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Distributed Event-Triggered Dual Decomposition Method for Cooperative One-Way Car-Sharing Control

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Abstract— **In this paper, we present cooperative rebalancing control of a one-way car-sharing service, where several service providers operate vehicles independently while sharing the common rental stations. The objective of service providers is to reduce the number of deadhead vehicles considering limited parking slots at stations. To this end, we propose a rebalancing control method by a distributed dual decomposition algorithm. Each provider transmits the estimation of the dual optimizers to the neighboring providers in an event-triggered manner. A numerical example shows that all service providers can find an optimal rebalancing solution while effectively reducing the number of communications.**

I. INTRODUCTION

Car-sharing is a sharing economy service allowing users to rent a car conveniently. The market of the car-sharing industry is predicted to proliferate around the world as mobility objectives become diverse [1]. In the car-sharing service, pick-up or drop-off sites are called stations. The car-sharing service is divided into two types: one-way and round-trip sharing services. A round-trip sharing service requires users to return a vehicle to the station where it was initially rented. In contrast, a one-way sharing service allows users to begin and end their trips at different stations. However, operating a one-way car-sharing service is generally more challenging due to an imbalance of vehicle allocation. The imbalance must be reduced by rebalancing control. As the rebalancing control, deadheading, that is, transferring vehicles between stations, is carried out to reduce the gap between supply and demand. However, the deadheading involves extra costs for service providers. To solve this issue, several providers can operate the service cooperatively.

The rebalancing control with several service providers has been investigated by different approaches such as an integer programming method [2], a game-theoretic approach [3], and a learning-based approach [4], [5]. These studies assume that the rebalancing control problem is solved in a centralized manner, which has the issue of scalability for large-scale car-sharing services. Recently, distributed optimization methods have been gaining significant attention [6], [7]. In distributed approaches, users or service providers act as independent decision-makers that collaboratively find an optimal solution [8]–[10]. However, service providers must communicate at each iteration in these distributed optimization methods, which may waste communication resources [11], [12]. To save limited communication resources, distributed event-triggered methods have been considered in many control and optimization applications [13]–[16].

In this paper, we present cooperative control of a one-way car-sharing service. Each service provider is modeled as an agent, and a group of agents operates vehicles independently but shares rental stations. The objective of service providers is to reduce the cost of the deadheading as much as possible, considering limited parking slots at stations. The cooperative rebalancing control to minimize the deadheading cost is formulated as a distributed optimization problem with a coupling inequality. The optimization problem can be solved by consensus-based algorithms such as a primal-dual algorithm [17], a dual decomposition algorithm [18], and an ADMM-based algorithm [19], [20]. In this paper, we extend the dual decomposition algorithm [18] to distributed event-triggered optimization. The communication between agents is conducted only when the discrepancy between the latest estimation transmitted to the neighbors and the current estimation exceeds a predefined threshold.

The event-triggered approach for cooperative car-sharing optimization was considered in [21]. However, this approach is based on the primal-dual algorithm, and the cost function of the rebalancing control is assumed to be strongly convex. On the other hand, we consider an approach based on the dual decomposition and the proximal algorithm [18]. Thus, the proposed method does not require strong convexity of the cost function and can be applied to a broader class of rebalancing control problems such as sparse optimization scheduling. Moreover, for a cost function with a quadratic form, the proximal maximization step in the dual update can be reduced to the projection onto the positive orthant. In general, the projection step in the primal-dual algorithm [17], [22] is computationally demanding. Therefore, the service providers can effectively find an optimal solution by applying the proposed algorithm combining dual decomposition with event-triggered communication. We also derive the convergence rate of the distributed dual decomposition algorithm with the event-triggered communication, which is not addressed quantitatively even for the time-triggered algorithm in [18].

The remainder of this paper is organized as follows. In Section II, we address the problem formulation of a cooperative car-sharing control problem and propose a distributed event-triggered dual decomposition algorithm. The

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convergence analysis of the proposed algorithm is conducted in Section III. The numerical simulation is shown in Section IV. Finally, concluding remarks are given in Section V.

II. PROPOSED ALGORITHM

A. Model of Rebalancing Control

Let $\mathbb{R}_{\geq 0}$ and $\mathbb{N}$ are the sets of the nonnegative real numbers and the nonnegative integers. We consider S car-stations that is shared by N service providers. Although the number of vehicles takes an integer value in real car-sharing systems, this paper considers a control problem with the expected number of vehicles. Suppose that $\gamma_{i,j}[k] \in \mathbb{R}_{\geq 0}$ and $h_{i,j\ell}[k] \in \mathbb{R}_{\geq 0}$ are the expected numbers of vehicles of provider $i \in \mathcal{V}$ at station $j \in \mathcal{S}$ and vehicles that move from the station ℓ to j at time $k \in \mathcal{T} = \{1, 2, \ldots, T\}$, where $\mathcal{V} = \{1, 2, \ldots, N\}$ and $\mathcal{S} = \{1, 2, \ldots, S\}$ are the sets of the service providers and the stations. Suppose also that $\vartheta_{i,j\ell}[k] \in \mathbb{R}_{\geq 0}$ and $\omega_{i,j\ell}[k] \in \mathbb{R}_{\geq 0}$ are the expected rates of the numbers of deadheading vehicles and demands of provider i from station ℓ to j.

