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Minimal realizations of input-output behaviors by LPV state-space representations with affine dependence

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Abstract— **The paper makes the first steps towards a behavioral theory of LPV state-space representations with only affine dependence on the scheduling signal, by characterizing minimality of such state-space representations. We show that minimality is equivalent to observability, and that minimal realizations of the same behavior are isomorphic. Finally, we establish a formal relationship between minimality of LPV state-space representations with an affine dependence on the scheduling signal and minimality of LPV state-space representations with a dynamic and meromorphic dependence on the scheduling signal.**

I. Introduction

Linear parameter-varying (LPV) systems represent a system class which is more general than *linear time-invariant* (LTI) systems and which can capture *nonlinear* and *time-varying* behavior. LPV systems are modeled by linear difference or differential equations, where the coefficients are functions of a time-varying *scheduling* signal. LPV systems are widely used in control ([3]–[6]) and in system identification, ([7]–[13]).

Despite these advances, there are still gaps in the theory of LPV systems, in particular in their *realization theory*. Realization theory aims at characterizing the relationship between the input-output behavior and certain classes of state-space representations (linear time-invariant, bilinear, etc.), see [14], [15]. Since realization theory is used in system identification, model reduction, and data-driven control [16], filling this gap is important.

Prior work on realization theory and motivation: Realization theory of LPV systems was first addressed in [1], [2], where, through the behavioral theory, concepts of minimality and equivalence for so general LPV state-space representation (*LPV-SS* for short) under *meromorphic* and *dynamical dependence* of the model coefficients on the scheduling have been established. We will refer to this class of LPV-SSs as *meromorphic LPV-SS*. A major drawback of meromorphic LPV-SSs is that, for practical applications, it is often preferable to use LPV-SSs with a static and affine dependence on the scheduling variable (*LPV-SSA* for short), i.e., LPV-SS whose matrices are affine functions of the instantaneous value of the scheduling variable. However, whereas LPV-SSAs are a subclass of meromorphic LPV-SSs, the system-theoretic transformations (passing from input-output to state-space representation, transforming a representation to a minimal one, etc.) of [1], [2] result in meromorphic LPV-SSs, even if applied to LPV-SSAs, see [17], [18]. In [19], realizability of LPV input-output equations by LPV-SSs with a general (non affine) dependence on the scheduling variable was investigated. However, it is not clear that all behaviors of interest admit the LPV input-output representations from [19], and [19] does not address minimality. In [20], a Kalman-style realization theory for LPV-SSAs was developed. A drawback of [20] lies in the use of input-output functions, which captures the input-output behavior only from a certain fixed initial state. In contrast, for control synthesis, the initial state is not fixed.

That is, [1], [2], [19] do not address behavioral realization theory for LPV-SSAs as a closed class.

Contributions: In this paper, we make the first step towards a behavioral approach directly for LPV-SSAs. Similarly to [1], we use the concept of manifest behavior from [21] to formalize the input-output behavior of LPV-SSAs. We show that the following counterparts of the well-known results for LTI behaviors [21] hold:

- An LPV-SSA is a minimal realization of a given behavior, if and only if it is observable, and all minimal realization of the same manifest behavior are related by a linear (constant) isomorphism.
- A behavior is controllable, if and only if its minimal LPV-SSAs is span-reachable from the zero initial state.

We also formulate a computionally effective minimization procedure for LPV-SSAs. Furthermore, we show that under some assumptions, a minimal LPV-SSA realization of a behavior is also minimal if viewed as a meromorphic LPV-SS [1]. The latter is interesting, as in contrast to meromorphic LPV-SSs, there are computationally effective algorithms for minimization and checking minimality of LPV-SSAs.

Outline: In Section II, we present the necessary background on LPV-SSA representations and then formalize several system theoretic concepts. In Section III, the main results are introduced, while Section IV gathers the proofs.

II. Preliminaries

Let $\mathbb{T} = \mathbb{R}_0^+ = [0, +\infty)$ be the time axis in the *continuous-time* (CT) case and $\mathbb{T} = \mathbb{N}$ in the *discrete-time* (DT) case. Let ξ be the differentiation operator $\frac{d}{dt}$ (in CT) and the forward

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time-shift operator q (in DT), *i.e.*, if $z : \mathbb{T} \to \mathbb{R}^n$, then $(\xi z)(t) = \frac{d}{dt}z(t)$ for CT and $(\xi z)(t) = z(t+1)$ for DT.

Define an LPV state-space representations with *affine* dependence on the *scheduling variable* (*LPV-SSA*) as

$$\Sigma \begin{cases} \xi x(t) &= A(p(t))x(t) + B(p(t))u(t), \\ y(t) &= C(p(t))x(t) + D(p(t))u(t), \end{cases} \quad (1)$$

where $x : \mathbb{T} \to \mathbb{R}^{n_\mathrm{x}}$ is the state trajectory, $y : \mathbb{T} \to \mathbb{R}^{n_\mathrm{y}}$ is the (measured) output trajectory, $u : \mathbb{T} \to \mathbb{R}^{n_\mathrm{u}}$ is the (control) input signal and $p : \mathbb{T} \to \mathbb{P} \subseteq \mathbb{R}^{n_\mathrm{p}}$ is the so called *scheduling signal* of the system represented by Σ. Moreover, A, B, C, D are matrix valued affine functions defined on $\mathbb{P}$, i.e., there exists matrices $A_i \in \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{x}}$, $B_i \in \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{u}}$, $C_i \in \mathbb{R}^{n_\mathrm{y} \times n_\mathrm{x}}$ and $D_i \in \mathbb{R}^{n_\mathrm{y} \times n_\mathrm{u}}$ for all $i = 0, 1, \ldots, n_\mathrm{p}$, such that

$$A(\mathbf{p}) = A_0 + \sum_{i=1}^{n_\mathrm{p}} A_i \mathbf{p}_i, \; B(\mathbf{p}) = B_0 + \sum_{i=1}^{n_\mathrm{p}} B_i \mathbf{p}_i,$$

$$C(\mathbf{p}) = C_0 + \sum_{i=1}^{n_\mathrm{p}} C_i \mathbf{p}_i, \; D(\mathbf{p}) = D_0 + \sum_{i=1}^{n_\mathrm{p}} D_i \mathbf{p}_i,$$

for every[1] $\mathbf{p} = [\; \mathbf{p}_1 \;\; \ldots \;\; \mathbf{p}_{n_\mathrm{p}} \;]^\top \in \mathbb{P}$.

Note that in LPV systems, the input and scheduling signals play the role of exogenous inputs. Furthermore, it is often assumed that the scheduling signals are bounded to ensure desirable properties, e.g., stability, hence in general $\mathbb{P} \neq \mathbb{R}^{n_\mathrm{p}}$.

