2023 62nd IEEE Conference on Decision and Control (CDC)
December 13-15, 2023. Marina Bay Sands, Singapore

H_∞ filter based functional observers for descriptor systems

Pabitra Kumar Tunga and Nutan Kumar Tomar, *Member, IEEE*

***Abstract*— This paper considers the H_∞ observers design problem for linear time-invariant descriptor systems. A sufficient condition is established for functional observers of order equal to the dimension of the vector to be estimated. This sufficient condition is milder than the other existing conditions in the literature. Furthermore, the observers are of the state space form, and the parameter matrices' existence is proved via elementary matrix theory. It is shown that the observer parameter matrices exist if a matrix equation is solvable. The solution of this matrix equation is not unique, and this non-uniqueness is utilized to meet other specifications of the observer via the solution theory of linear matrix inequalities (LMIs). The theoretical findings are illustrated by designing a functional observer for an electric circuit.**

I. INTRODUCTION

We study linear descriptor systems (also known as singular systems, generalized state space systems, or systems described by differential-algebraic equations (DAEs)) of the form:

$$E\dot{x}(t) = Ax(t) + Bu(t) + Fv(t) \tag{1a}$$

$$y(t) = Cx(t) + Gv(t) \tag{1b}$$

$$z(t) = Kx(t) + Hv(t) \tag{1c}$$

where E, $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times k}$, $F \in \mathbb{R}^{m\times q}$, $C \in \mathbb{R}^{p\times n}$, $G \in \mathbb{R}^{p\times q}$, $K \in \mathbb{R}^{r\times n}$, and $H \in \mathbb{R}^{r\times q}$ are known constant matrices. The vectors $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^k$, $v(t) \in \mathbb{R}^q$, $y(t) \in \mathbb{R}^p$, $z(t) \in \mathbb{R}^r$ are the semistate vector, the control input vector, the disturbance vector, the output vector, and the functional vector, respectively. The first order polynomial $\lambda E - A$ is called regular matrix pencil if $m = n$ and $\det(\lambda E - A)$ is not identically zero polynomial. This paper makes no special assumptions on the matrix pencil $\lambda E - A$. Instead, we assume the system designer has already defined the system matrices in such a way that the set of solutions to (1) is nonempty, cf. Definition 1.

Descriptor systems occur naturally when dynamical systems are subject to algebraic constraints. These systems have found a wide range of applications in various fields, including electrical circuits, mechanical systems, and chemical engineering; see e.g. [1]–[5] and the references therein. Due to the algebraic constraints on system dynamics, $x(t)$ does not satisfy the semigroup property of standard state space theory and cannot be initialized with an arbitrary initial condition; therefore, we call $x(t)$ the semistate vector instead of the state vector. The functional vector $z(t) \in \mathbb{R}^r$ contains unmeasured (output) variables, and therefore observers are required to estimate them. An observer that provides an estimate of z without observing the whole semistate vector x is said to be a functional (or partial state) observer. Moreover, any functional observer reduces to a full-state observer if K is the identity matrix of order n. Since the seminal work of numerous researchers in the 1980s [2], [6], [7], the full-state observers design problem for linear descriptor systems has been extensively studied; for a current and comprehensive discussion on the existence conditions for full-order observers, we refer the readers to Jaiswal *et al.* [8].

Notably, the full-state observers may estimate even those states that are either directly measurable or are of no use. To this end, functional observers can have significantly lower order because such observers eliminate the redundancy in full-state observers. Moreover, functional observers can be designed under considerably weaker assumptions than those which are necessary for full-order observers. For these reasons, functional observer design is an active area of research, even in the case of standard linear state space systems, see e.g. [9]. Moreover, as far as the descriptor systems are concerned, considerable attention has also been paid to the design of functional observers; see e.g. [10]–[13] and the references therein. Moreover, functional observers have also been designed for descriptor systems with unknown inputs [14]–[17]. Such unknown input observers essentially decouple the estimation error dynamics from the unknown inputs in the original system.

On the other hand, the filtering problem is also concerned with determining the internal (semistate) variables based on noisy input and output measurement data. However, filtering-based observers consider the effect of noise on the estimation error dynamics. Like the state space systems, one of the ways to deal with filtering problems for descriptor systems is the celebrated Kalman filtering approach [18], [19]. These filters are optimal in the sense that the covariance of the estimation error is minimized. It is important to note that Kalman filters are based on the assumption that the noises are with known statistics. When the noises are arbitrary but bounded in second norm, H_∞ filtering technique is well-known in state space systems theory since its inception in [20]. The H_∞ filtering problem for regular descriptor systems was initially investigated by Xu et al. in 2003 [21]. A reduced order H_∞ filter for regular descriptor systems is studied under the assumption that the system is impulse-free

This work is supported by the Science and Engineering Research Board, New Delhi, via Grant no. MTR/2019/000494.

