IEEE Control Systems Letters paper presented at
2024 IEEE 63rd Conference on Decision and Control (CDC)
December 16-19, 2024. MiCo, Milan, Italy

The Mandalay Derivative For Nonsmooth Systems: Applications To Nonsmooth Control Barrier Functions

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Abstract— **One of the most challenging aspects of nonsmooth analysis is to overcome nondifferentiability. A possible approach is to use the generalized notions of the classical gradient and directional derivatives. In this paper we define a generalized directional derivative, the Mandalay derivative, based on set-valued Lie derivatives. For this operator, we derive the analogues to the classical chain rule, superposition rule (for linear combinations of functions), product rule, and quotient rule in the form of inequalities, which facilitate the computation of the Mandalay derivative in the context of nonsmooth system analysis and design. Moreover, we demonstrate the application of the Mandalay derivative for both first and high-order nonsmooth Control Barrier Functions in multiple examples.**

I. INTRODUCTION

Nonsmooth functions have been explored in the optimization and controls community with the purpose of extending classical results to broader classes of systems and functions. Nonsmooth functions have been used in applications from stability analysis using nonsmooth Lyapunov functions [1] to multiagent robotic systems [2]–[4] and hybrid systems [5], [6] to name a few.

When the function of interest is differentiable, its gradient exists and the Lie derivative is a suitable tool to use. If the function is nonsmooth, some type of generalized directional derivative, satisfying the comparison lemma, is sought. One obvious choice is the Dini derivative. However, Dini derivatives are hard to compute because their associated subdifferentials lack some desirable properties - the subdifferential can be empty at some points even for locally Lipschitz functions. It also lacks containment properties, for example, the subdifferential of the sum of two locally Lipschitz functions is not contained in the sum of their subdifferentials. The chain rule for Dini derivatives (in inequality form) is only available if one of the functions is differentiable. In this work we use the Mandalay derivative, based on Clarke's subdifferentials, which are compact, convex and nonempty when the function is locally Lipschitz. Special cases of the Mandalay derivative have been used in the nonsmooth literature [2], [7], without naming it. Given that we are the first to consider taking higher order derivatives, we define the Mandalay[1] derivative as a generalization of these previous notions and establish a systematic and shared notation.

Additionally, we derive the chain rule, superposition rule, product rule and the quotient rule of Mandalay derivatives for locally Lipschitz functions. These rules compute a bound for the Mandalay derivative of a complex function in terms of Mandalay derivatives of simpler functions. For these simpler functions, their Mandalay derivatives consist of maximums and minimums of compact intervals. The Mandalay derivative is a well suited operator for important aspects of nonsmooth system analysis and design. For example, the Mandalay derivative can be used, instead of the classical derivative, to prove system stability using nonsmooth Lyapunov functions.

In this paper we will focus on its application to CBFs. CBFs [8]–[11] enforce forward invariance of the constraint set so that no trajectory initialized within the constraint set ever leaves or violates the constraint set. Nonsmooth CBFs functions naturally arise in applications such as multiagent systems, where they take maximums and minimums of continuously differentiable functions. In such cases, and with control-affine systems, we show that using a nonsmooth CBF allows us to obtain closed form solutions to CBF-based convex optimization programs. Moreover, we also demonstrate how the Mandalay derivative can be used to derive high-order nonsmooth CBFs.

The organization of this paper is the following: Section II includes key notations and definitions, background materials from set operations, as well as a review of generalized derivatives, dynamical systems, and smooth CBFs theory. Section III presents the main results of this work and its application in examples with nonsmooth CBFs both first and high-order. Lastly, Section IV summarizes the contributions of this work to the existing nonsmooth systems literature.

II. BACKGROUND MATERIALS

A. Abbreviations and Acronyms

For a set $S \subseteq \mathbb{R}$, we use $\overline{S} = \sup S$ and $\underline{S} = \inf S$ with $\inf, \sup : 2^{\mathbb{R}} \mapsto \overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = \mathbb{R} \cup \pm\infty$. For a function $h : \mathcal{D} \to \mathbb{R}$, $\mathcal{D} \subseteq \mathbb{R}^n$, if it is continuously differentiable, we use $\nabla h(x)$ to express the gradient of h and $\dot{h}(x) = \nabla h(x)\dot{x}$ when x is understood to be a function of time $t \in \mathbb{R}$ and $\dot{x}$ denotes derivative of x with respect to time. Otherwise, if h is not differentiable, $\partial h(x) \subseteq \mathcal{D}^*$ is used to express its

This research was supported in part by the Office of Naval Research under award N00014-21-1-2759 and the National Science Foundation under award #1749357.