In the sequel, we use the shorthand notation

$$\Sigma = (\mathbb{P}, \{A_i, B_i, C_i, D_i\}_{i=0}^{n_\mathrm{p}})$$

to denote an LPV-SSA of the form (1) and use $\dim(\Sigma) = n_\mathrm{x}$ to denote its state dimension.

To develop our results, we need to formalize the solution concept for LPV-SSAs. To this end, we define the sets $\mathcal{X}, \mathcal{Y}, \mathcal{U}, \mathcal{P}$ of state, output, input, and scheduling trajectories as follows. For a set X, let $X^\mathbb{N}$ denote the set of all functions of the form $f : \mathbb{N} \to X$. In DT, let $\mathcal{X} = (\mathbb{R}^{n_\mathrm{x}})^\mathbb{N}, \mathcal{Y} = (\mathbb{R}^{n_\mathrm{y}})^\mathbb{N}, \mathcal{U} = (\mathbb{R}^{n_\mathrm{u}})^\mathbb{N}, \mathcal{P} = \mathbb{P}^\mathbb{N}$. In CT, let us denote by $\mathcal{C}_\mathrm{p}(\mathbb{R}_0^+, \mathbb{R}^n)$ the set of all functions of the form $f : \mathbb{R}_0^+ \to \mathbb{R}^n$ which are piecewise-continuous. In addition let $\mathcal{C}_\mathrm{a}(\mathbb{R}_0^+, \mathbb{R}^n)$ be the set of all absolutely continuous functions of the form $f : \mathbb{R}_0^+ \to \mathbb{R}^n$. Then, in CT, let $\mathcal{X} = \mathcal{C}_\mathrm{a}(\mathbb{R}_0^+, \mathbb{R}^{n_\mathrm{x}})$, $\mathcal{Y} = \mathcal{C}_\mathrm{p}(\mathbb{R}_0^+, \mathbb{R}^{n_\mathrm{y}})$, $\mathcal{U} = \mathcal{C}_\mathrm{p}(\mathbb{R}_0^+, \mathbb{R}^{n_\mathrm{u}})$, $\mathcal{P} = \mathcal{C}_\mathrm{p}(\mathbb{R}_0^+, \mathbb{P})$.

By a solution of Σ, we mean a tuple of trajectories $(x, y, u, p) \in (\mathcal{X}, \mathcal{Y}, \mathcal{U}, \mathcal{P})$ satisfying (1) for almost all $t \in \mathbb{T}$ in the CT case, and for all $t \in \mathbb{T}$ in DT.

Note that for any input and scheduling signal $(u, p) \in \mathcal{U} \times \mathcal{P}$ and any initial state $x_\mathrm{o} \in \mathbb{R}^{n_\mathrm{x}}$, there exists a *unique* pair $(y, x) \in \mathcal{Y} \times \mathcal{X}$ such that (x, y, u, p) is a solution of (1) and $x(0) = x_\mathrm{o}$, see [1]. Next, inspired by [1], [21], we define the notion of manifest behaviors for LPV-SSAs.

Definition 1: A *manifest behavior* is a subset $\mathcal{B} \subseteq \mathcal{Y} \times \mathcal{U} \times \mathcal{P}$. The *manifest behavior* $\mathcal{B}(\Sigma)$ of an LPV-SSA Σ is defined as

$$\mathcal{B}(\Sigma) = \big\{(y, u, p) \in \mathcal{Y} \times \mathcal{U} \times \mathcal{P} \mid \exists x \in \mathcal{X} \\ \text{s.t. } (x, y, u, p) \text{ is a solution of (1)}\big\}.$$

The LPV-SSA Σ is a *realization* of a manifest behavior $\mathcal{B} \subseteq \mathcal{Y} \times \mathcal{U} \times \mathcal{P}$, if $\mathcal{B} = \mathcal{B}(\Sigma)$. □

That is, the manifest behavior of an LPV-SSA Σ is the set of all tuples (y, u, p) such that Σ generates the output y for some initial state, if Σ is fed by the input u and scheduling p. The corresponding definition of minimality is as follows.

Definition 2: An LPV-SSA Σ is a *minimal realization* of a manifest behavior $\mathcal{B}$, if it is a realization of $\mathcal{B}$, and for any LPV-SSA Σ' such that Σ' is a realization of $\mathcal{B}$, $\dim \Sigma \leq \dim \Sigma'$. We say that Σ is *minimal*, if Σ is a minimal realization of its own manifest behavior $\mathcal{B}(\Sigma)$. □

Manifest behaviors are a natural formalization of the intuition behind input-output behaviors of LPV-SSAs. However, input-output behaviors can also be formalized using input-output functions. The latter was used in [20] for proposing a Kalman-style realization theory for LPV-SSAs. The principal definitions are as follows:

Definition 3: Let $x_\mathrm{o} \in \mathbb{R}^{n_\mathrm{x}}$ be an initial state of Σ. Define the *input-output (i/o) function* $\mathfrak{Y}_{\Sigma, x_\mathrm{o}} : \mathcal{U} \times \mathcal{P} \to \mathcal{Y}$, induced by the initial state x_o as follows: for any $(u, p) \in \mathcal{U} \times \mathcal{P}$, $y = \mathfrak{Y}_{\Sigma, x_\mathrm{o}}(u, p)$ holds if and only if there exists a solution (x, y, u, p) of (1) such that $x(0) = x_\mathrm{o}$. □

Definition 4: An LPVS-SSA Σ is a *realization* of an i/o function $\mathfrak{F} : \mathcal{U} \times \mathcal{P} \to \mathcal{Y}$ from the initial state $x_\mathrm{o} \in \mathbb{R}^{n_\mathrm{x}}$, if $\mathfrak{F}$ coincides with the i/o function of Σ induced by x_o, i.e. $\mathfrak{F} = \mathfrak{Y}_{\Sigma, x_\mathrm{o}}$. We say Σ is a *realization* of $\mathfrak{F}$, if it is a realization of $\mathfrak{F}$ from some initial state. Additionally, the LPV-SSA Σ is a *minimal realization of* $\mathfrak{F}$ if it is a realization of $\mathfrak{F}$, and for every LPV-SSA Σ' which is a realization of $\mathfrak{F}$, $\dim(\Sigma) \leq \dim(\Sigma')$. □

A drawback of using i/o functions instead of manifest behaviors is that the former capture the input-output behavior for one choice of initial states. However, we can account for all initial states by using families of i/o functions.