Pabitra Kumar Tunga thanks to Council of Scientific and Industrial Research, New Delhi, India, for the award of SRF through grant number 09/1023(0032)/2019-EMR-I.

The authors are with the Department of Mathematics, Indian Institute of Technology Patna, India (e-mail: pabitra_1921ma02@iitp.ac.in; nktomar@iitp.ac.in).

Corresponding author: Nutan Kumar Tomar.

[22]. Darouach has proved the existence of an H_∞ filter for rectangular descriptor systems of type (1) if the system is impulsive observable [23]. Osorio-Gordillo et al. [24] have appled the idea of H_∞ filters to estimate a functional vector of semistate and unknown input variables under the same conditions as in [23].

The purpose of the present paper is to provide milder sufficient conditions for the existence of H_∞ filter based functional observers than in the above mentioned works. We do not assume that the system is regular or even square. A general solution theory, based on behavioral approach, is adopted for descriptor systems and a rigorous definition is introduced for H_∞ filter based functional observers. We design observers of the form:

$$\dot{w}(t) = Nw(t) + \begin{bmatrix} H & L \end{bmatrix} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}, \tag{2a}$$

$$\hat{z}(t) = w(t) + \begin{bmatrix} M_1 & M_2 \end{bmatrix} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}, \tag{2b}$$

where $w(t) \in \mathbb{R}^r$ and $N,\ H,\ L,\ M_1,\ M_2$ are parameter matrices with appropriate dimensions. Notably, the observers (2) are of state space form where the dynamics is governed only by ordinary differential equations (ODEs). Such observers are easy to implement by using standard ODE solvers, e.g. in MATLAB.

We use the following notation: 0 and I stand for appropriate dimensional zero and identity matrices, respectively. In a block partitioned matrix, all missing blocks are zero matrices of appropriate dimensions. Sometimes, for more clarity, the identity matrix of size $n \times n$ is denoted by I_n. The set of complex numbers is denoted by $\mathbb{C}$ and $\bar{\mathbb{C}}^+ := \{s \in \mathbb{C} \mid \mathrm{Re}\, s \geq 0\}$. The symbols $A^\top$, A^+, $\ker A$, and $\mathrm{Row}(A)$ denote the transpose, Moore-Penrose inverse (MP-inverse), kernel, and row space of a matrix A, respectively. For any square matrix A, we write $A > 0$ ($A < 0$) if, and only if, A is positive (negative) definite. A block diagonal matrix having diagonal elements $A_1, \ldots, A_k$ is represented by $\text{blk-diag}\{A_1, \ldots, A_k\}$. $\mathscr{L}^1_{loc}$ and $\mathcal{AC}_{loc}$ represent the set of locally Lebesgue integrable functions and the set of locally absolute continuous functions, respectively.

II. Preliminaries

We adopt the following behavioral approach to define solutions of (1):

Definition 1: The tuple (x, u, v, y, z) is said to be a solution of (1), if it belongs to the set

$$\mathscr{B} := \{(x,u,v,y,z) \in \mathscr{L}^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^{n+k+q+p+r}) \mid Ex \in \mathcal{AC}_{loc}(\mathbb{R};\mathbb{R}^m) \text{ and } (x,u,v,y,z) \text{ satisfies (1) for almost all } t \in \mathbb{R}\}.$$

The set $\mathscr{B}$ is called *behavior* in [25]. Moreover, this behavior set $\mathscr{B}$ has been used to define various observability concepts for (1) in [26] and for proving existence conditions for functional observers in [10]–[17]. We now exploit $\mathscr{B}$ to define H_∞ functional observer for (1)

Definition 2: System (2) is said to be a H_∞ functional observer for (1), if for every $(x,u,v,y,z) \in \mathscr{B}$ there exists $w \in \mathcal{AC}_{loc}(\mathbb{R};\mathbb{R}^q)$ and $\hat{z} \in \mathcal{L}^1_{\text{loc}}(\mathbb{R};\mathbb{R}^r)$ such that $(w,u,y,\hat{z})$ satisfy (2) for almost all $t \in \mathbb{R}$, and for all such $w, \hat{z}$ the following properties hold:

(a) If $v, z, \hat{z} \in \mathcal{L}^2_{\text{loc}}(\mathbb{R};\mathbb{R}^{q+2r})$ and $e = \hat{z} - z$, then

$$\sup_{v \neq 0} \frac{||e||_2}{||v||_2} < \gamma,$$

where γ is a given positive scalar.

(b) If $v \overset{\text{a.e.}}{=} 0$ and $e = \hat{z} - z$, then
   (i) $e(t) \to 0$ for $t \to \infty$,
   (ii) if $e(0) = 0$, then $e \overset{\text{a.e.}}{=} 0$.

We now present some fundamental results which play an important role in further discussion. The following basic result can be found in any standard textbook on linear algebra.