In this paper, we consider a probabilistic model for the demands based on a Poisson distribution [23]:

$$\mathbb{P}[W_{i,j\ell}[k] = \tau] = \frac{(\omega_{i,j\ell}[k])^\tau}{\tau!} e^{-\omega_{i,j\ell}[k]}, \quad \forall i \in \mathcal{V},\ \forall j, \ell \in \mathcal{S},\ \forall k \in \mathcal{T},$$

where $W_{i,j\ell}[k]$ is the number of vehicles departing from station ℓ to j corresponding to the demand at time k. The expectation of $W_{i,j\ell}[k]$ is given as $\mathbb{E}[W_{i,j\ell}[k]] = \omega_{i,j\ell}[k]$.

Let $\xi_i[k] = [\gamma_{i,1}[k], \gamma_{i,2}[k], \ldots, \gamma_{i,S}[k], h_{i,12}[k], h_{i,13}[k], \ldots, h_{i,S,S-1}[k]]^\top \in \mathbb{R}^n$, $u_i[k] = [\vartheta_{i,21}[k], \vartheta_{i,31}[k], \ldots, \vartheta_{i,S-1,S}[k]]^\top \in \mathbb{R}^m$, and $\omega_i[k] = [\omega_{i,21}[k], \omega_{i,31}[k], \ldots, \omega_{i,S-1,S}[k]]^\top \in \mathbb{R}^m$ be provider i's state, control input, and external input, where $m = S(S-1)$ and $n = S^2$. Then, the dynamics of each provider is represented as $\xi_i[k+1] = A_i \xi_i[k] + B_i u_i[k] + B_i \omega_i[k]$, where $A_i \in \mathbb{R}^{n \times n}$ and $B_i \in \mathbb{R}^{n \times m}$ describe the dynamics of sharing service, and $\xi_i[0] = \xi_{i0} \in \mathbb{R}_{\geq 0}$ [21].

B. Formulation of Cooperative Rebalancing Control

Let $u_i = [u_i^\top[0],\ u_i^\top[1],\ \ldots,\ u_i^\top[T-1]]^\top \in \mathbb{R}^{mT}$ be the stacked vector for the number of deadheading vehicles of provider i. The constraint on the parking space is represented as $\Psi \sum_{i=1}^N \xi_i[k] \leq \psi[k]$ for all $k \in \mathcal{T}$, where $\Psi = [I_S\ O_{S \times (n-S)}] \in \mathbb{R}^{S \times n}$ and $\psi[k] \in \mathbb{R}^S$ is the maximum number of vehicles at stations at time k. The inequality relation between vectors stands for the component-wise inequality relation. Furthermore, the constraint for the number of deadheading vehicles is given as $u_i \in \mathcal{U}_i = \{u' \in \mathbb{R}^{mT} \mid 0 \leq u' \leq \bar{u}_i\}$ for all $i \in \mathcal{V}$, where $\bar{u}_i \in \mathbb{R}^{mT}_{\geq 0}$. Since $\mathcal{U}_i$ is bounded and closed, there exists $C_u > 0$ such that $\|u'\| \leq C_u$ for all $u' \in \mathcal{U}_i$ and $i \in \mathcal{V}$.

Then, with a similar discussion as in [21], the rebalancing control problem is formulated as follows:

$$\underset{\{u_i\}_{i=1}^N}{\text{minimize}} \quad \sum_{i=1}^N f_i(u_i) \tag{1a}$$