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[1]Mandalay is the name of a historical city in Myanmar, the birth place of one of the authors. The authors chose the name, Mandalay, to encourage participation in control research from underrepresented countries such as Myanmar.

generalized gradient set as defined in [12, Pg. 10], where $\mathcal{D}^*$ is the dual space of $\mathcal{D}$. If S is a set, ∂S denotes the boundary of S instead. Lastly, $\mathbb{R}_{>0}$ is used to express the positive real numbers and $\emptyset$ refers to the empty set.

B. Operations on Sets

The ***scalar multiplication*** of a nonempty set $S \subseteq \mathbb{R}$ with $\lambda \in \mathbb{R}$ is defined as $\lambda S = \{\lambda s \in \mathbb{R} | s \in S\}$. The ***sum of two nonempty sets*** $S_1, S_2 \subseteq \mathbb{R}$ is the Minkowski sum, $S_1 + S_2 = \{s_1 + s_2 \in \mathbb{R} | s_1 \in S_1, s_2 \in S_2\}$. The ***Linear combination of nonempty sets*** is such that given $\lambda_1, \ldots, \lambda_m$, $m \in \mathbb{N}$,

$$\sum_{i=1}^{m} \lambda_i S_i = \left\{ \sum_{i=1}^{m} \lambda_i s_i \in \mathbb{R} | s_1 \in S_i, \ldots, s_m \in S_m \right\}. \quad (1)$$

Let $\Delta = \{\{\lambda_i\}_{i=1}^m | m \in \mathbb{N}, \lambda_i \geq 0, \sum_{i=1}^m \lambda_i = 1, \forall 1 \leq i \leq m\}$. The ***convex hull*** of $S \subseteq \mathbb{R}^n$ is defined as

$$co\{S\} = \left\{ \sum_{i=1}^{m} \lambda_i s_i \in \mathbb{R}^n | \{\lambda_i\}_{i=1}^m \in \Delta, s_i \in S \right\}. \quad (2)$$

The ***product of two nonempty sets*** $X, Y \subseteq \mathbb{R}$ is defined as $X \cdot Y = \{xy \in \mathbb{R} \mid x \in X, y \in Y\}$.

Proposition 1: [13] If $a = [\underline{a}, \overline{a}]$ and $b = [\underline{b}, \overline{b}]$ are nonempty compact intervals on $\mathbb{R}$, then

$$a + b = [\underline{a} + \underline{b}, \overline{a} + \overline{b}] \quad (3)$$

$$a \cdot b = [\min\{\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b}\}, \max\{\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b}\}]. \quad (4)$$

C. Regularity, General Derivatives and Dynamical Systems

Definition 1: [12, Def. 2.3.4] Let X, Y be Banach spaces. The function $V : X \to Y$ is said to be ***regular*** at x provided that for all v, the following limit exists and coincides with the generalized directional derivative $V^\circ(x, v)$ [12, Pag. 10 Eq. 1], i.e, $\lim_{\alpha \to 0^+} \frac{V(x+\alpha v) - V(x)}{\alpha} = V^\circ(x, v)$.

Definition 2: Let $\mathcal{F} : \mathbb{R}^n \times \mathbb{R}^m \to 2^{\mathbb{R}^n}$ be a set-valued map and the function $h : \mathcal{D} \to \mathbb{R}$, $\mathcal{D} \subseteq \mathbb{R}^n$, be locally Lipschitz. The ***Lower Mandalay derivative*** of h with respect to $\mathcal{F}$ at (x', u'), $x' \in \mathcal{D}$, $u' \in \mathbb{R}^m$, is defined as

$$\underline{M}_{\mathcal{F}} h(x', u') = \inf L_{\mathcal{F}}^W h(x', u').$$

When h is both regular and locally Lipschitz, the ***Strong Lower Mandalay derivative*** is defined as