Definition 5: If Σ is an LPV-SSA with state-space $\mathbb{R}^{n_\mathrm{x}}$, then the set $\mathbb{F}(\Sigma) = \{\mathfrak{Y}_{\Sigma, x_\mathrm{o}} \mid x_\mathrm{o} \in \mathbb{R}^{n_\mathrm{x}}\}$ of all i/o functions of Σ induced by some initial state of Σ is called the *family of i/o functions* of Σ. A family Φ of i/o functions of the form $\mathfrak{F} : \mathcal{U} \times \mathcal{P} \to \mathcal{Y}$ is *realized* by Σ, if $\Phi = \mathbb{F}(\Sigma)$. □

It is natural to ask if using families of i/o functions are equivalent to using LPV manifest behaviors. Clearly, if $\mathbb{F}(\Sigma) = \mathbb{F}(\hat{\Sigma})$, then $\mathcal{B}(\Sigma) = \mathcal{B}(\hat{\Sigma})$ holds. In fact, the example below shows that the converse is not true.

Example 1: Consider the LPV-SSAs Σ and Σ'

$$\Sigma \begin{cases} x_1(t+1) = x_1(t) + p(t)x_2(t), \; x_2(t+1) = 0, \\ y(t) = x_1(t) + p(t)x_2(t), \end{cases}$$

$$\Sigma' \begin{cases} z(t+1) = z(t), \; y(t) = z(t), \end{cases}$$

with the scheduling space $\mathbb{P} = \mathbb{R}$. A straightforward calculation reveals that $\mathcal{B}(\Sigma') = \mathcal{B}(\Sigma)$, see [17] for details. However, for $x_\mathrm{o} = \begin{bmatrix}1 & 1\end{bmatrix}^\top$ and any $x_\mathrm{o}' \in \mathbb{R}$, $\mathfrak{Y}_{\Sigma, x_\mathrm{o}} \neq \mathfrak{Y}_{\Sigma', x_\mathrm{o}'}$. To see this, it is enough to evaluate the i/o functions involved for any two scheduling signals p and p' such that $p(0) = 0$ and $p'(0) = 1$, see [17] for details. Hence, $\mathbb{F}(\Sigma) \neq \mathbb{F}(\Sigma')$.

That is, realization theory of manifest behaviors is not equivalent to that of i/o functions. In particular, the minimality

[1]Note that in the sequel we use italic letters to denote scheduling signals and boldface letters to distinguish elements of $\mathbb{P}$.

results of [20] do not apply in the behavioral setting. It is then natural to ask if similarly to [20], observability and span-reachability characterize minimality in the behavioral setting. The following definitions are recalled.

Definition 6: Let Σ be an LPV-SSA of the form (1). Σ is *span-reachable* from an initial state $x_{\mathrm{o}} \in \mathbb{R}^{n_{\mathrm{x}}}$, if the linear span of all states reachable from x_{o} equals the whole state-space $\mathbb{R}^{n_{\mathrm{x}}}$, i.e., $\mathrm{Span}\{x(t) \mid (x,y,u,p) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{U} \times \mathcal{P}, (x,y,u,p)$ is a solution of (1), $t \in \mathbb{T},\ x(0) = x_{\mathrm{o}}\} = \mathbb{R}^{n_{\mathrm{x}}}$. The LPV-SSA Σ is *observable* if any two distinct initial states induce distinct i/o functions, i.e. $\forall x_1, x_2 \in \mathbb{R}^{n_{\mathrm{x}}}$: $x_1 \neq x_2 \implies \mathfrak{Y}_{\Sigma,x_1} \neq \mathfrak{Y}_{\Sigma,x_2}$. □

Observability and span-reachability can be characterized by rank conditions [20]. Finally, similarly to [20], we would like to have minimal realizations of the same manifest behavior to be isomorphic. The latter notion is defined below.

Definition 7: Let Σ be of the form (1) and let $\Sigma' = (\mathbb{P}, \{A_i', B_i', C_i', D_i'\}_{i=0}^{n_{\mathrm{p}}})$ be an LPV-SSA with $\dim(\Sigma) = \dim(\Sigma') = n_{\mathrm{x}}$. A nonsingular matrix $T \in \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{x}}}$ is an *isomorphism* from Σ to Σ', if for all $i = 0, 1, \ldots, n_{\mathrm{p}}$,

$$A_i'T = TA_i \quad B_i' = TB_i \quad C_i'T = C_i \quad D_i' = D_i$$

□

Note that the matrix T in the definition above does not depend on the scheduling signal, and it acts only on the states of the LPV-SSAs involved. In particular, the LPV-SSAs Σ and Σ' have the same inputs and outputs and are defined over the same set of scheduling signals.

Problem formulation: in this paper we will address the following questions.

(1) If two LPV-SSAs have the same manifest behavior, do they have the same set of i/o functions?

(2) Can we characterize minimal LPV-SSAs in terms of observability and span-reachability?

(3) Are minimal LPV-SSA realizations of the same manifest LPV behavior isomorphic?

(4) Is there an algorithm for transforming an LPV-SSA to a minimal LPV-SSA realization of its manifest behavior?

(5) Are minimal LPV-SSAs also minimal as meromorphic LPV-SS from [1], [2]?

III. Main results

In this section, we present the main results of the paper, which answer the questions formulated above.

A. Input-output functions vs. behaviors

We start by clarifying the relationship between manifest behaviors and i/o functions of LPV-SSAs. To this end, we need the following definition.

Definition 8: An LPV-SSA of the form (1) is said to satisfy the *regularity certificate (RC)* if **(i)** $\mathbb{P}$ is convex with a non-empty interior, and, in addition, **(ii)** in the DT case, the matrix $A(\mathbf{p})$ is invertible for all $\mathbf{p} \in \mathbb{P}$. □

In CT, satisfaction of the RC depends only on $\mathbb{P}$, and it is satisfied if $\mathbb{P}$ is a Cartesian product of intervals, e.g., $\mathbb{P} = [a,b]^{n_{\mathrm{p}}}$, $a < b$. In DT, the RC condition is more restrictive.

Theorem 1: Let Σ and $\hat{\Sigma}$ be two LPV-SSAs which satisfy the RC. Then Σ and $\hat{\Sigma}$ have the same family of i/o functions, i.e., $\mathbb{F}(\Sigma) = \mathbb{F}(\hat{\Sigma})$, if and only if their manifest behavior is the same, i.e., $\mathcal{B}(\Sigma) = \mathcal{B}(\hat{\Sigma})$. □

The proof of Theorem 1 is presented in Section IV. The theorem above says that manifest behaviors and families of i/o function are equivalent formalizations of input-output behaviors of LPV-SSAs satisfying the RC. Note that Theorem 1 is no longer true if we drop the RC, see Example 1.

B. Minimality

Theorem 1 and an extension of the results of [20] to families of i/o functions lead to the following characterization of minimal realizations of manifest behaviors.