$$\text{subject to} \quad \sum_{i=1}^N g_i(u_i) \leq 0, \tag{1b}$$

$$u_i \in \mathcal{U}_i, \quad i \in \mathcal{V}. \tag{1c}$$

In the optimization problem (1), $f_i(u_i) = u_i^\top H_i u_i$ is the cost function of provider i, where $H_i = \sum_{\tau=0}^{T-1} E^\top[\tau] R_i E[\tau] \in \mathbb{R}^{mT \times mT}$, $E[k] = [O_{m \times mk},\ I_m,\ O_{m \times m(T-k-1)}] \in \mathbb{R}^{m \times mT}$, and $R_i \in \mathbb{R}^{m \times m}$ is a positive-definite matrix. Moreover, $g_i(u_i) = \Xi_i u_i + \zeta_i$ is provider i's constraint function, where $\Xi_i = \left[-\hat{\Phi}_i^\top, \check{\Phi}_i^\top\right]^\top \in \mathbb{R}^{q \times mT}$, $\zeta_i = \left[-(\hat{A}_i \xi_i[0])^\top, \mu_i^\top\right]^\top \in \mathbb{R}^q$, $\mu_i \in \mathbb{R}^{ST}$ is a coefficient vector, and $q = (n+S)T$. The matrices $\hat{\Phi}_i$, $\check{\Phi}_i$, and $\hat{A}_i$ are defined as $\hat{\Phi}_i = [\Phi_i^\top[1], \Phi_i^\top[2], \ldots, \Phi_i^\top[T]]^\top \in \mathbb{R}^{nT \times mT}$, $\check{\Phi}_i = [(\Psi\Phi_i[1])^\top, (\Psi\Phi_i[2])^\top, \ldots, (\Psi\Phi_i[T])^\top]^\top \in \mathbb{R}^{ST \times mT}$, $\hat{A}_i = [A_i^\top, (A_i^2)^\top, \ldots, (A_i^T)^\top]^\top \in \mathbb{R}^{nT \times n}$, and $\Phi_i[k+1] = [A_i^k B_i,\ A_i^{k-1} B_i,\ \ldots,\ B_i,\ O_{n \times m(T-k-1)}] \in \mathbb{R}^{n \times mT}$ with $\Phi_i[0] \in O_{n \times mT}$.

The Lagrange function for (1) is given as $\mathcal{L}(u, \lambda) = \sum_{i=1}^N \mathcal{L}_i(u_i, \lambda)$, where $u = [u_1^\top, u_2^\top, \ldots, u_N^\top]^\top \in \mathcal{U} \subset \mathbb{R}^{mNT}$. Let $\varpi_i(\lambda) = \min_{u_i \in \mathcal{U}_i} \mathcal{L}_i(u_i, \lambda)$ be the dual function for provider i. We consider the following dual problem for (1):

$$\begin{aligned} &\text{maximize} \quad \sum_{i=1}^N \varpi_i(\lambda) \\ &\text{subject to} \quad \lambda \geq 0. \end{aligned} \tag{2}$$

In this paper, we assume that Slater's qualification for (1) is satisfied [17]. Then, we obtain $\Lambda^* \neq \emptyset$ and $f^* = \varpi^*$, where Λ^* is the set of the dual optimizers, and f^* and ϖ^* are the optimal values of (1) and (2).

C. Distributed Event-Triggered Dual Decomposition Algorithm

Let $u_i^{(t)} \in \mathbb{R}^{mT}$ and $\lambda_i^{(t)} \in \mathbb{R}^q$ be the estimations of a primal optimizer of (1) and a dual optimizer of (2) at iteration t. An undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ represents a communication network of service providers, where $\mathcal{E}$ is the set of communication edges. The estimation $\lambda_i^{(t)}$ is sent to the neighboring provider when $\|\lambda_i^{(t)} - \check{\lambda}_i^{(t)}\| \geq e_i^{(t)}$ holds, where $e_i^{(t)}$ is provider i's threshold of the event-triggered communication and

$$\check{\lambda}_i^{(t)} = \begin{cases} \lambda_i^{(t)}, & \text{if } t \text{ is a triggered iteration of } i, \\ \check{\lambda}_i^{(t-1)}, & \text{otherwise.} \end{cases} \tag{3}$$

The event-triggered dual decomposition algorithm of provider i is given by

$$w_i^{(t)} = \lambda_i^{(t)} + \sum_{j=1}^{N} a_{ij}(\check{\lambda}_j^{(t)} - \check{\lambda}_i^{(t)}), \tag{4}$$

$$u_i^{(t+1)} = \arg\min_{u_i \in \mathcal{U}_i} \left\{ f_i(u_i) + (w_i^{(t)})^\top g_i(u_i) \right\}, \tag{5}$$

$$\lambda_i^{(t+1)} = \left[w_i^{(t)} + \alpha^{(t)} g_i(u_i^{(t+1)}) \right]_+, \tag{6}$$

where $\alpha^{(t)} > 0$ is a stepsize, $[\cdot]_+$ is the projection onto the positive orthant, and a_{ij} is the edge weight such that $a_{ij} \in [a, 1)$ for $\{i, j\} \in \mathcal{E}$, $a_{ij} = 0$ for $\{i, j\} \notin \mathcal{E}$ and $i \neq j$, and $a_{ii} \geq a$ for all $i \in \mathcal{V}$ with a positive constant a.

The proposed event-triggered algorithm can effectively reduce the number of communications compared with the time-triggered algorithm [18], which requires information exchange at every iteration. In addition, different from the consensus-based primal-dual algorithm [21], [22], the proposed dual decomposition approach does not require strong convexity of the cost function.

III. CONVERGENCE ANALYSIS

To establish convergence of the proposed event-triggered dual decomposition algorithm, we make the following assumptions.

Assumption 1: The graph $\mathcal{G}$ is connected.