Lemma 1: System $XA = B$ has solution for X if and only if $\mathrm{rank} \begin{bmatrix} A \\ B \end{bmatrix} = \mathrm{rank}\, A$. Moreover,

$$X = BA^+ - Z(I - AA^+),$$

where Z is an arbitrary matrix with an appropriate dimension.

The following two results for LMIs are extracted from [27].

Lemma 2: Suppose that Q, M, and R are matrices and that M and Q are symmetric. Then the following are equivalent

1) The matrix inequalities $Q > 0$ and $M - RQ^{-1}R^\top > 0$ hold.
2) The matrix inequality
$$\begin{bmatrix} M & R \\ R^\top & Q \end{bmatrix} > 0$$
is satisfied.

Lemma 3: Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times k}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times k}$ be given, and the matrix $P = P^\top \in \mathbb{R}^{n\times n}$ be the variable. Then the LMI

$$P > 0, \quad \begin{bmatrix} A^\top P + PA + C^\top C & PB + C^\top D \\ B^\top P + D^\top C & D^\top D - \gamma^2 I \end{bmatrix} \leq 0 \tag{3}$$

is feasible if and only if the linear system

$$\dot{x}(t) = Ax(t) + Bu$$
$$y(t) = Cx(t) + Du(t)$$

is nonexpensive, i.e.,

$$\int_0^T y(t)^\top y(t) dt \leq \gamma^2 \int_0^T u(t)^\top u(t) dt. \tag{4}$$

We conclude this section by providing the following decomposition for the coefficient matrices of any given system (1).

Lemma 4: Let $E \in \mathbb{R}^{m\times n}$, $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times k}$, and $F \in \mathbb{R}^{m\times q}$. Then there exist two orthogonal matrices $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ such that

$$UEV = \begin{bmatrix} E_{11} & E_{12} \\ 0 & E_{22} \\ 0 & 0 \end{bmatrix}, \ UAV = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ 0 & A_{32} \end{bmatrix}, \tag{5a}$$

$$UB = \begin{bmatrix} B_{11} \\ B_{21} \\ 0 \end{bmatrix}, \text{ and } UF = \begin{bmatrix} F_{11} \\ F_{21} \\ 0 \end{bmatrix}, \tag{5b}$$

where

1) E_{11} has full row rank,
2) $\begin{bmatrix} E_{11} & E_{12} & B_{11} & F_{11} \\ 0 & E_{22} & B_{21} & F_{21} \end{bmatrix}$ has full row rank,
3) A_{32} has full column rank.

The proof of the above lemma is given in [17] (see Lemma 6 in [17]). However, for the sake of completeness, a complete method to obtain the matrices U and V is summarized in Algorithm 1 below.

Algorithm 1 Computational steps for U and V of Lemma 4

---

1) Compute $U_1 \in \mathbb{R}^{m\times m}$ such that

$$U_1\begin{bmatrix} E & B & F \end{bmatrix} = \begin{matrix} & n & k & q & \\ & \left[\begin{matrix} E_1 \\ 0 \end{matrix}\right. & \begin{matrix} B_1 \\ 0 \end{matrix} & \left.\begin{matrix} F_1 \\ 0 \end{matrix}\right] & \begin{matrix} r_1 \\ r_2 \end{matrix} \end{matrix}$$

where $\operatorname{rank}\begin{bmatrix} E & B & F \end{bmatrix} = r_1$ and $r_1 + r_2 = m$.

2) Denote $U_1 A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \begin{matrix} r_1 \\ r_2 \end{matrix}$ and compute an orthogonal matrix $V_1 \in \mathbb{R}^{n\times n}$ such that

$$A_2 V_1 = \begin{matrix} c_1 & c_2 & \\ \left[0 \right. & \left. A_{32} \right] & r_2 \end{matrix},$$

where $\operatorname{rank} A_{32} = c_2$ and $c_1 + c_2 = n$.

3) Denote $E_1 V_1 = \begin{matrix} c_1 & c_2 & \\ \left[\tilde{E}_1 \right. & \left. \bar{E}_1 \right] & r_1 \end{matrix}$, and compute an orthogonal matrix $U_2 \in \mathbb{R}^{r_1 \times r_1}$ such that

$$U_2 \tilde{E}_1 = \begin{bmatrix} E_{11} \\ 0 \end{bmatrix},$$

where E_{11} has full row rank.