$$\underline{M}_{\mathcal{F}}^S h(x', u') = \inf L_{\mathcal{F}}^S h(x', u').$$

$L_{\mathcal{F}}^W h(x', u') = \{a \in \mathbb{R} : \exists v \in \mathcal{F}(x', u'), \exists \xi \in \partial h(x') s.t. \langle \xi, v \rangle = a\}$ and $L_{\mathcal{F}}^S h(x', u') = \{a \in \mathbb{R} : \exists v \in \mathcal{F}(x', u'), s.t. \langle \xi, v \rangle = a, \forall \xi \in \partial h(x')\}$, with $x' \in \mathcal{D}$, $u' \in \mathbb{R}^m$, are the weak and strong ***set-valued Lie derivatives*** [2] [1], [2]. Note that in the remainder of this paper, we will introduce a slight change of notation for $L_{\mathcal{F}}^W$ and use instead:

$$L_{\mathcal{F}}^W h(x', u') = \{\langle \xi, v \rangle \in \mathbb{R} \mid v \in \mathcal{F}(x', u'), \xi \in \partial h(x')\}. \quad (5)$$

[2] In [1], [2] $\mathcal{F}$ is only a function of x'. In this work $\mathcal{F}$ is a function of both x' and u' and therefore the arguments of the strong and weak set-valued Lie derivatives have been modified accordingly.

When $\inf$ is replaced by $\sup$, they will be referred to as ***Upper Mandalay derivative*** and ***Upper Strong-Mandalay derivative*** and denoted by $\overline{M}_{\mathcal{F}} h$ and $\overline{M}_{\mathcal{F}}^S h$, respectively. When $\underline{M}_{\mathcal{F}} h(x', u') = \overline{M}_{\mathcal{F}} h(x', u')$, the function is said to be Mandalay differentiable with respect to $\mathcal{F}$ and its ***Mandalay Derivative*** is denoted by $M_{\mathcal{F}} h(x', u')$. When $\underline{M}_{\mathcal{F}}^S h(x', u') = \overline{M}_{\mathcal{F}}^S h(x', u')$ the function is said to be strongly Mandalay differentiable and its ***Strong Mandalay Derivative*** will be denoted by $M_{\mathcal{F}}^S h(x', u')$. If there is no explicit dependence on u', these generalized derivatives will simply be denoted as $\underline{M}_{\mathcal{F}} h(x')$ and $\underline{M}_{\mathcal{F}}^S h(x')$, $\overline{M}_{\mathcal{F}} h(x')$ and $\overline{M}_{\mathcal{F}}^S h(x')$.

Remarks: When $\mathcal{F}$ is a singleton f and if h is continuously differentiable, the Mandalay derivative is the classical directional derivative, which can be written in the Lie derivative notation as, $M_f h(x, u) = L_f h(x, u)$. Note that both the upper and lower Mandalay derivatives can be potentially taken iteratively, e.g. $\underline{M}_{\mathcal{F}}\{\underline{M}_{\mathcal{F}} h(x)\}$ is well-defined if $\underline{M}_{\mathcal{F}} h(x)$ is locally Lipschitz. We will denote by $\underline{M}_{\mathcal{F}}^k h$ the k^{th} successive Mandalay derivative of h with respect to $\mathcal{F}$. We allow $\overline{M}_{\mathcal{F}}$ and $\underline{M}_{\mathcal{F}}$ to take values in the extended reals, therefore $\overline{M}_{\mathcal{F}}$ and $\underline{M}_{\mathcal{F}}$ always exist albeit they may take infinite values. However, when h is locally Lipschitz and $\mathcal{F}$ is compact, $L_{\mathcal{F}}^W h$ is compact and nonempty, and $\overline{M}_{\mathcal{F}}$ and $\underline{M}_{\mathcal{F}}$ are finite.

Definition 3: For $X : \mathbb{R}^d \to \mathbb{R}^d$, define the ***Filippov set-valued map*** [14, Eq. 19] $K[X] : \mathbb{R}^d \to 2^{\mathbb{R}^d}$ by

$$K[X](x) \triangleq \bigcap_{\delta > 0} \bigcap_{\mu(N)=0} \overline{co}\{X(B(x, \delta) \backslash N)\}, \quad (6)$$

where $\bigcap_{\mu(N)=0}$ denotes the intersection over all sets N of Lebesgue measure zero, $\overline{co}$ denotes convex closure, and $B(x, \delta)$ is the ball of radius δ centered at x. Note that $\overline{co}$ and co are equivalent for any compact subset of $\mathbb{R}^n$.