Assumption 2: $\sum_{j=1}^{N} a_{ij} = 1$ for any $i \in \mathcal{V}$ and $\sum_{i=1}^{N} a_{ij} = 1$ for any $j \in \mathcal{V}$.

Assumption 3: The stepsize satisfies $\lim_{t\to\infty} \alpha^{(t)} = 0$, $\sum_{t=0}^{\infty} (\alpha^{(t)})^2 < \infty$, and $\sum_{t=0}^{\infty} \alpha^{(t)} = \infty$. Moreover, the trigger threshold satisfies $\lim_{t\to\infty} E^{(t)} = 0$ and $\sum_{t=0}^{\infty} E^{(t)} < \infty$, where $E^{(t)}$ is the upper bound of the threshold at iteration t such that $e_i^{(t)} \leq E^{(t)}$ for all $i \in \mathcal{V}$.

Without loss of generality, we assume that there exists a positive constant E such that $E^{(t)} < E$ for all $t \in \mathbb{N}$.

The next result shows that the estimation by the proposed algorithm converges to a dual optimizer.

Proposition 1: Under Assumptions 1–3, we have $\lim_{t\to\infty} \|\lambda_i^{(t)} - \lambda^*\| = 0$ for all $i \in \mathcal{V}$, where $\lambda^* \in \Lambda^*$.

Proof: We define the weighted sum of the trigger errors by $\hat{e}_i^{(t)} = \sum_{j=1}^{N} a_{ij}(e_j^{(t)} - e_i^{(t)})$. From Assumptions 1–3, we have

$$\begin{aligned} \sum_{i=1}^{N} \hat{e}_i^{(t)} &= \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(e_j^{(t)} - e_i^{(t)}) \\ &= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(e_j^{(t)} - e_i^{(t)}) \\ &\quad + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ji}(e_j^{(t)} - e_i^{(t)}) = 0 \end{aligned} \tag{7}$$

and

$$\begin{aligned} \|\hat{e}_i^{(t)}\| &\leq \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\|e_j^{(t)} - e_i^{(t)}\| \\ &\leq 2\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}E^{(t)} = 2NE^{(t)}. \end{aligned} \tag{8}$$

Then, by an argument similar to Theorem 1 in [18], Proposition 1 can be proven based on the result for the event-triggered subgradient method [13]. ■

The following theorem is the main result of this paper that guarantees the convergence by the proposed algorithm.

Theorem 1: Under Assumptions 1–3, we have $\lim_{t\to\infty} \mathrm{dist}(\hat{u}^{(t)}, \mathcal{U}^*) = 0$, where $\mathrm{dist}(\hat{u}^{(t)}, \mathcal{U}^*) = \min_{v\in\mathcal{U}^*} \|\hat{u}^{(t)} - v\|$, $\hat{u}^{(t)} = [(\hat{u}_1^{(t)})^\top, (\hat{u}_2^{(t)})^\top, \ldots, (\hat{u}_N^{(t)})^\top]^\top$ and $\hat{u}_i^{(t+1)} = (\sum_{r=0}^{t} \alpha^{(r)} u_i^{(r+1)}) / \sum_{r=0}^{t} \alpha^{(r)}$.

Proof: From the convexity of the constraint function, we have

$$\begin{aligned} \sum_{i=1}^{N} g_i(\hat{u}_i^{(t+1)}) &\leq \sum_{i=1}^{N} \frac{\sum_{r=0}^{t} \alpha^{(r)} g_i(u_i^{(r+1)})}{\sum_{r=0}^{t} \alpha^{(r)}} \\ &= \frac{\sum_{r=0}^{t} \alpha^{(r)} \sum_{i=1}^{N} g_i(u_i^{(r+1)})}{\sum_{r=0}^{t} \alpha^{(r)}}. \end{aligned} \tag{9}$$

Since $\lambda_i^{(t+1)} = \left[w_i^{(t)} + \alpha^{(t)} g_i(u_i^{(t+1)}) \right]_+ \geq w_i^{(t)} + \alpha^{(t)} g_i(u_i^{(t+1)})$, we have

$$\begin{aligned} \sum_{r=0}^{t} \alpha^{(r)} \sum_{i=1}^{N} g_i(u_i^{(r+1)}) &\leq \sum_{r=0}^{t}\sum_{i=1}^{N} (\lambda_i^{(r+1)} - w_i^{(r)}) \\ &= \sum_{r=0}^{t}\sum_{i=1}^{N} (\lambda_i^{(r+1)} - \lambda_i^{(r)}) \\ &= \sum_{i=1}^{N} (\lambda_i^{(t+1)} - \lambda_i^{(0)}) < \infty, \end{aligned} \tag{10}$$

where the first equality follows from Assumption 2 and (7), and the last inequality follows from the fact that $\{\lambda_i^{(t)}\}$ is bounded. Since $\sum_{r=0}^{\infty} \alpha^{(r)} = \infty$, from (9) and (10), we have

$$\limsup_{t\to\infty} \sum_{i=1}^{N} g_i(\hat{u}_i^{(t+1)}) \leq 0. \tag{11}$$

From the convexity of $\mathcal{L}_i(u_i, \lambda)$ with respect to u_i, we have

$$2\sum_{i=1}^{N} \mathcal{L}_i(\hat{u}_i^{(t+1)}, \lambda^*) \leq \frac{\sum_{r=0}^{t} 2\alpha^{(r)} \mathcal{L}(u^{(r+1)}, \lambda^*)}{\sum_{r=0}^{t} \alpha^{(r)}}. \tag{12}$$