Theorem 2: An LPV-SSA which satisfies the RC is minimal, if and only if it is observable. Furthermore, any two minimal LPV-SSAs, which satisfy the RC and are realizations of the same manifest behavior, are isomorphic. □

The proof of Theorem 2 is presented in Section IV. Note that minimality of LPV-SSAs does not require span-reachability. This is in contrast with minimal LPV-SSA realizations of i/o functions, but this is consistent with the classical results for LTI systems [21].

Theorem 2 suggests a minimization procedure which is based on the observability reduction procedure from [20], [22]. We recall the latter below. Let Σ be an LPV-SSA of the form (1) and recall from [20] the definition of extended n-step observability matrices $\mathcal{O}_n$ of Σ, $n \in \mathbb{N}$,

$$\mathcal{O}_0 = \begin{bmatrix} C_0^\top & \cdots & C_{n_{\mathrm{p}}}^\top \end{bmatrix}^\top,$$
$$\mathcal{O}_{n+1} = \begin{bmatrix} \mathcal{O}_n^\top & A_0^\top \mathcal{O}_n^\top & \cdots & A_{n_{\mathrm{p}}}^\top \mathcal{O}_n^\top \end{bmatrix}^\top.$$

By [20], Σ is observable, if and only if $\mathrm{rank}\,(\mathcal{O}_{n_{\mathrm{x}}-1}) = n_{\mathrm{x}}$.

Procedure 1 (Observability reduction): Consider the matrix $T = \begin{bmatrix} b_1 & b_2 & \ldots & b_{n_{\mathrm{x}}} \end{bmatrix}^{-1}$, where $\{b_i\}_{i=1}^{n_{\mathrm{x}}} \subset \mathbb{R}^{n_{\mathrm{x}}}$ is a basis such that $\mathrm{Span}\{b_{o+1}, \ldots, b_{n_{\mathrm{x}}}\} = \mathrm{Ker}\{\mathcal{O}_{n_{\mathrm{x}}-1}\}$. Then it can be shown that

$$TA_iT^{-1} = \begin{bmatrix} A_i^{\mathrm{O}} & 0 \\ A_i' & A_i'' \end{bmatrix},\ TB = \begin{bmatrix} B_i^{\mathrm{O}} \\ B_i' \end{bmatrix},\ C_iT^{-1} = \begin{bmatrix} C_i^{\mathrm{O}} & 0 \end{bmatrix},$$

where $A_i^{\mathrm{O}} \in \mathbb{R}^{o \times o}$, $B_i^{\mathrm{O}} \in \mathbb{R}^{o \times n_{\mathrm{u}}}$ and $C_i^{\mathrm{O}} \in \mathbb{R}^{n_{\mathrm{y}} \times o}$. Define $\Sigma^{\mathrm{O}} = (\mathbb{P}, \{A_i^{\mathrm{O}}, B_i^{\mathrm{O}}, C_i^{\mathrm{O}}, D_i\}_{i=0}^{n_{\mathrm{p}}})$. □

Procedure 1 is similar to the well-known observability reduction for LTI/bilinear systems, and it can readily be implemented numerically, e.g., see [22, Remark 2].

Remark 1: By [20], Σ^{O} is observable. Let $\Pi \in \mathbb{R}^{o \times n_{\mathrm{x}}}$ be such that Πz is formed by the first o elements of Tz. Then (x,y,u,p) is a solution of Σ, if and only if $(\Pi x, y, u, p)$ is a solution of Σ^{O}. Hence, $\mathcal{B}(\Sigma) = \mathcal{B}(\Sigma^{\mathrm{O}})$. Moreover, for any initial state x_{o} of Σ, $\mathfrak{Y}_{\Sigma,x_{\mathrm{o}}} = \mathfrak{Y}_{\Sigma^{\mathrm{O}},\Pi(x_{\mathrm{o}})}$. Furthermore, if Σ satisfies the RC, then so does Σ^{O}. For CT, there is nothing to show. For DT, notice that $A^{\mathrm{O}}(\mathbf{p})$, $\mathbf{p} \in \mathbb{P}$, is the upper left block of the triangular matrix $TA(\mathbf{p})T^{-1}$, hence if $A(\mathbf{p})$ is invertible, then so is $A^{\mathrm{O}}(\mathbf{p})$.

Corollary 1 (Minimization): If Σ satisfies the RC, then Σ^{O} returned by Procedure 1 satisfies the RC, it is minimal and it has the same manifest behavior as Σ. □

As in the LTI case, span-reachability is necessary for minimality of LPV-SSA realizations of *controllable* behaviors. The latter is defined similarly to [21].

Definition 9: The manifest behavior $\mathcal{B}$ is *controllable*, if for any two trajectories $(y_1, u_1, p_1), (y_2, u_2, p_2) \in \mathcal{B}$ and any time instance $t \in \mathbb{T}$, there exists a $(y, u, p) \in \mathcal{B}$ and a $\mathbb{T} \ni \tau > 0$, such that $(y|_{[0,t]}, u|_{[0,t]}, p|_{[0,t]}) = (y_1|_{[0,t]}, u_1|_{[0,t]}, p_1|_{[0,t]})$, and for all $s \in \mathbb{T}$, $s \geq t + \tau$, $(y(s), u(s), p(s)) = (y_2(s-t-\tau), u_2(s-t-\tau), p_2(s-t-\tau))$. □

Intuitively, a behavior is controllable, if any i/o trajectory generated by the system up to some time can be continued by any other admissible i/o trajectory.

Theorem 3: Let $\mathcal{B}$ be a manifest behavior, and let Σ be an LPV-SSA satisfying the RC. If $\mathcal{B}$ is controllable, then Σ is a minimal realization of $\mathcal{B}$ if and only if Σ is span-reachable from zero and observable. Conversely, if Σ is span-reachable from zero, then its manifest behavior $\mathcal{B}(\Sigma)$ is controllable. □

The proof of Theorem 3 is presented in Section IV. Recall from [20] that an LPV-SSA is a minimal realization of an i/o function from the zero initial state, if and only if it is observable and span-reachable from zero. Theorem 3 says that LPV-SSAs which satisfy the RC are minimal realizations of an i/o function from zero initial state, if and only if they are minimal realizations of their own manifest behaviors.

C. Relationship with the prior results

Below we show that Theorems 2-3 are consistent with the results of [1]. To this end, recall that LPV-SSAs are special cases of meromorphic LPV-SSs. Recall from [1] the notions of structural state-observability and structural state-reachability and state-trimness and minimality.

Theorem 4: If Σ is an LPV-SSA which satisfies the RC, then it is state-trim and the following holds.