Definition 4: [1, Def. 6] Given a differential equation with discontinuous right hand side of the form

$$\dot{x} = X(x), \quad (7)$$

a Filippov solution of (7) on a nondegenerate interval $I \subseteq \mathbb{R}$ is a function $\varphi : I \to \mathbb{R}^n$ such that $\varphi(\cdot)$ is absolutely continuous on any interval $[t_1, t_2] \subseteq I$ and

$$\dot{\varphi}(\cdot) \in K[X](\varphi(t)) \text{ for almost all } t \in I. \quad (8)$$

According to this definition, Filippov solutions replace the right hand side of (7) by a differential inclusion defined using the operator K.

Proposition 2: [14, Prop. 3] Let $X : \mathbb{R}^d \to \mathbb{R}^d$ be measurable and locally essentially bounded, that is, bounded on a bounded neighborhood of every point, excluding sets of measure zero. Then for all $x_0 \in \mathbb{R}^d$, there exists a Filippov solution of (7) with initial condition $x(0) = x_0$.

D. Barrier Functions and Control Barrier Functions

Suppose now a closed set S defined as $S = \{x \mid h(x) \geq 0\}$, with boundary $\partial S = \{x \mid h(x) = 0\}$, for some continuously differentiable function $h(x)$, called a ***Barrier Function***, with the property that $h(x) = 0$ implies $\nabla h(x) \neq$

0. The set S is ***forward invariant*** for the system $\dot{x} = f(x)$, $x \in \mathbb{R}^n$, if for all $T > 0$, all $x_0 \in S$, and all Filippov solutions $x(t)$ on $[0, T]$ satisfying $x(0) = x_0$, it holds that $x(t) \in S$ for all $t \in [0, T]$. If, further, f is Lipschitz continuous, it holds for all $x \in \partial S$ that

$$S \text{ is forward invariant } \iff \dot{h}(x) = \nabla h(x)^T f(x) \geq 0$$

which is classically known as Nagumo's Theorem. In the barrier function literature, the righthand condition is often strengthened to

$$\dot{h}(x) \geq -\alpha(h(x)) \quad \text{for all } x \in \mathbb{R}^n \tag{9}$$

for some locally Lipschitz function $\alpha : \mathbb{R} \to \mathbb{R}$ satisfying $\alpha(0) = 0$. The advantage is that this condition, which must hold for all x rather than only on the boundary of S, more readily leads to control design techniques. For example, consider the controlled system

$$\dot{x} = f(x) + g(x)u, \tag{10}$$

with input $u \in \mathbb{R}^m$, and the goal of designing a feedback controller $\sigma(x)$ such that S is forward invariant. Then, condition (9) leads to the design criterion that any Lipschitz continuous feedback controller $\sigma(x) \in U(x)$ where

$$U(x) = \{u \mid \nabla h(x)^T (f(x) + g(x)u) \geq -\alpha(h(x))\} \tag{11}$$

ensures forward invariance of S. Notably, the inequality in the definition of $U(x)$ is affine in u and, therefore, can be included in convex optimization programs that compute a feedback controller $\sigma(x)$, possibly online at runtime. If such a feedback controller exists, then $h(x)$ is called a (classical) ***Control Barrier Function (CBF)***.

E. High-Order Control Barrier Functions

A common challenge in standard CBF-based control design is that, for many physically meaningful systems, $\nabla h(x)^T g(x)$ can be identically zero so that $U(x)$ becomes empty for some x. A possible solution is to use the theory of ***High-Order Control Barrier Functions (HO-CBF)*** [15]–[17] that systematically constructs an alternative barrier function as follows: initialize $\psi_1(x) = h(x)$ and, as long as $\nabla \psi_i(x)^T g(x) \equiv 0$, iteratively set $\psi_{i+1}(x) = \nabla \psi_i(x)^T f(x) + \alpha_i(\psi_i(x))$ for some user-chosen Lipschitz functions $\alpha_i(\cdot)$. Suppose the process terminates after r iterations. Then the resulting final $\psi_r(x)$ can often (*e.g.*, when the system has a well-defined uniform relative degree) be used as a CBF that guarantees forward invariance of $\cap_{1 \leq i \leq r} \{x \mid \psi_i(x) \geq 0\}$, which is a subset of S.