From (23) and (24) in Appendix, we also obtain

$$
\begin{aligned}
&2\alpha^{(t)}\mathcal{L}(u^{(t+1)},\lambda^*) \\
&\le 2\alpha^{(t)}\mathcal{L}(u^*,\bar{\lambda}^{(t+1)}) \\
&\quad + \sum_{i=1}^{N}(\|\lambda_i^{(t)}-\lambda^*\|^2 - \|\lambda_i^{(t+1)}-\lambda^*\|^2) \\
&\quad + \frac{4NG^2}{\delta}(\alpha^{(t)})^2 + 2G\alpha^{(t)}\sum_{i=1}^{N}\|\lambda_i^{(t+1)}-\bar{\lambda}^{(t+1)}\| \\
&\quad + 8N^2C_\lambda E^{(t)} + 4N^3(E^{(t)})^2 \\
&\le 2\alpha^{(t)}\mathcal{L}(u^*,\lambda^*) + \sum_{i=1}^{N}(\|\lambda_i^{(t)}-\lambda^*\|^2 - \|\lambda_i^{(t+1)}-\lambda^*\|^2) \\
&\quad + \frac{4NG^2}{\delta}(\alpha^{(t)})^2 + 2G\alpha^{(t)}\sum_{i=1}^{N}\|\lambda_i^{(t+1)}-\bar{\lambda}^{(t+1)}\| \\
&\quad + 8N^2C_\lambda E^{(t)} + 4N^3(E^{(t)})^2. \qquad (13)
\end{aligned}
$$

From (12) and (13), we have

$$
\begin{aligned}
&2\mathcal{L}(\hat{u}^{(t+1)},\lambda^*) \\
&\le 2\mathcal{L}(u^*,\lambda^*) \\
&\quad + \frac{1}{\sum_{r=0}^{t}\alpha^{(r)}}\left(\sum_{i=1}^{N}\|\lambda_i^{(0)}-\lambda^*\|^2 + \frac{4NG^2}{\delta}\sum_{r=0}^{t}(\alpha^{(r)})^2\right. \\
&\qquad + 2G\sum_{r=0}^{t}\alpha^{(r)}\sum_{i=1}^{N}\|\lambda_i^{(r+1)}-\bar{\lambda}^{(r+1)}\| \\
&\qquad \left. +8N^2C_\lambda\sum_{r=0}^{t}E^{(r)} + 4N^3\sum_{r=0}^{t}(E^{(r)})^2\right). \qquad (14)
\end{aligned}
$$

Thus, from (21) and Assumption 3, we have $\limsup_{t\to\infty}\mathcal{L}(\hat{u}^{(t+1)},\lambda^*) \le \mathcal{L}(u^*,\lambda^*)$. On the other hand, from (24), $\mathcal{L}(u^*,\lambda^*) \le \mathcal{L}(\hat{u}^{(t+1)},\lambda^*)$ holds. This yields that

$$\lim_{t\to\infty}\mathcal{L}(\hat{u}^{(t+1)},\lambda^*) = \mathcal{L}(u^*,\lambda^*). \qquad (15)$$

Let $\{\hat{u}^{(t_k)}\}$ be a subsequence of $\{\hat{u}^{(t)}\}$. Here, we assume that $\lim_{t\to\infty}\mathrm{dist}(\hat{u}^{(t)},\mathcal{U}^*) = 0$ does not hold. Then, there exists a positive constant ε such that

$$\lim_{t\to\infty}\mathrm{dist}(\hat{u}^{(t_k)},\mathcal{U}^*) \ge \varepsilon. \qquad (16)$$

Since $\mathcal{U}_i$ is bounded and closed, $\{\hat{u}^{(t_k)}\}$ has a convergent subsequence. Moreover, from (11) and (15), the limit point of the subsequence is feasible and belongs to the set of the optimal primal solutions, which contradicts (16). ■

Theorem 1 shows that each service provider can find an optimal control input for the rebalancing control by the weighted time-averaged estimation.

The next proposition evaluates the convergence rate of the proposed algorithm.

Proposition 2: Suppose that the stepsize is given as $\alpha^{(t)} = 1/t^b$ and Assumptions 1–3 hold, where $1/2 < b < 1$. Then, we obtain $(1/K)\sum_{t=1}^{K} f(u^{(t+1)}) - f^* \le C_r/K^{1-b}$, where $f(u) = \sum_{i=1}^{N} f_i(u_i)$ and C_r is a positive constant.