4)
$$U = \begin{bmatrix} U_2 & 0 \\ 0 & I_{m-r_1} \end{bmatrix} U_1 \text{ and } V = V_1.$$

III. H_∞ OBSERVER DESIGN

We assume the following rank condition on (1)

$$\operatorname{rank}\Gamma = \operatorname{rank}\Psi, \tag{6}$$

where

$$\Gamma = \begin{bmatrix} E & A & B & 0 & 0 & F \\ 0 & E & 0 & A & 0 & 0 \\ 0 & 0 & 0 & E & A & 0 \\ 0 & 0 & 0 & C & 0 & 0 \\ 0 & 0 & 0 & 0 & C & 0 \\ 0 & 0 & 0 & 0 & K & 0 \end{bmatrix}$$

and

$$\Psi = \begin{bmatrix} E & A & B & 0 & 0 & F \\ 0 & E & 0 & A & 0 & 0 \\ 0 & 0 & 0 & E & A & 0 \\ 0 & 0 & 0 & C & 0 & 0 \\ 0 & 0 & 0 & 0 & C & 0 \\ 0 & 0 & 0 & 0 & K & 0 \\ 0 & 0 & 0 & K & 0 & 0 \end{bmatrix}.$$

System transformation and Observer system: Assume that the given system (1) satisfies (6). First, transform system (1) into a new coordinate system by using the matrices U and V in Lemma 4. In view of the decomposition (5), system (1) can be written as

$$E_{11}\dot{x}_1 + E_{12}\dot{x}_2 = A_{11}x_1 + A_{12}x_2 + B_{11}u + F_{11}v, \tag{7a}$$
$$E_{22}\dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_{21}u + F_{21}v, \tag{7b}$$
$$0 = A_{32}x_2, \tag{7c}$$
$$y = C_1x_1 + C_2x_2 + Gv, \tag{7d}$$
$$z = K_{11}x_1 + K_{12}x_2 + Hv, \tag{7e}$$

where $x = V\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $CV = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$, $KV = \begin{bmatrix} K_{11} & K_{12} \end{bmatrix}$, and the number of columns in C_1 and K_{11} are the same as in E_{11}. The fact A_{32} has full column rank implies $x_2 = 0$ and hence system (7) reduces to

$$E_{11}\dot{x}_1 = A_{11}x_1 + B_{11}u + F_{11}v, \tag{8a}$$
$$\bar{y} = \bar{C}_1x_1 + G_1v, \tag{8b}$$
$$z = K_{11}x_1 + Hv, \tag{8c}$$

where $\bar{y} = \begin{bmatrix} -B_{21}u \\ y \end{bmatrix}$, $\bar{C}_1 = \begin{bmatrix} A_{21} \\ C_1 \end{bmatrix}$, and $G_1 = \begin{bmatrix} F_{21} \\ G \end{bmatrix}$.

We consider the following observer system:

$$\dot{w}(t) = Nw(t) + TB_{11}u(t) + L\bar{y}(t), \tag{9a}$$
$$\hat{z}(t) = w(t) + M\bar{y}(t), \tag{9b}$$

where $w(t) \in \mathbb{R}^r$ is the state vector of observer and $\hat{z}(t)$ represents the estimate of $z(t)$.

Remark 1: In (9), N, T, L and M are parameter matrices for the observer, and we aim to design these matrices so that system (9) become an H_∞ functional observer for system (8), cf. Definition 2. Notably, the functional vector remains the same in transforming system (1) to system (8). Therefore, any functional observer for (8) is also one for system (1).

Before determining the observer parameter matrices, we transform the assumptions (6) from system (1) to system (8). Define non-singular matrices:

$$\tilde{U} = \text{blk-diag}(U, U, U, I_p, I_p, I_r) \text{ and}$$
$$\tilde{V} = \text{blk-diag}(V, V, I_k, V, V, I_q).$$

Since the rank of a matrix does not change by pre- and post-multiplication of invertible matrices, we obtain

$$\operatorname{rank}\Gamma = \operatorname{rank}\tilde{U}\Gamma\tilde{V} \tag{10}$$

and

$$\operatorname{rank}\Psi = \operatorname{rank}\begin{bmatrix} \tilde{U} & \\ & I_r \end{bmatrix}\Psi\tilde{V}. \tag{11}$$

Now, by writing Γ and Ψ in terms of the system coefficients E, A, B, F, and C on the right-hand side of (10) and (11), respectively, and then following the steps similar to the mathematical operations applied in [17] (see Eqns. (17)-(20) in [17]), it is straightforward that

$$\operatorname{rank}\Gamma = \operatorname{rank}\begin{bmatrix} E_{11} & E_{12} & B_{11} & F_{11} \\ 0 & E_{22} & B_{21} & F_{21} \end{bmatrix} + 3\operatorname{rank}A_{32} + \operatorname{rank}E_{11} + \operatorname{rank}\Gamma_1, \tag{12}$$

and

$$\operatorname{rank}\Psi = \operatorname{rank}\begin{bmatrix} E_{11} & E_{12} & B_{11} & F_{11} \\ 0 & E_{22} & B_{21} & F_{21} \end{bmatrix} + 3\operatorname{rank} A_{32} + \operatorname{rank} E_{11} + \operatorname{rank}\Psi_1, \tag{13}$$

where $\Gamma_1 = \begin{bmatrix} E_{11} & A_{11} \\ \bar{C}_1 & 0 \\ 0 & \bar{C}_1 \\ 0 & K_{11} \end{bmatrix}$ and $\Psi_1 = \begin{bmatrix} E_{11} & A_{11} \\ \bar{C}_1 & 0 \\ 0 & \bar{C}_1 \\ 0 & K_{11} \\ K_{11} & 0 \end{bmatrix}$.