III. Main Results

In this paper we focus on systems like (7) and feedback control laws $\sigma(x)$ that make the right hand side piecewise continuous in x. Under these assumptions, Proposition 2 guarantees the existence of Filippov solutions and its Filippov set-valued map takes compact and convex values. Moreover, if $x(t)$ is a Filippov solution, it is absolutely continuous in time. As discussed previously, potential applications of the Mandalay derivative are nonsmooth CBFs and nonsmooth HO-CBFs. Given a locally Lipschitz $h : \mathcal{D} \to \mathbb{R}$, $\mathcal{D} \subseteq \mathbb{R}^n$ at x, $h(x(t))$ is also absolutely continuous in t. Under these conditions [2, Thm. 2] shows that h is a valid non-smooth CBF if there exists a $\beta \in \mathbb{R}_{>0}$ such that

$$\underline{M}_{\mathcal{F}} h(x, u) \geq -\beta h(x). \tag{12}$$

The Mandalay derivative is in general a nonlinear operator, and therefore, depending on the function h, computing the left hand side of (12) may not be straightforward. One possible solution is to express h as a composite function and bound it in terms of the Mandalay derivative of simpler component functions. To do so, in the remainder of this section, and as the main results of this paper, we derive the analogues of the classical chain rule, superposition rule (for linear combinations of functions), product rule, and quotient rule in the form of inequalities. We also demonstrate that these tools allow us to derive closed form solutions for piecewise continuous control laws under mild assumptions and even Lipschitz control laws in some special cases.

A. Mandalay Derivative Of Composition Of Two Locally Lipschitz Functions: The Chain Rule.

Lemma 1: Suppose $\mathcal{F}$ is defined as in Definition 2 and takes only nonempty, compact and convex values, and $h_1 : \mathbb{R} \to \mathbb{R}, h_2 : \mathbb{R}^n \to \mathbb{R}$ are locally Lipschitz near x. Then the product, $L^W_{\mathcal{F}} h_2(x, u) \cdot \partial h_1(h_2(x))$, is a nonempty compact interval.

Proof: If h_1, h_2 are locally Lipschitz near x, it is known that $\partial h_2(x)$ and $\partial h_1(h_2(x))$ are nonempty, compact and convex [14, Prop. 6]. Since $\partial h_1(h_2(x)) \subseteq \mathbb{R}$, it is a compact interval. If $\mathcal{F}$ takes only compact and convex values, $L^W_{\mathcal{F}} h_2(x, u)$ is the set of inner products of the elements in two compact and connected subsets of $\mathbb{R}^n$. Thus, $L^W_{\mathcal{F}} h_2(x, u)$ is compact and connected. Since $L^W_{\mathcal{F}} h_2(x, u) \subseteq \mathbb{R}$, it is a compact interval. As a product of two nonempty compact intervals, $L^W_{\mathcal{F}} h_2(x, u) \cdot \partial h_1(h_2(x))$ is a nonempty compact interval. ■

*Theorem 1 (***Chain Rule***):* Let $h_1 : \mathbb{R} \to \mathbb{R}$ and $h_2 : \mathcal{D} \to \mathbb{R}$, $\mathcal{D} \subseteq \mathbb{R}^n$, be locally Lipschitz. Define $h(x) = h_1(h_2(x))$. If $\mathcal{F}$ is defined as in Definition 2 and takes only nonempty, compact and convex values, then, the following holds $\forall x \in \mathcal{D}$

$$\begin{aligned} \underline{M}_{\mathcal{F}} h(x, u) \geq \min\{ &\underline{M}_{\mathcal{F}} h_2(x, u)\ \underline{\partial h}_1(h_2(x)), \\ &\underline{M}_{\mathcal{F}} h_2(x, u)\ \overline{\partial h}_1(h_2(x)), \\ &\overline{M}_{\mathcal{F}} h_2(x, u)\ \underline{\partial h}_1(h_2(x)), \\ &\overline{M}_{\mathcal{F}} h_2(x, u)\ \overline{\partial h}_1(h_2(x))\}. \end{aligned} \tag{13}$$

Moreover, the equality holds if $\mathcal{F}$ is a singleton, h_2 is continuously differentiable and h_1 is regular.