Proof: We note that (23) still holds by replacing λ^* with 0 because (23) is valid for any feasible dual solution. Since $\bar{\lambda}^{(t)} \ge 0$ and $g(u^*) \le 0$ for any $t \in \mathbb{N}$, we further have $2\alpha^{(t)}(\mathcal{L}(u^*,\bar{\lambda}^{(t+1)}) - \mathcal{L}(u^{(t+1)},0)) \le 2\alpha^{(t)}(f(u^*) - f(u^{(t+1)}))$, where $g(u) = \sum_{i=1}^{N} g_i(u_i)$. Thus, for $t \in \{0,1,\ldots,K\}$, we obtain

$$
\begin{aligned}
&f(u^{(t+1)}) - f(u^*) \\
&\le \frac{1}{\alpha^{(K)}}\left\{\sum_{i=1}^{N}(\|\lambda_i^{(t)}\|^2 - \|\lambda_i^{(t+1)}\|^2) + \frac{4NG^2}{\delta}(\alpha^{(t)})^2\right. \\
&\quad + 2G\alpha^{(t)}\sum_{i=1}^{N}\|\lambda_i^{(t+1)}-\bar{\lambda}^{(t+1)}\| \\
&\quad \left. + 8N^2C_\lambda E^{(t)} + 4N^3(E^{(t)})^2\right\}. \qquad (17)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\frac{1}{K}\sum_{t=1}^{K}(f(u^{(t+1)}) - f(u^*)) \\
&\le \frac{1}{K\alpha^{(K)}}\left\{\sum_{i=1}^{N}\|\lambda_i^{(0)}\|^2 + \frac{4NG^2}{\delta}\sum_{t=1}^{K}(\alpha^{(t)})^2\right. \\
&\quad + 2G\sum_{t=1}^{K}\alpha^{(t)}\sum_{i=1}^{N}\|\lambda_i^{(t+1)}-\bar{\lambda}^{(t+1)}\| \\
&\quad \left. +8N^2C_\lambda\sum_{t=1}^{K}E^{(t)} + 4N^3\sum_{t=1}^{K}(E^{(t)})^2\right\}. \qquad (18)
\end{aligned}
$$

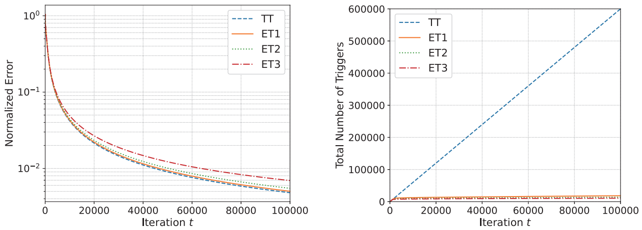
Thus, the proof is concluded with (22) and Lemma 1. ■

Proposition 2 implies that the convergence rate in a finite horizon is sublinear of $O(1/K^{1-b})$. The investigation for the event-triggered algorithms for faster convergence [20] is a part of future work.

IV. NUMERICAL EXAMPLE

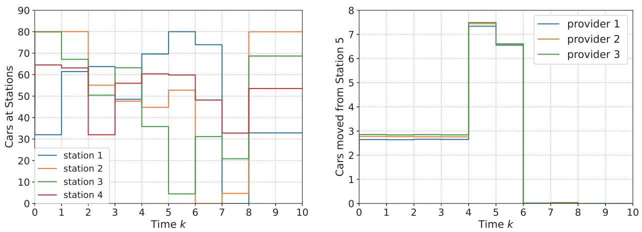
We consider a rebalancing control problem with $N = 3$ and $S = 4$. The maximum number of parking vehicles is given by $\xi_j[k] = 80$ for all $j \in \mathcal{S}$ and $k \in \mathcal{T}$. The total number of operating vehicles is 194 for provider 1, 165 for provider 2, and 178 for provider 3. In this example, we consider the case when the calls of users from station 1 to 2 increases at times 6 and 7. The stepsize is $\alpha^{(t)} = 1/(t+10^5)^{0.51}$ and the trigger threshold is $e_i^{(t)} = c/(t+1)$ for all $i \in \mathcal{V}$ with $c > 0$. The weight matrix R_i is given as the identity matrix for all $i \in \mathcal{V}$.

Figures 1(a) and 1(b) show the normalized error $(\sum_{i=1}^{N}|f(\hat{u}_i^{(t)}) - f^*|)/(\sum_{i=1}^{N}|f(\hat{u}_i^{(0)}) - f^*|)$ and the total number of communications for the different thresholds, where f^* is the cost computed by CVXPY [24]. In Figs. 1(a) and 1(b), TT represents the result by the time-triggered algorithm in [18], and ET1, ET2, and ET3 are the results by the event-triggered algorithm with $c = 0.05$, 0.1, and 0.2, respectively. These results show that the convergence by the event-triggered algorithm is comparable to that by the time-triggered algorithm. Moreover, the number of communications by the event-triggered algorithm is efficiently



(a) Normalized errors (b) Number of communications

Fig. 1. The normalized errors and the total number of communications.



