0.16	0.050
	m_4	0.22	0.061	0.20	0.050	0.11	0.050	0.16	0.050

will belong to $\mathcal{Q}_{\mathrm{str}}$. In particular, condition (23) of Corollary 1 guarantees that, even in the worst case where all the agents (except for the pinner) influencing agent i have the opposite opinion of the pinner with strength c/d, the control gain κ_ι is strong enough to ensure assumption (14) of Theorem 1 is fulfilled. Condition (24) of Corollary 2 shows that, different from layer 1, in the other layers the control gain has only an indirect influence. Indeed, for a given level l, κ_ι may affect the cardinality $|\mathcal{B}_{l-1}|$ of the set of neighbors of node i that take the same action and have an opinion at least as strong as that of the pinner. The larger $|\mathcal{B}_{l-1}|$, the more nodes in layer l will belong to $\mathcal{Q}_{\mathrm{str}}$, since b_i in (24) will be larger.

IV. PINNING SELECTION STRATEGIES

Corollaries 1 and 2 can be used in an algorithmic fashion to identify a set of nodes $\widetilde{\mathcal{B}}_q$ that we can guarantee will belong to $\mathcal{B}_q$ for any $|x_i(0)| \leq c/d$. More specifically, condition (23) can be used to compute $\widetilde{\mathcal{B}}_1$, and then condition (24) can be iteratively applied to sequentially compute $\widetilde{\mathcal{B}}_2, \ldots, \widetilde{\mathcal{B}}_q$.

Noting that $\mathcal{B}_q \subseteq \mathcal{Q}_{\mathrm{str}}$, we can then use $\tilde{q}(\mathcal{D}_1, \kappa_\iota) := |\widetilde{\mathcal{B}}_q|$, evaluated algorithmically through the two corollaries, as a proxy for the effectiveness of the choice of the set of pinned nodes $\mathcal{D}_1$ with a given cardinality M in solving problem (7), for a given selection of the control gain κ_ι. In what follows, assuming we can freely select κ_ι, we propose a greedy heuristic that solves in polynomial time the NP-hard problem of selecting $\mathcal{D}_1$ with the goal of maximizing $\tilde{q}_\infty(\mathcal{D}_1) := \lim_{\kappa_\iota \to +\infty} \tilde{q}(\mathcal{D}_1, \kappa_\iota)$. We compare the effectiveness of the solution with respect to both Problems (5) and (7) against alternative choices of the pinned nodes based on centrality metrics, similar to what has been done in [21], [22].

A. Heuristic strategy for selecting $\mathcal{D}_1$

Starting from an empty set of pinned nodes, namely $\mathcal{D}_1 = \varnothing$, our greedy strategy sequentially adds nodes so that, at every iteration, $\tilde{q}_\infty$ is maximized given the current cardinality of $\mathcal{D}_1$. The heuristic stops as soon as $|\mathcal{D}_1| = M$. Defining

$$\bar{\kappa}_\iota := \max_{i \in \mathcal{V}} \frac{|\tilde{\lambda}| + \epsilon + \frac{c}{d}\delta_i^{\mathrm{in}}}{|\bar{x}_\iota|},$$

as the control gain ensuring, from Corollary 1, that any pinned node belongs to $\mathcal{Q}_{\mathrm{str}}$ the steps of our algorithm are:

1) initialize $\mathcal{D}_1 = \varnothing$, and set $\kappa_\iota > \bar{\kappa}_\iota$;
2) using Corollaries 1 and 2 compute $\tilde{q}(\mathcal{D}_1 \cup \{i\}, \kappa_\iota)$ for all $i \in \mathcal{V} \setminus \mathcal{D}_1$;
3) randomly select $i^\star$ in the set

$$\arg\max_{i \in \mathcal{V} \setminus \mathcal{D}_1} \tilde{q}(\mathcal{D}_1 \cup \{i\}, \kappa_\iota), \tag{28}$$

and update $\mathcal{D}_1 = \mathcal{D}_1 \cup \{i^\star\}$;
4) if $|\mathcal{D}_1| < M$, go to step 2, otherwise stop the algorithm.