Thus, from (12) and (13), it follows that system (1) satisfies (6) if and only if the system (8) satisfies the condition

$$\operatorname{rank}\Gamma_1 = \operatorname{rank}\Psi_1. \tag{14}$$

Observer error dynamics and parameter matrices: Let $e = \hat{z} - z$ be the error between the actual and estimated functional vectors. Define $e_1 = w - TE_{11}x_1$. Then from (8) and (9), we obtain

$$\begin{aligned} e &= (w + M\bar{y}) - (K_{11}x_1 + Hv) \\ &= e_1 + (TE_{11} + M\bar{C}_1 - K_{11})x_1 + (MG_1 - H)v \end{aligned}$$

and

$$\begin{aligned} \dot{e}_1 &= \dot{w} - TE_{11}\dot{x}_1 \\ &= Ne_1 + (NTE_{11} + L\bar{C}_1 - TA_{11})x_1 + (LG_1 - TF_{11})v. \end{aligned}$$

Thus the error vector is governed by the equations

$$\dot{e}_1 = Ne_1 + (LG_1 - TF_{11})v \tag{15a}$$

$$e = e_1 + (MG_1 - H)v, \tag{15b}$$

if and only if the observer parameter matrices N, T, M, and L satisfy the matrix equations:

$$TE_{11} + M\bar{C}_1 = K_{11}, \tag{16a}$$

$$NTE_{11} + L\bar{C}_1 - TA_{11} = 0. \tag{16b}$$

Notably, Eq. (16b) is nonlinear in the unknown matrices. Therefore, by substituting (16a) into (16b), we reduce (16b) into a linear equation

$$TA_{11} + P\bar{C}_1 - NK_{11} = 0, \tag{17}$$

where

$$P = NM - L. \tag{18}$$

Clearly, Eqs. (16a) and (17) can be rewritten in matrix form

$$\begin{bmatrix} T & M & P & N \end{bmatrix}\Sigma = \Theta, \tag{19}$$

where $\Sigma = \begin{bmatrix} E_{11} & A_{11} \\ \bar{C}_1 & 0 \\ 0 & \bar{C}_1 \\ 0 & -K_{11} \end{bmatrix}$ and $\Theta = \begin{bmatrix} K_{11} & 0 \end{bmatrix}$.

Now, Lemma 1 reveals that Eq. (19) is solvable for the unknowns N, T, M, and L if and only if (14) holds. Moreover,

$$\begin{bmatrix} T & M & P & N \end{bmatrix} = \Theta\Sigma^{+} - Z(I - \Sigma\Sigma^{+}), \tag{20}$$

where Z is an arbitrary matrix of appropriate dimension. Therefore,

$$T = T_1 - ZT_2, \tag{21a}$$

$$M = M_1 - ZM_2, \tag{21b}$$

$$P = P_1 - ZP_2, \tag{21c}$$

$$N = N_1 - ZN_2, \tag{21d}$$

where

$$T_1 = \Theta\Sigma^{+}\begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix}, T_2 = (I - \Sigma\Sigma^{+})\begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix}, M_1 = \Theta\Sigma^{+}\begin{bmatrix} 0 \\ I \\ 0 \\ 0 \end{bmatrix},$$

$$M_2 = (I - \Sigma\Sigma^{+})\begin{bmatrix} 0 \\ I \\ 0 \\ 0 \end{bmatrix}, P_1 = \Theta\Sigma^{+}\begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix},\ N_1 = \Theta\Sigma^{+}\begin{bmatrix} 0 \\ 0 \\ 0 \\ I \end{bmatrix},$$

$$P_2 = (I - \Sigma\Sigma^{+})\begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix}, \text{ and } N_2 = (I - \Sigma\Sigma^{+})\begin{bmatrix} 0 \\ 0 \\ 0 \\ I \end{bmatrix}.$$

Finally, our remaining task is to find Z in such a way that the system (9) with the above parameter matrices satisfy all conditions for H_∞ as defined in Definition 2.