(a) Number of vehicles at stations (b) Number of vehicles departed from station 5

Fig. 2. The number of vehicles at stations and departing vehicles.

reduced compared with that by the time-triggered algorithm. As the iterations progress, the estimations of agents approach the optimal solution, leading to fewer significant deviations from previous states. This results in fewer triggers for communication. Despite the reduced communication, the convergence performance is less affected as seen in Fig. 1(a). This is because the diminishing step size ensures that each agent makes smaller adjustments, which are more likely to be in the right direction near the optimal solution. For example, the convergence performance by the event-triggered algorithm ET1 with $c = 0.05$ is almost the same as that by the time-triggered algorithm, while the total number of communications is significantly reduced by more than 95%.

Figures 2(a) and 2(b) show the number of vehicles at each station and the number of vehicles that depart from station 5 for $c = 0.2$. In this example, the vehicles move from station 1 to station 2 from time 6 to time 8 to respond to the surge in demand. To meet the surging demand of station 2, the number of vehicles departing from station 5 at times 4 and 5 is increased as shown in Fig. 2(b). In addition, the constraint on the maximum number of vehicles that can be parked $\xi_j[k] = 80$ has been met for all stations at all times. These results illustrate that service providers can respond flexibly to fluctuations in demand under the constraint of available cars and parking spaces.

V. Conclusion

This paper considered a distributed rebalancing control problem for a one-way car-sharing service. Several service providers cooperatively provide services to achieve less deadheading while sharing parking spaces in common stations. We proposed a distributed dual decomposition algorithm with event-triggered communication. We showed that each provider could find an optimal solution to reduce the deadheading vehicles as much as possible. The numerical example showed that service providers could achieve optimal deadheading under the constraint of available parking spaces. The extension to other cooperative sharing economy services is a line of the future direction of this research article.

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APPENDIX

Lemma 1: Under Assumptions 1–3, we have

$$\sum_{t=1}^{\infty}\sum_{i=1}^{N}\|\phi_i^{(t)}\|^2 < \infty, \tag{19}$$

$$\lim_{t\to\infty}\|\phi_i^{(t)}\| = 0, \quad \forall i \in \mathcal{V}, \tag{20}$$

$$\sum_{t=1}^{\infty}\alpha^{(t)}\sum_{i=1}^{N}\|\lambda_i^{(t+1)} - \bar{\lambda}^{(t+1)}\| < \infty, \tag{21}$$

where $\phi_i^{(t)} = \lambda_i^{(t+1)} - w_i^{(t)}$ and $\bar{\lambda}^{(t)} = (1/N)\sum_{i=1}^{N}\lambda_i^{(t)}$.

Proof: From Proposition 1 in [8], under Assumptions 1 and 2, for all $j, \ell \in \mathcal{V}$ and $r, r' \in \mathbb{N}$ with $r \geq r'$, there exist constants $0 < \beta < 1$ and $C > 0$ such that $\left|[A^{r-r'+1}]_{j\ell} - 1/N\right| \leq C\beta^{r-r'}$, where $A \in \mathbb{R}^{N\times N}$ is the weight matrix whose (j,ℓ)-th element is the edge weight $a_{j\ell}$.

Since $\mathcal{U}_i$ is closed and convex for all $i \in \mathcal{V}$, there exists a positive constant G such that $\|g_i(u_i)\| \leq G$ for all $u_i \in \mathcal{U}_i$. Moreover, from Proposition 1, the sequence of the estimation $\{\lambda_i^{(t)}\}$ is convergent, and hence, there exists a positive constant C_λ such that $\|\lambda_i^{(t)}\| \leq C_\lambda$ for all $i \in \mathcal{V}$ and $t \in \mathbb{N}$.

By following the similar argument of Lemmas 1 and 2 in [18], we have

$$\begin{aligned}
&2\sum_{t=1}^{K}\alpha^{(t)}\sum_{i=1}^{N}\|\lambda_i^{(t+1)} - \bar{\lambda}^{(t+1)}\| \\
&\quad \leq \delta C_1\sum_{t=1}^{K}\sum_{j=1}^{N}\|\phi_j^{(t)}\|^2 + C_2\sum_{t=1}^{K}(\alpha^{(t)})^2 + C_3
\end{aligned} \tag{22}$$