Proof: From Clarke's first Chain Rule [12, Thm. 2.3.9]

$$\partial h(x) \subseteq \overline{co}\{\alpha\zeta \mid \alpha \in \partial h_1(h_2(x)), \zeta \in \partial h_2(x)\}. \tag{14}$$

Note that $\overline{co}$ and co are equivalent for any compact subset of $\mathbb{R}^n$. Since $\partial h_1(h_2(x))$ and $\partial h_2(x)$ are compact, the set $\{\alpha\zeta \mid \alpha \in \partial h_1(h_2(x)), \zeta \in \partial h_2(x)\}$ is a continuous image of a compact set. Therefore, it is a compact set. Define now the set, $\mathbb{W}$, where $\mathbb{W} = \{\langle v, \theta \rangle \in \mathbb{R} \mid v \in \mathcal{F}(x,u), \theta \in co\{\alpha\zeta \mid \alpha \in \partial h_1(h_2(x)), \zeta \in \partial h_2(x)\}\}$, and note that $L^W_{\mathcal{F}} h(x,u) \subseteq \mathbb{W}$. Using (2) $\mathbb{W}$ can also be expressed as

$$\begin{aligned}\mathbb{W} =& \{\langle v, \sum_{i=1}^{m} \lambda_i \alpha_i \zeta_i \rangle \in \mathbb{R} \mid \\ & \{\lambda_i\} \in \Delta, v \in \mathcal{F}(x,u), \alpha_i \in \partial h_1(h_2(x)), \zeta_i \in \partial h_2(x)\} \\ =& \{\sum_{i=1}^{m} \lambda_i \langle v, \zeta_i \rangle \alpha_i \in \mathbb{R} \mid \\ & \{\lambda_i\} \in \Delta, v \in \mathcal{F}(x,u), \alpha_i \in \partial h_1(h_2(x)), \zeta_i \in \partial h_2(x)\}.\end{aligned} \tag{15}$$

In (15), α_i is a scalar and it can scale the bilinear product. By definition of the co, (15) is equivalent to

$$\begin{aligned}\mathbb{W} =& \bigcup_{v \in \mathcal{F}} co\{\langle v, \zeta \rangle \alpha \in \mathbb{R} \mid \alpha \in \partial h_1(h_2(x)), \zeta \in \partial h_2(x)\} \\ \subseteq & \, co\{\langle v, \zeta \rangle \alpha \in \mathbb{R} \mid v \in \mathcal{F}(x,u), \alpha \in \partial h_1(h_2(x)), \zeta \in \partial h_2(x)\}\end{aligned} \tag{16}$$

$$\begin{aligned}&= co\{\{\langle v, \zeta \rangle \in \mathbb{R} \mid v \in \mathcal{F}(x,u), \zeta \in \partial h_2(x)\} \cdot \partial h_1(h_2(x))\} \\ &= co\{L^W_{\mathcal{F}} h_2(x,u) \cdot \partial h_1(h_2(x))\}.\end{aligned} \tag{17}$$

Taking now the infimum of both sides in (17)

$$\begin{aligned}\inf \mathbb{W} \geq & \inf\{co\{L^W_{\mathcal{F}} h_2(x,u) \cdot \partial h_1(h_2(x))\}\} \\ =& \inf\{L^W_{\mathcal{F}} h_2(x,u) \cdot \partial h_1(h_2(x))\} \\ =& \min\{\underline{M}_{\mathcal{F}} h_2(x,u)\, \underline{\partial h}_1(h_2(x)), \\ & \underline{M}_{\mathcal{F}} h_2(x,u)\, \overline{\partial h}_1(h_2(x)), \\ & \overline{M}_{\mathcal{F}} h_2(x,u)\, \underline{\partial h}_1(h_2(x)), \\ & \overline{M}_{\mathcal{F}} h_2(x,u)\, \overline{\partial h}_1(h_2(x))\}.\end{aligned} \tag{18}$$

Equation (18) follows from Lemma 1 and Proposition 1. Since $L^W_{\mathcal{F}} h(x,u) \subseteq \mathbb{W}$, $\underline{M}_{\mathcal{F}} h(x,u) = \inf\{L^W_{\mathcal{F}} h(x,u)\} \geq \inf \mathbb{W}$, the result follows. If $\mathcal{F}$ is a singleton, equality is obtained in (16). If h_1 is regular and h_2 is continuously differentiable, by [12, Thm. 2.3.9], equality holds in (14) and $L^W_{\mathcal{F}} h(x,u) = \mathbb{W}$. ∎

Proposition 3: Consider the system in (10), with piecewise continuous feedback controller $\sigma(x)$, and let $\mathcal{F}(x, \sigma(x))$ be the Filippov operator generated. Suppose $\dot{x}_i = f_i(x)$ are continuous $\forall i \in I \subset \{1, \ldots, n\}$, and also suppose $h : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable in x and $\frac{\partial h}{\partial x_i} = 0, \forall i \in I^C$, then h is Mandalay differentiable with respect to $\mathcal{F}$ and $M_{\mathcal{F}} h(x) = \dot{h}$.