Theorem 1: Consider a system (1) which satisfies the rank condition (6). Then (9) is an H_∞ functional observer with system parameter matrices (21) and error dynamics (15), if there exist matrices $Y > 0$ and Y_1 such that

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{12}^{\top} & R_{22} \end{bmatrix} < 0, \tag{22}$$

where

$$R_{11} = (YN_1 + N_1^{\top}Y) - (Y_1\mathcal{N}_2 + \mathcal{N}_2^{\top}Y_1^{\top}) + I, \tag{23a}$$

$$R_{12} = YL_1 - Y_1\mathcal{L}_2 + \bar{H}, \tag{23b}$$

$$R_{22} = \bar{H}^{\top}\bar{H} - \gamma^2 I, \tag{23c}$$

$$Y_1 = YZ, \tag{23d}$$

$$\mathcal{N}_2 = (I - \bar{M}_2\bar{M}_2^{+})N_2, \tag{23e}$$

$$\bar{M}_2 = M_2G_1, \tag{23f}$$

$$L_1 = (N_1M_1 - P_1)G_1 - T_1F_{11}, \tag{23g}$$

$$\mathcal{L}_2 = (I - \bar{M}_2\bar{M}_2^{+})\{(N_2M_1 - P_2)G_1 - T_2F_{11}, \tag{23h}$$

$$\bar{H} = M_1G_1 - H. \tag{23i}$$

Proof: In light of Remark (1), it is sufficient to show that (9) is an H_∞ functional observer for (8). Let $v, z, \hat{z} \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^{q+2r})$ and $e = \hat{z} - z$. Then we first show that (9) with the coefficient matrices (21) satisfies property (a) in Definition 2. Define $\bar{M}_2 := M_2G_1$ and $Z := Z_1(I - \bar{M}_2\bar{M}_2^{+})$, where Z_1 is a matrix of appropriate dimension, M_2 and G_1 are the same as in (21b) and (8b), respectively. Then (21d) reduces to

$$N = N_1 - Z_1\mathcal{N}_2, \tag{24}$$

where $\mathcal{N}_2 = (I - \bar{M}_2\bar{M}_2^{+})N_2$. (cf. (23e))
Now, assume that $L_1 = (N_1M_1 - P_1)G_1 - T_1F_{11}$, $\mathcal{L}_2 = (I -$

$\bar{M}_2\bar{M}_2^+)\}\{(N_2M_1 - P_2)G_1 - T_2F_{11}$, and $\bar{H} = M_1G_1 - H$, then it is straightforward to calculate that

$$\begin{aligned} &LG_1 - TF_{11} \\ &= (N_1 - ZN_2)MG_1 - (P_1 - ZP_2)G_1TF_{11}, \\ &= (N_1M - P_1)G_1 - Z(N_2M - P_2)G_1 - TF_{11}, \\ &= (N_1M_1 - P_1)G_1 - Z(N_2M_1 - P_2)G_1 - TF_{11}, \\ &= L_1 - Z_1\mathcal{L}_2, \end{aligned} \tag{25}$$

and

$$\begin{aligned} MG_1 - H &= (M_1 - ZM_2)G_1 - H \\ &= M_1G_1 - Z_1(I - \bar{M}_2\bar{M}_2^+)M_2G_1 - H \\ &= M_1G_1 - Z_1(I - \bar{M}_2\bar{M}_2^+)\bar{M}_2 - H \\ &= M_1G_1 - H \\ &= \bar{H}, \text{(cf. (23i))}. \end{aligned} \tag{26}$$

Thus, by substituting the expressions from (24), (25), and (26) into (15), we obtain

$$\dot{e}_1 = (N_1 - Z_1\mathcal{N}_2)e_1 + (L_1 - Z_1\mathcal{L}_2)v \tag{27a}$$

$$e = e_1 + \bar{H}v \tag{27b}$$

It is now a direct consequence of Lemma 3 that if the LMI (22) holds, then for any $\gamma > 0$ and error dynamics system (27), we have that

$$\sup_{v \neq 0} \frac{||e||_2}{||v||_2} < \gamma.$$

Let $v \overset{\text{a.e.}}{=} 0$. Then the error dynamics (27) becomes

$$\dot{e} = Ne, \tag{28}$$

which gives that

$$e(t) = \exp(Nt)e(0). \tag{29}$$

In this case, we have to show that (9) satisfies property (b) in Definition 2. Clearly, the property (b) holds if the matrix N is stable. It is a simple consequence of Lemma 2 that (22) holds if and only if

$$\begin{bmatrix} YN + N^\top Y & YL & I \\ L^\top Y & -\gamma I & \bar{H} \\ I & \bar{H}^\top & -\gamma I \end{bmatrix} < 0,$$

which implies that $YN + N^\top Y < 0$, i.e. N is stable [27, Chapter 5]. ■

Based on Theorems 1, we now summarize the H_∞ functional observer design procedure in the form of Algorithm 2 below.

IV. NUMERICAL ILLUSTRATION

In this section, we implement the proposed functional ODE observer design methods on a simple electrical circuit as shown in Figure 1 [2], where voltage source $V(t)$ is the driver (control input), R, L, and C stand for the resistor, inductor, and capacity, respectively, as well as their quantities, and their voltages are denoted by $V_R(t)$, $V_L(t)$, $V_C(t)$, respectively. Here we assume that the input voltage

Algorithm 2 Computational steps to construct H_∞ functional observer (9) for system (1)

1) Compute U and V such that (1) converted into the form (8).
2) Extract N_1 and N_2 from $\Theta\Sigma^+$ and $I - \Sigma\Sigma^+$, respectively by using (21d).
3) Compute L_1, $\mathcal{L}_2$, and $\bar{H}$ from (23g), (23h), and (23i), respectively.
4) Solve for Y and Y_1 such that (22) becomes negative definite.
5) Compute $Z_1 = Y^{-1}Y_1$ and $Z = Z_1(I - \bar{M}_2\bar{M}_2^+)$.
6) Compute T, M, P, and N from (21).
7) Compute $L = NM - P$.