and

$$\begin{aligned}
&\sum_{i=1}^{N}\|\lambda_i^{(t+1)} - \lambda^*\|^2 \\
&\leq \sum_{i=1}^{N}\|\lambda_i^{(t)} - \lambda^*\|^2 - (1-\delta)\sum_{i=1}^{N}\|\phi_i^{(t)}\|^2 + \frac{4NG^2}{\delta}(\alpha^{(t)})^2 \\
&\quad + 2G\alpha^{(t)}\sum_{i=1}^{N}\|\lambda_i^{(t+1)} - \bar{\lambda}^{(t+1)}\| \\
&\quad + 2\alpha^{(t)}(\mathcal{L}(u^*, \bar{\lambda}^{(t+1)}) - \mathcal{L}(u^{(t+1)}, \lambda^*)) \\
&\quad + 8N^2C_\lambda E^{(t)} + 4N^3(E^{(t)})^2,
\end{aligned} \tag{23}$$

where C_1, C_2, and C_3 are positive constants, $0 < \delta < 1/(1+C_1)$, $u^{(t)} = [(u_1^{(t)})^\top, (u_2^{(t)})^\top, \ldots, (u_N^{(t)})^\top]^\top$, and $u^* = [(u_1^*)^\top, (u_2^*)^\top, \ldots, (u_N^*)^\top]^\top$ is the stacked vector of the primal optimizers.

Since (u^*, λ^*) is the optimal primal-dual pair, for all $u \in \mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \cdots \times \mathcal{U}_N$ and $\lambda \in \mathbb{R}^q_{\geq 0}$, we have

$$\mathcal{L}(u^*, \lambda) \leq \mathcal{L}(u^*, \lambda^*) \leq \mathcal{L}(u, \lambda^*). \tag{24}$$

This yields that $\mathcal{L}(u^*, \bar{\lambda}^{(t+1)}) - \mathcal{L}(u^{(t+1)}, \lambda^*) = (\mathcal{L}(u^*, \bar{\lambda}^{(t+1)}) - \mathcal{L}(u^*, \lambda^*)) + (\mathcal{L}(u^*, \lambda^*) - \mathcal{L}(u^{(t+1)}, \lambda^*)) \leq 0$. Then, from (22) and (23), we have

$$\begin{aligned}
&\sum_{t=1}^{K}\sum_{i=1}^{N}\|\lambda_i^{(t+1)} - \lambda^*\|^2 \\
&\leq \sum_{t=1}^{K}\sum_{i=1}^{N}\|\lambda_i^{(t)} - \lambda^*\|^2 - (1-\delta)\sum_{t=1}^{K}\sum_{i=1}^{N}\|\phi_i^{(t)}\|^2 \\
&\quad + \delta C_1 G\sum_{t=1}^{K}\sum_{j=1}^{N}\|\phi_j^{(t)}\|^2 + C_2 G\sum_{t=1}^{K}(\alpha^{(t)})^2 + C_3 G \\
&\quad + \frac{4NG^2}{\delta}\sum_{t=1}^{K}(\alpha^{(t)})^2 + 8N^2C_\lambda\sum_{t=1}^{K}E^{(t)} \\
&\quad + 4N^3\sum_{t=1}^{K}(E^{(t)})^2 \\
&= \sum_{t=1}^{K}\sum_{i=1}^{N}\|\lambda_i^{(t)} - \lambda^*\|^2 \\
&\quad - (1 - \delta(1 + C_1 G))\sum_{t=1}^{K}\sum_{i=1}^{N}\|\phi_i^{(t)}\|^2 \\
&\quad + \left(\frac{4NG^2}{\delta} + C_2 G\right)\sum_{t=1}^{K}(\alpha^{(t)})^2 + 8N^2C_\lambda\sum_{t=1}^{K}E^{(t)} \\
&\quad + 4N^3\sum_{t=1}^{K}(E^{(t)})^2 + C_3 G.
\end{aligned} \tag{25}$$

It follows that

$$\begin{aligned}
&(1 - \delta(1 + C_1 G))\sum_{t=1}^{K}\sum_{i=1}^{N}\|\phi_i^{(t)}\|^2 \\
&\leq \sum_{t=1}^{K}\sum_{i=1}^{N}(\|\lambda_i^{(t)} - \lambda^*\|^2 - \|\lambda_i^{(t+1)} - \lambda^*\|^2) \\
&\quad + \left(\frac{4NG^2}{\delta} + C_2 G\right)\sum_{t=1}^{K}(\alpha^{(t)})^2 + 8N^2C_\lambda\sum_{t=1}^{K}E^{(t)} \\
&\quad + 4N^3\sum_{t=1}^{K}(E^{(t)})^2 + C_3 G \\
&\leq \|\lambda_i^{(1)} - \lambda^*\|^2 + \left(\frac{4NG^2}{\delta} + C_2 G\right)\sum_{t=1}^{K}(\alpha^{(t)})^2 \\
&\quad + 8N^2C_\lambda\sum_{t=1}^{K}E^{(t)} + 4N^3\sum_{t=1}^{K}(E^{(t)})^2 + C_3 G.
\end{aligned} \tag{26}$$

Therefore, from Assumption 3, (19) and (20) hold. Then, from (19) and (22), we have (21). ■