- If Σ is observable, then it is structurally state-observable.
- If Σ is span-reachable from $x_o = 0$, then it is structurally state-reachable.
- If Σ is a minimal, then it is a minimal dimensional meromorphic LPV-SS in the sense of [1]. □

The proof is presented in Section IV. Note that Theorem 4 ceases to be true, if Σ does not satisfy the RC, see [18, Example 4.1] for a counter-example. In general, there is a tradeoff between the dimensionality of LPV-SSs and the dependence on the scheduling variable (meromorphic, affine), [18, page 180]. However, for LPV-SSAs which satisfy the RC, there is no such tradeoff, i.e., the algorithms of [1], [2] will not result in smaller state-space representations when applied to such LPV-SSAs. However, they may still introduce meromorphic dependencies on the scheduling signal, see [17, Example 2].

IV. Proofs of the result

A. Auxiliary results: observability revealing scheduling

The proofs rely on the observation that for any observable LPV-SSA which satisfies the RC, there exists a scheduling signal such that the output response to that scheduling signal, under zero input, determines the initial state uniquely.

Theorem 5: Let Σ be an observable LPV-SSA which satisfies the RC. Then there exist a scheduling signal $p_o \in \mathcal{P}$ and a time instant $t_o \in \mathbb{T}$ such that for any two initial states $x_{1,o}, x_{2,o}$ of Σ,

$$\mathfrak{Y}_{\Sigma, x_{1,o}}(0, p_o)|_{[0,t_o]} = \mathfrak{Y}_{\Sigma, x_{2,o}}(0, p_o)|_{[0,t_o]} \implies x_{1,o} = x_{2,o}$$

and, in CT, p_o is analytic. □

The proof relies on viewing LPV-SSAs with zero input as bilinear systems whose inputs are the scheduling signals. Then the existence of p_o follows from the existence of a universal input for bilinear systems [23], [24]. The RC condition is necessary for using [23], [24].

Proof: [Proof of Theorem 5] Let Σ be of the form (1). It is enough to show that there exists $t_o \in \mathbb{T}$, $p_o \in \mathcal{P}$, such that if $\mathfrak{Y}_{\Sigma, x_{1,o}}(0, p_o)|_{[0,t_o]} = \mathfrak{Y}_{\Sigma, x_{2,o}}(0, p_o)|_{[0,t_o]}$, then $\mathfrak{Y}_{\Sigma, x_{1,o}}(0, p) = \mathfrak{Y}_{\Sigma, x_{2,o}}(0, p)$ for all $p \in \mathcal{P}$. Indeed, the latter equality implies $\mathfrak{Y}_{\Sigma, x_{1,o}} = \mathfrak{Y}_{\Sigma, x_{2,o}}$, and hence by observability, $x_{1,o} = x_{2,o}$, as $\mathfrak{Y}_{\Sigma, x_{i,o}}(u, p) = \mathfrak{Y}_{\Sigma, x_{i,o}}(0, p) + \mathfrak{Y}_{\Sigma, 0}(u, p)$ for all $i = 1, 2$, $u \in \mathcal{U}$. To this end, consider the bilinear system

$$\xi \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} A(p(t)) & 0 \\ C(p(t)) & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}, \quad s(t) = z(t) \tag{2}$$

with input $p \in \mathcal{P}$ and output $s \in \mathcal{Y}$. For any $p \in \mathcal{P}$, denote by $s((x_o, z_o), p)$ the output trajectory of (2) generated from the initial state $(x_o^\top, z_o^\top)^\top \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_y}$ under input p.

In CT, let us take any $t_o \in \mathbb{T}$, and let p_o be the analytic universal input from [24, Theorem 2.11] applied to (2). Note that if $\mathbb{P}$ is a convex set with a non-empty interior, then $\mathbb{P}$ satisfies [24, Condition H4] by [25, Corollary 2.3.9]. For the DT case, let p_o and t_o be such that $p_o|_{[0,t_o]}$ is the universal input $\bar{\nu}$ from [23, page 1120, proof of Theorem 5.3], applied to (2). The proof of [23, Theorem 5.3] requires observability of (2) and the following property. For any two distinct initial states of (2), and any input $p \in \mathcal{P}$, and time $t \in \mathbb{T}$, if the outputs generated from these two initial states are equal on $[0, t]$, then the states of (2) at time t reached from these initial states are distinct. The latter is assured by invertability of $A(\mathbf{p})$, $\mathbf{p} \in \mathbb{P}$. Observability of (2) follows from that of Σ.

Then, for any two initials states of (2), if the outputs from those initial states are the same on $[0, t_o]$ for the input p_o, then the outputs are the same for any input $p \in \mathcal{P}$ and any time interval. Since $\xi s((x_o, z_o)), p) = \mathfrak{Y}_{\Sigma, x_o}(0, p)$ for any $(x_o^\top, z_o^\top)^\top \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_y}$ and $p \in \mathcal{P}$, it follows that p_o and t_o satisfy the statement of the theorem. ■

Corollary 2: Assume that Σ, $\hat{\Sigma}$ satisfy the RC. There exists $p_o \in \mathcal{P}$ and $t_o \in \mathbb{T}$ such that

$$\mathfrak{Y}_{\Sigma, x_o}(0, p_o)|_{[0,t_o]} = \mathfrak{Y}_{\hat{\Sigma}, \hat{x}_o}(0, p_o)|_{[0,t_o]} \implies \Big(\forall p \in \mathcal{P} : \mathfrak{Y}_{\Sigma, x_o}(0, p) = \mathfrak{Y}_{\hat{\Sigma}, \hat{x}_o}(0, p)\Big)$$

for any initial states x_o and $\hat{x}_o$ of Σ and $\hat{\Sigma}$ respectively, □

Proof: [Proof of Corollary 2] Let Σ be as in (1) and $\hat{\Sigma} = (\mathbb{P}, \{\hat{A}_i, \hat{B}_i, \hat{C}_i, \hat{D}_i\}_{i=0}^{n_p})$. Consider the LPV-SSA

$$\Sigma_c = (\mathbb{P}, \{\begin{bmatrix} A_i & 0 \\ 0 & \hat{A}_i \end{bmatrix}, \begin{bmatrix} B_i \\ \hat{B}_i \end{bmatrix}, \begin{bmatrix} C_i & -\hat{C}_i \end{bmatrix}, D_i - \hat{D}_i\}_{i=0}^{n_p}).$$