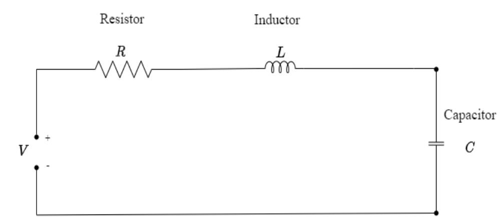

Fig. 1. An electronic circuit

is affected by additive disturbance. Then from Kirchhoff's laws, we have the circuit equations in the form of (1), where

$$E = \begin{bmatrix} L & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{C} & 0 & 0 & 0 \\ -R & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix},$$

$F = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^\top$, $x = \begin{bmatrix} i & V_L & V_C & V_R \end{bmatrix}^\top$, and $u = V$.

For simulation purposes, we take the circuit parameters $C = 4\ mF,\ R = 4\ \Omega,\ L = 5\ H$. It can be checked easily that the system satisfies assumptions (6).

System transformation: By applying Algorithm 1, we obtain the matrices

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, V = \begin{bmatrix} 0 & 0 & 0.2425 & -0.9701 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.9701 & 0.2425 \end{bmatrix},$$

such that system (1) convert into system (2), where coefficient matrices of system (2) are

$$E_{11} = \begin{bmatrix} 0 & 0 & 1.2127 \\ 0 & -1 & 0 \end{bmatrix}, \; A_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -0.0606 \end{bmatrix},$$

$$B_{11} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; F_{11} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; \bar{C}_1 = \begin{bmatrix} -1 & -1 & -0.9701 \\ 0 & 1 & 0 \end{bmatrix},$$

$$G_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \text{ and } K_{11} = \begin{bmatrix} 0 & 0 & 0.2425 \end{bmatrix}.$$

Observer parameter determination: By applying our algorithm for $\gamma = 0.8$, we get the coefficient matrices for H_∞

functional observer (9) given by

$$N = \begin{bmatrix} -0.8930 \end{bmatrix}, \ T = \begin{bmatrix} 0.2000 & 0.3722 \end{bmatrix},$$
$$L = \begin{bmatrix} -0.2000 & -0.5324 \end{bmatrix}, \text{ and } M = \begin{bmatrix} 0.0000 & 0.3722 \end{bmatrix}.$$

The true and estimated values of z have been plotted by taking $x(0) = \begin{bmatrix} 2 & -1 & -1.5 \end{bmatrix}^\top$, $w(0) = 1$, input vector $u(t) = cos(t)$ and the disturbance vector

$$v(t) = \begin{cases} 1.2sin(2\pi t) & : 10 \leq t \leq 15 \\ 0 & : \text{elsewhere} \end{cases}$$

in Figure 2.

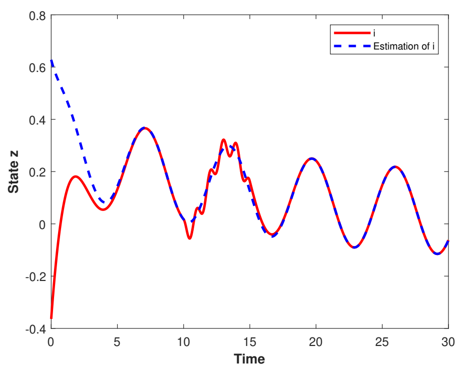

Fig. 2. Time responses of the actual and estimated i.

V. Conclusion

The solution to the H_∞ observers design problem is presented for linear time-invariant descriptor systems. A precise mathematical definition for H_∞ filtering based observers is established. Existence conditions for observers are given in terms of a simple rank condition on system (plant) matrices and one linear matrix inequality (LMI). The main advantages of our results are as follows. We do not require any assumptions on the matrix pencil associated with the (plant) descriptor systems (except for the well-posedness of the system in the sense that there exits an admissible pair of input output variables). The existence conditions are milder than the existing theories for H_∞ observers. The proposed observer design algorithm has computational advantages due to the arbitrariness of the observer parameter matrix Z, which is designed by using solution of an LMI. Furthermore, all the system transformations during the observer design procedure are performed by orthogonal matrices.

Possible directions to which our results can be extended include the general control problem for discrete-time systems and linear time-varying systems. Moreover, designing conditions and algorithm for the smallest possible order observer is a challenging problem that goes beyond the scope of this paper.