B. Performance in a sample retweet network from Twitter

We compare the proposed heuristic both against chance, that is, a random selection of the set $\mathcal{D}_1$, and alternative topological strategies, which consist in encompassing in $\mathcal{D}_1$ the nodes with maximum (or minimum) outdegree, indegree, and betweenness centrality [29]. To do so, we extracted a directed unweighted subgraph of 580 nodes of a retweet graph from [30]. Then, we set the number of pinned nodes to $M = 0.05N$, and evaluated the following metrics to assess the performance of the proposed heuristic:

- $m_1 = |\mathcal{Q} \cap \mathcal{D}|/|\mathcal{D}|$ and $m_2 = |\mathcal{Q}|/|\mathcal{V}|$, that is, the fraction of nodes in $\mathcal{D}$ and in $\mathcal{V}$, respectively, that take the same action of the pinner;
- $m_3 = |\mathcal{Q}_{\mathrm{str}} \cap \mathcal{D}|/|\mathcal{D}|$ and $m_4 = |\mathcal{Q}_{\mathrm{str}}|/|\mathcal{V}|$, that is, the fraction of nodes in $\mathcal{D}$ and $\mathcal{V}$, respectively, that take the same action of the pinner and have an opinion that is at least as strong as that of the pinner;

Note that m_1 and m_3 focus on the nodes that, given the selection $\mathcal{D}_1$, are directly affected by the pinner, whereas m_2 and m_4 evaluate the effectiveness of $\mathcal{D}_1$ for all the agents in $\mathcal{V}$. We evaluated these metrics for initial opinions

1) drawn from a uniform distribution in $[-c/d, c/d]$,
2) furthest from that of the pinner, i.e. $x_i(0) = -c/d$ as we set $\bar{y}_\iota = 1$.

For case 1), the results are averaged over 1000 random selections of the initial conditions. Table I shows that the proposed heuristic outperforms the alternative strategies. Also, metrics $m_{1,2}$ and $m_{3,4}$ are equivalent when all agents start with opinions that are furthest from the pinner, and the ranking of the strategies does not change depending on the metric. Moreover, we observe that the maximization of the out-degree is the topological strategy that more closely matches the performance of the proposed heuristic.

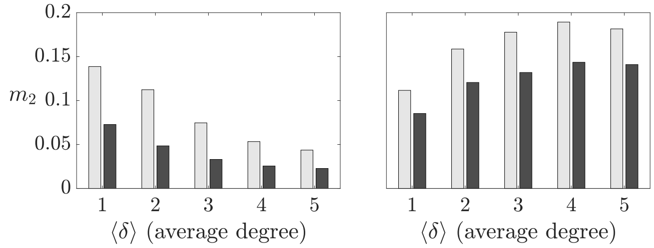

Fig. 4. Comparing the proposed heuristic against pinning the nodes with maximum out-degree in terms of the fraction m_2 of nodes taking the same action of the pinner. The left and right panel refer to ER and SF networks, respectively. The probability of ER graphs is equal to $\langle\delta\rangle/N$, the exponent of the power law of SF network is equal to 2.6. Each data point is averaged over 100 realizations of the graph topology.

C. Performance in synthetic networks

As pinning the nodes with maximum out-degree leads to performance close to the proposed heuristic strategy, we performed a comprehensive numerical analysis on Erdös-Rényi (ER) and Scale Free (SF) graphs, generated by means of the configuration model, to assess whether the proposed heuristic yielded a significant improvement. For both ER and SF topologies, and for each value of the average degree, varied between 1 and 5 with step 1, we generated 100 graphs of $N = 500$ nodes, and we computed the average values of $m_1, \ldots, m_4$ setting $x_i(0) = -c/d$ for all i. As shown in Figure 4, in all synthetic networks the proposed heuristic outperforms pinning the nodes with maximum out-degree, and a t-test confirms that the difference is significant, with a p-value smaller than 0.001.

V. Conclusions

In this letter, we used pinning control to maximize the effect of an influencer in social groups interacting according to a nonlinear opinion dynamics model. We derived sufficient conditions on the topology and control gain so that individuals take the same action and have an opinion at least as strong as that of the pinner. We leveraged our results to design a heuristic to select the nodes where inputs should be injected so to maximize the influence of the pinner, given a constraint on the number of nodes to pin. Future work should be devoted to test our results on real world data borrowed from online social media, and investigating the more realistic cases where the agents' parameters are heterogeneous or there is more than one pinner competing for influencing the other agents.

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