Then Σ_c satisfies the RC, and the i/o functions of Σ_c are the differences between the i/o functions of Σ and those of $\hat{\Sigma}$. That is, for $x_{\rm o,c} = \begin{bmatrix} x_{\rm o}^\top & \hat{x}_{\rm o}^\top \end{bmatrix}^\top$, $\mathfrak{Y}_{\Sigma_c, x_{\rm o,c}} = \mathfrak{Y}_{\Sigma, x_{\rm o}} - \mathfrak{Y}_{\hat{\Sigma}, \hat{x}_{\rm o}}$. Let $\Sigma_c^{\rm O}$ be the result of applying Procedure 1 to $\Sigma_{\rm c}$. By Remark 1, $\mathfrak{Y}_{\Sigma_{\rm c}, x_{\rm o,c}} = \mathfrak{Y}_{\Sigma_c^{\rm O}, \Pi x_{\rm o,c}}$ and $\Sigma_c^{\rm O}$ satisfies the RC. Let us apply Theorem 5 to $\Sigma_c^{\rm O}$. We claim that $p_{\rm o}$, $t_{\rm o}$ satisfy the conclusion of Theorem 2. Indeed, assume that $\mathfrak{Y}_{\Sigma, x_{\rm o}}(0, p_{\rm o})|_{[0,t_{\rm o}]} = \mathfrak{Y}_{\hat{\Sigma}, \hat{x}_{\rm o}}(0, p_{\rm o})|_{[0,t_{\rm o}]}$ holds. Then the output $\mathfrak{Y}_{\Sigma_c, x_{\rm o,c}}(0, p_{\rm o})$ of $\Sigma_{\rm c}$ is zero on $[0, t_{\rm o}]$, where $x_{\rm o,c} = \begin{bmatrix} x_{\rm o}^\top & \hat{x}_{\rm o}^\top \end{bmatrix}^\top$. Hence, $\mathfrak{Y}_{\Sigma_c^{\rm O}, \Pi x_{\rm o,c}}(0, p_{\rm o})|_{[0,t_{\rm o}]} = 0 = \mathfrak{Y}_{\Sigma_c^{\rm O}, 0}(0, p_{\rm o})|_{[0,t_{\rm o}]}$. From Theorem 5, $\Pi x_{\rm o,c} = 0$. Hence, $\mathfrak{Y}_{\Sigma, x_{\rm o}}(0, p) - \mathfrak{Y}_{\Sigma, \hat{x}_{\rm o}}(0, p) = \mathfrak{Y}_{\Sigma_c, x_{\rm o,c}}(0, p) = \mathfrak{Y}_{\Sigma_c^{\rm O}, \Pi x_{\rm o,c}}(0, p) = \mathfrak{Y}_{\Sigma_c^{\rm O}, 0}(0, p) = 0$ for any $p \in \mathcal{P}$. ∎

B. Behaviors vs. i/o functions

Proof: [Proof of Theorem 1] The implication $\mathbb{F}(\Sigma) = \mathbb{F}(\hat{\Sigma}) \implies \mathcal{B}(\hat{\Sigma}) = \mathcal{B}(\Sigma)$ is obvious. We show the reverse implication. Assume that $\mathcal{B}(\hat{\Sigma}) = \mathcal{B}(\Sigma)$. Consider the scheduling signal $p_{\rm o}$ and $t_{\rm o} > 0$ from Corollary 2.

First, we show that $\mathfrak{Y}_{\Sigma,0} = \mathfrak{Y}_{\hat{\Sigma},0}$. Consider any $p \in \mathcal{P}$, $u \in \mathcal{U}$ and $\mathbb{T} \ni \tau > 0$ and define $\tilde{u}$ and $\tilde{p}$ such that for all $s \in \mathbb{T}$, $\tilde{u}(s + t_{\rm o} + \tau) = u(s)$, $\tilde{p}(s + t_{\rm o} + \tau) = p(s)$, and $\tilde{u}|_{[0,t_{\rm o}+\tau)} = 0$ and $\tilde{p}|_{[0,t_{\rm o}]} = p_{\rm o}$. Let $\tilde{y} = \mathfrak{Y}_{\Sigma,0}(\tilde{u}, \tilde{p})$ be the output of Σ generated by the zero initial state for input $\tilde{u}$ and scheduling signal $\tilde{p}$. Since $(\tilde{y}, \tilde{u}, \tilde{p})$ belongs to $\mathcal{B}(\hat{\Sigma}) = \mathcal{B}(\Sigma)$, there exists an initial state $\hat{x}_{\rm o}$ such that $\tilde{y} = \mathfrak{Y}_{\hat{\Sigma}, \hat{x}_{\rm o}}(\tilde{u}, \tilde{p})$. Since $\tilde{u}$ equals zero on $[0, t_{\rm o}]$ and $\tilde{y}$ is the output of Σ generated from the zero initial state, $\tilde{y}$ must be zero on $[0, t_{\rm o}]$. Since $\tilde{p}$ equals $p_{\rm o}$ on $[0, t_{\rm o}]$, $\mathfrak{Y}_{\hat{\Sigma}, \hat{x}_{\rm o}}(0, p_{\rm o})|_{[0,t_{\rm o}]} = \tilde{y}|_{[0,t_{\rm o}]} = 0 = \mathfrak{Y}_{\Sigma,0}(0, p_{\rm o})|_{[0,t_{\rm o}]} = 0$. From Corollary 2, it follows that $\mathfrak{Y}_{\hat{\Sigma}, \hat{x}_{\rm o}}(0, \bar{p}) = \mathfrak{Y}_{\Sigma,0}(0, \bar{p}) = 0$ for all $\bar{p} \in \mathcal{P}$. Hence, $\mathfrak{Y}_{\hat{\Sigma}, \hat{x}_{\rm o}}(\bar{u}, \bar{p}) = \mathfrak{Y}_{\hat{\Sigma}, \hat{x}_{\rm o}}(0, \bar{p}) + \mathfrak{Y}_{\hat{\Sigma},0}(\bar{u}, \bar{p}) = \mathfrak{Y}_{\hat{\Sigma},0}(\bar{u}, \bar{p})$ for all $\bar{u} \in \mathcal{U}$.

In particular, $\mathfrak{Y}_{\hat{\Sigma},0}(\tilde{u}, \tilde{p}) = \mathfrak{Y}_{\Sigma,0}(\tilde{u}, \tilde{p}) = \tilde{y}$. Let $x_f, \hat{x}_f$ be the states of Σ and $\hat{\Sigma}$ reached from the zero initial state at time $t_{\rm o} + \tau$ under input $\tilde{u}$ and scheduling signal $\tilde{p}$. Since $\tilde{u}(s + \tau + t_{\rm o}) = u(s)$, $\tilde{p} = (s + \tau + t_{\rm o}) = p(s)$, it follows that $\mathfrak{Y}_{\Sigma, x_f}(u, p)(s) = \tilde{y}(s + t_{\rm o} + \tau) = \mathfrak{Y}_{\hat{\Sigma}, \hat{x}_f}(u, p)(s)$ for all $s \in \mathbb{T}$. However, the restriction of $\tilde{u}$ to $[0, t_{\rm o} + \tau)$ is zero, and hence x_f and $\hat{x}_f$ are zero. Hence, $\mathfrak{Y}_{\hat{\Sigma},0}(u, p) = \mathfrak{Y}_{\Sigma,0}(u, p)$, and as u and p are arbitrary, $\mathfrak{Y}_{\hat{\Sigma},0} = \mathfrak{Y}_{\Sigma,0}$ follows.