References

[1] S. L. V. Campbell, *Singular systems of differential equations II*. Pitman London, vol. 61, 1982.
[2] L. Dai, *Singular control systems*. Springer, vol. 118, 1989.
[3] P. Kunkel and V. Mehrmann, Differential-algebraic equations: analysis and numerical solution. European Mathematical Society, vol. 2, 2006.
[4] R. Riaza, Differential-algebraic systems: Analytical aspects and circuit applications. World Scientific, 2008.
[5] G.-R. Duan, *Analysis and design of descriptor linear systems*. Springer Science & Business Media, vol. 23, 2010.
[6] M. El-Tohami, V. Lovass-Nagy, and R. Mukundan, "On the design of observers for generalized state space systems using singular value decomposition," *International Journal of Control*, vol. 38, no. 3, pp. 673–683, 1983.
[7] F. L. Lewis, "A survey of linear singular systems," Circuits, systems and signal processing, vol. 5, no. 1, pp. 3–36, 1986.
[8] J. Jaiswal, M. K. Gupta, and N. K. Tomar, "Necessary and sufficient conditions for ODE observer design of descriptor systems," *Systems & Control Letters*, vol. 151, p. 104916, 2021.
[9] M. Darouach and T. Fernando, "Functional Detectability and Asymptotic Functional Observer Design," IEEE Transactions on Automatic Control, vol. 68, no. 2, pp. 975-990, Feb. 2023.
[10] J. Jaiswal and N. K. Tomar, "Existence conditions for ODE functional observer design of descriptor systems," *IEEE Control Systems Letters*, vol. 6, pp. 355–360, 2022.
[11] J. Jaiswal, P. K. Tunga, and N. K. Tomar, "Functional ODE observers for a class of descriptor systems," in *Eighth Indian Control Conference (ICC), Chennai, India*. IEEE, 2022, pp. 97–102.
[12] J. Jaiswal, M. K. Gupta, and N. K. Tomar, "On functional observers for descriptor systems," in *2021 American Control Conference(ACC)*. IEEE, 2021, pp. 4093–4098.
[13] J. Jaiswal, T. Berger, and N. K. Tomar, "Partial detectability and generalized functional observer design for linear descriptor systems," arXiv preprint arXiv:2301.10109, 2023.
[14] J. Lan and R. J. Patton, "Robust fault-tolerant control based on a functional observer for linear descriptor systems," *IFAC-papersonline*, vol. 48, no. 14, pp. 138–143, 2015.
[15] J. Zhang, Z. Wang, M. Chadli, and Y. Wang, "On prescribed-time functional observers of linear descriptor systems with unknown input," *International Journal of Control*, pp. 1–11, 2021.
[16] T. Berger, "Disturbance decoupled estimation for linear differential-algebraic systems," *International Journal of Control*, vol. 92, no. 3, pp. 593–612, 2019.
[17] P. K. Tunga, J. Jaiswal and N. K. Tomar, "Functional Observers for Descriptor Systems With Unknown Inputs," IEEE Access, vol. 11, pp. 19680-19689, 2023.
[18] L. Dai, "State estimation schemes for singular systems," in IFAC Proceedings Volumes, vol. 20, no. 5, pp. 209–213, 1987.
[19] M. Darouach, M. Zasadzinski, and D. Mehdi, "State estimation of stochastic singular linear systems," International Journal of Systems Science, vol. 24, no. 2, pp. 345–354, 1993.
[20] G. Zames, "Feedback and optimal sensitivity: Model reference transformations, multiplicative seminorms, and approximate inverses," IEEE Transactions on Automatic Control, vol. 26, no. 2, pp. 301-320, April 1981.
[21] S. Xu, J. Lam and Y. Zou, "H_∞ filtering for singular systems," IEEE Transactions on Automatic Control, vol. 48, no. 12, pp. 2217-2222, Dec. 2003.
[22] S. Xu and J Lam, "Reduced-order H_∞ filtering for singular systems," Systems & Control Letters, vol. 56, no. 1, pp. 48-57, 2007.
[23] M. Darouach, "H_∞ Unbiased Filtering for Linear Descriptor Systems via LMI," IEEE Transactions on Automatic Control, vol. 54, no. 8, pp. 1966-1972, Aug. 2009.
[24] G. L. Osorio-Gordillo, M. Darouach, and C. M. Astorga-Zaragoza. "H_∞ dynamical observers design for linear descriptor systems. Application to state and unknown input estimation," European Journal of Control, vol. 26, pp. 35-43, 2015.
[25] S. Trenn, "Solution concepts for linear DAEs: A survey," in Surveys in Differential-Algebraic Equations I. Springer, pp. 137–172, 2013.
[26] T. Berger, T. Reis, and S. Trenn, "Observability of linear differential-algebraic systems: A survey," in Surveys in Differential-Algebraic Equations IV, Cham: Springer International Publishing, pp. 161–219, 2017.
[27] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, Linear Matrix Inequalities in System and Control Theory, SIAM studies in applied mathematics, vol. 15, 1994.