Next, we show that $\mathbb{F}(\Sigma) \subseteq \mathbb{F}(\hat{\Sigma})$. To this end, let $x_{\rm o}$ be an initial state of Σ. If $y = \mathfrak{Y}_{\Sigma, x_{\rm o}}(0, p_{\rm o})$, then $(y, 0, p_{\rm o}) \in \mathcal{B}(\Sigma) = \mathcal{B}(\hat{\Sigma})$, and thus there exists an initial state $\hat{x}_{\rm o}$ of $\hat{\Sigma}$ such that $y = \mathfrak{Y}_{\hat{\Sigma}, \hat{x}_{\rm o}}(0, p_{\rm o})$. From Corollary, 2 it follows that $\mathfrak{Y}_{\Sigma, x_{\rm o}}(0, p) = \mathfrak{Y}_{\hat{\Sigma}, \hat{x}_{\rm o}}(0, p)$ for all $p \in \mathcal{P}$. Since $\mathfrak{Y}_{\Sigma, x_{\rm o}}(u, p) = \mathfrak{Y}_{\Sigma, x_{\rm o}}(0, p) + \mathfrak{Y}_{\Sigma,0}(u, p)$, $\mathfrak{Y}_{\hat{\Sigma}, \hat{x}_{\rm o}}(u, p) = \mathfrak{Y}_{\hat{\Sigma}, x_{\rm o}}(0, p) + \mathfrak{Y}_{\hat{\Sigma},0}(u, p)$, and $\mathfrak{Y}_{\hat{\Sigma},0} = \mathfrak{Y}_{\Sigma,0}$, this implies $\mathfrak{Y}_{\Sigma, x_{\rm o}} = \mathfrak{Y}_{\hat{\Sigma}, \hat{x}_{\rm o}}$. The inclusion $\mathbb{F}(\hat{\Sigma}) \subseteq \mathbb{F}(\Sigma)$ can be shown similarly. ∎

C. Minimality results

Proof: [Proof of Theorem 2] First, we show that if Σ is a minimal realization of $\mathcal{B}$, then it is observable. Assume that Σ is not observable. Let us apply Procedure 1 to Σ. Then, the resulting LPV-SSA $\Sigma^{\rm O}$ has a smaller state-space dimension than Σ. By remark 1, $\Sigma^{\rm O}$ is also a realization of $\mathcal{B}$. This contradicts the minimality of Σ.

We prove that observability implies minimality. Consider two LPV-SSA realizations Σ and $\hat{\Sigma}$ of $\mathcal{B}$ which both satisfy the RC. By Theorem 1, $\Phi = \mathbb{F}(\Sigma) = \mathbb{F}(\hat{\Sigma})$. Let $\mathfrak{S}(\Sigma)$ and $\mathfrak{S}(\hat{\Sigma})$ be the linear switched systems associated with Σ and $\hat{\Sigma}$ respectively as defined in [20, Appendix, Subsection B]. Recall from [20, Definition 2] the definition of the switched i/o function $\mathfrak{S}(\mathfrak{F})$ associated with an i/o function $\mathfrak{F} \in \Phi$, and recall that the mapping $\mathfrak{F} \mapsto \mathfrak{S}(\mathfrak{F})$ is injective. Let $\mathfrak{S}(\Phi) = \{\mathfrak{S}(\mathfrak{F}) \mid \mathfrak{F} \in \Phi\}$, $n_{\rm x} = \dim(\Sigma)$ and $\hat{n}_{\rm x} = \dim(\hat{\Sigma})$ and let $\mu : \mathfrak{S}(\Phi) \to \mathbb{R}^{n_{\rm x}}$ and $\hat{\mu} : \mathfrak{S}(\Phi) \to \mathbb{R}^{\hat{n}_{\rm x}}$ be such that for any $\mathfrak{F} \in \Phi$, the i/o functions of Σ and $\hat{\Sigma}$ induced by the initial states $\mu(\mathfrak{S}(\mathfrak{F}))$ and $\hat{\mu}(\mathfrak{S}(\mathfrak{F}))$ respectively equal $\mathfrak{F}$, i.e., $\mathfrak{Y}_{\Sigma, \mu(\mathfrak{S}(\mathfrak{F}))} = \mathfrak{Y}_{\hat{\Sigma}, \hat{\mu}(\mathfrak{S}(\mathfrak{F}))} = \mathfrak{F}$. Then, $(\mathfrak{S}(\Sigma), \mu)$, $(\mathfrak{S}(\hat{\Sigma}), \hat{\mu})$ are realizations of $\mathfrak{S}(\Phi)$ in the sense[2] of [26]. Assume that Σ is observable. Then by [20, Theorem 4], $\mathfrak{S}(\Sigma)$ is observable. Moreover, μ is surjective. Indeed, for any initial state $x_{\rm o}$, the i/o functions generated by $\mu(\mathfrak{S}(\mathfrak{Y}_{\Sigma, x_{\rm o}}))$ and $x_{\rm o}$ are the same, and hence by observability of Σ, $x_{\rm o} = \mu(\mathfrak{S}(\mathfrak{Y}_{\Sigma, x_{\rm o}}))$. Then $(\mathfrak{S}(\Sigma), \mu)$ is span-reachable and by [26], $(\mathfrak{S}(\Sigma), \mu)$ is a minimal realization of $\mathfrak{S}(\Phi)$. Hence, $\dim \mathfrak{S}(\Sigma) = \dim \Sigma \leq \dim \mathfrak{S}(\hat{\Sigma}) = \dim \hat{\Sigma}$.

Assume that Σ and $\hat{\Sigma}$ are minimal realizations of $\mathcal{B}$. By the first part of the theorem, they are observable, and hence the linear switched systems $\mathfrak{S}(\Sigma)$ and $\mathfrak{S}(\hat{\Sigma})$ are observable. From the argument of the previous paragraph, it follows that $(\mathfrak{S}(\hat{\Sigma}), \hat{\mu})$ and $(\mathfrak{S}(\Sigma), \mu)$ are minimal realizations of $\mathfrak{S}(\Phi)$. Hence, by [26], [27], they are isomorphic, and by [20, Theorem 4], Σ and $\hat{\Sigma}$ are isomorphic too. ∎

In order to prove Theorem 3, we need to relate controllability and observability of LPV-SSAs with that of *linear-time varying state-space* (LTV-SS) representations obtained from the LPV-SSA by fixing a a particular scheduling signal p. The latter LTV-SS is denoted by $\Sigma(p)$ and it is defined as

$$\Sigma(p) \begin{cases} \xi x(t) = A(t)x(t) + B(t)u(t) \\ y(t) = C(t)x(t) + D(t) \end{cases}$$

where $A(t) = A(p(t))$, $B(t) = B(p(t))$, $C(t) = C(p(t))$, $D(t) = D(p(t))$. Note that $\Sigma(p)$ depends on the scheduling signal p. Recall from [28, Section 8.3 and Section 8d.3] the notion of observability and controllability of LTV-SSs on a time interval. Then, Theorem 5 implies the following results:

Corollary 3: With the assumptions and notations of Theorem 5, the LTV-SS $\Sigma(p_{\rm o})$ is observable on $[0, t_{\rm o}]$.

Corollary 4: If Σ is span-reachable from $x_{\rm o} = 0$ and Σ satisfies the RC, then there exists $p_{\rm r} \in \mathcal{P}$ and $t_{\rm r} \in \mathbb{T}$, such that the LTV-SS $\Sigma(p_{\rm r})$ is controllable on $[0, t_{\rm r}]$.

Corollary 4 follows from Corollary 3 by duality, see [17]

Proof: [Proof of Theorem 3] If Σ is observable and span-reachable from zero, then it is minimal by Theorem 2. Conversely, assume that Σ is a minimal realization of $\mathcal{B}$. By Theorem 2 is Σ is observable. Let $x_{\rm o}$ be an initial state of Σ. Let $p_{\rm o}$, $t_{\rm o}$ be the scheduling signal and time instance from Theorem 5. Let $y = \mathfrak{Y}_{\Sigma, x_{\rm o}}(0, p_{\rm o})$.

[2] The results of [26] can readily be extended to DT by using the relationship from [27] between linear switched systems in DT and rational representations.

Then, $(y, 0, p_o) \in \mathcal{B}$. Note that $(0, 0, p_o) \in \mathcal{B}$, as $0 = \mathfrak{Y}_{\Sigma,0}(0, p_o)$. From controllability of $\mathcal{B}$ it follows that there exists $0 < \tau \in \mathbb{T}$, and an element $(\tilde{y}, \tilde{u}, \tilde{p}) \in \mathcal{B}$, such that $(\tilde{y}|_{[0,t_o]}, \tilde{u}(t)|_{[0,t_o]}, \tilde{p}|_{[0,t_o]}) = (0, 0, p_o|_{[0,t_o]})$, and $(\tilde{y}(s + t_o + \tau), \tilde{u}(s + t_o + \tau), \tilde{p}(s + t_o + \tau)) = (y(s), 0, p_o(s))$, $s \in \mathbb{T}$. Since Σ is a realization of $\mathcal{B}$, there exists an initial state $\hat{x}_o$ of Σ such that $\mathfrak{Y}_{\Sigma,\hat{x}_o}(\tilde{u}, \tilde{p}) = \tilde{y}$. It then follows that $\mathfrak{Y}_{\Sigma,\hat{x}_o}(0, p_o)|_{[0,t_o]} = 0 = \mathfrak{Y}_{\Sigma,0}(0, p_o)|_{[0,t_o]}$, and hence $\hat{x}_o = 0$. Consider the state $\hat{x}_f$ of Σ reached from the zero initial state at time $t_o + \tau$ using $\tilde{u}$, $\tilde{p}$. Then $\mathfrak{Y}_{\Sigma,\hat{x}_f}(0, p_o)(s) = \mathfrak{Y}_{\Sigma,0}(\tilde{u}, \tilde{p})(s+t_o+\tau) = \mathfrak{Y}_{\Sigma,x_o}(0, p_o)(s)$, $s \in \mathbb{T}$. By Theorem 5 $x_o = \hat{x}_f$, i.e., x_o is reachable from zero. Hence, Σ is span-reachable from zero.

Let Σ be a realization of $\mathcal{B}$ which is span-reachable from zero and which satisfies the RC. Consider $p_r \in \mathcal{P}$, $t_r \in \mathbb{T}$ from Corollary 4. Consider $(u_1, p_1, y_1), (u_2, p_2, y_2) \in \mathcal{B}$ and let $t \in \mathbb{T}$. Then, there exist initial states $x_{o,1}, x_{o,2}$ such that $y_i = \mathfrak{Y}_{\Sigma,x_{o,i}}(u_i, p_i)$, $i = 1, 2$. Let x_o be the state of Σ reached from $x_{o,1}$ at time t^+ under u_1 and p_1, where $t^+ = t$ in CT and $t^+ = t + 1$ in DT. Let $u_c \in \mathcal{U}$ and $\tau \in [0, t_r]$ be such that $x_{o,2}$ is the state of Σ reached from x_o at time τ under u_c and p_r. Since $\Sigma(p_r)$ is controllable on $[0, t_r]$, such u_c and τ exist. Define $\tilde{u}, \tilde{p}$ such that $\tilde{u}|_{[0,t]} = u_1|_{[0,t]}, \tilde{p}|_{[0,t]} = p_1|_{[0,t]}$, $\tilde{u}(s) = u_c(s - t^+), \tilde{p}(s) = p_r(s - t^+)$, $s \in (t, t + \tau)$, and $\tilde{u}(s + t + \tau) = u_2(s), \tilde{p}(s + t + \tau) = p_2(s)$, $s \in \mathbb{T}$. Let $\tilde{y} = \mathfrak{Y}_{\Sigma,x_{o,1}}(\tilde{u}, \tilde{p})$. Then, $(\tilde{u}, \tilde{p}, \tilde{y}) \in \mathcal{B}$ and $\tilde{y}|_{[0,t]} = y_1|_{[0,t]}$ and $\tilde{y}(s + t + \tau) = \mathfrak{Y}_{\Sigma,x_{o,2}}(u_2, p_2)(s) = y_2(s)$, $s \in \mathbb{T}$. Hence, $\mathcal{B}$ is controllable. ■

D. Relationship with prior work

Proof: [Proof of Theorem 4] We prove the statement for observability, the statement on span-reachability follows by duality. By Corollary 3, the LTV-SS $\Sigma(p_o)$ is observable on $[0, t_o]$. From [28, Section 8d.3, Theorem 12] it follows that in DT the t_o-step observability matrix of $\Sigma(p_o)$ is full column. From [29, Theorem 3], it follows that the $n_x - 1$-step observability matrix of $\Sigma(p_o)$ is full column rank for almost all t on $(0, t_o)$ in CT. From [1, Definition 3.34], it then follows that Σ is structurally observable. The last statement of the theorem follows from [1, Theorem 3.14]. ■

V. Conclusions

In this paper, a characterization of minimal LPV-SSA realizations of LPV behaviors in terms of observability has been presented. It has also been shown that minimal LPV-SSA realizations of the same behavior are isomorphic. These results represent the first steps towards a behavioral approach directly for LPV-SSAs. Future work will be directed towards developing i/o partitioning and kernel representations for manifest behaviors of LPV-SSAs.

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