Proof: For any $v \in \mathcal{F}$,

$$\begin{aligned}\langle v, \nabla h \rangle =& \sum_{i=1}^{n} v_i \frac{\partial h}{\partial x_i} = \sum_{i \in I} \dot{x}_i \frac{\partial h}{\partial x_i} + \sum_{i \in I^C} v_i \cancelto{0}{\frac{\partial h}{\partial x_i}} \\ =& \sum_{i=1}^{n} \dot{x}_i \frac{\partial h}{\partial x_i} = \nabla h \dot{x} = \dot{h}.\end{aligned} \tag{19}$$

As h is continuously differentiable in x, $\partial h(x) = \{\nabla h(x)\}$. The weak set-valued Lie derivative (5) of h equals

$$L^W_{\mathcal{F}} h(x) = \{\langle v, \nabla h(x) \rangle \in \mathbb{R} \mid v \in \mathcal{F}(x, \sigma(x))\}.$$

From (19) $\langle v, \nabla h(x) \rangle = \dot{h}$ for all $v \in \mathcal{F}(x, \sigma(x))$. Thus, $L^W_{\mathcal{F}} h(x) = \{\dot{h}\}$, which yields $\underline{M}_{\mathcal{F}} h(x) = \overline{M}_{\mathcal{F}} h(x) = M_{\mathcal{F}} h(x) = \dot{h}$. ∎

Thanks to Proposition 3, Theorem 1 can be now simplified as shown in Corollary 1.

Corollary 1: In Theorem 1, if h_2 is Mandalay differentiable, $\forall x \in \mathcal{D}$

$$M_{\mathcal{F}} h(x,u) \geq \begin{cases} M_{\mathcal{F}} h_2(x,u)\, \underline{\partial h}_1(h_2(x)), & M_{\mathcal{F}} h_2(x,u) \geq 0 \\ M_{\mathcal{F}} h_2(x,u)\, \overline{\partial h}_1(h_2(x)), & M_{\mathcal{F}} h_2(x,u) < 0. \end{cases}$$

Proof: If h_2 is Mandalay differentiable, $\underline{M}_{\mathcal{F}} h_2(x) = \overline{M}_{\mathcal{F}} h_2(x) = M_{\mathcal{F}} h_2(x)$. Thus, by Theorem 1, $\forall x \in \mathcal{D}$,

$$\begin{aligned}M_{\mathcal{F}} h(x,u) \geq & \min\{M_{\mathcal{F}} h_2(x,u)\, \underline{\partial h}_1(h_2(x)), \\ & M_{\mathcal{F}} h_2(x,u)\, \overline{\partial h}_1(h_2(x))\}\end{aligned}$$

$$M_{\mathcal{F}} h(x,u) \geq \begin{cases} M_{\mathcal{F}} h_2(x,u)\, \underline{\partial h}_1(h_2(x)), & M_{\mathcal{F}} h_2(x,u) \geq 0 \\ M_{\mathcal{F}} h_2(x,u)\, \overline{\partial h}_1(h_2(x)), & M_{\mathcal{F}} h_2(x,u) < 0. \end{cases}$$

∎

B. Application: Lipschitz CBF Using Chain Rule

Consider the system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u \tag{20}$$

which satisfies Proposition 3. Consider as well a candidate nonsmooth CBF $h = L - x_2 - \mid x_1^2 - x_1 \mid$, with $L \in \mathbb{R}_{>0}$. h can also be expressed as $h = h_0 + h_1(h_2)$, where $h_0 = L - x_2$ is continuously differentiable and $h_1(h_2) = - \mid x_1^2 - x_1 \mid = - \mid h_2 \mid$, which satisfies the set up of Theorem 1. Thus, at points where $x_1^2 - x_1 = 0$, $\partial h_1 = [-1, 1]$, $\partial h_2 = \nabla h_2$, and $\partial h_0 = \nabla h_0$. Also note that h_2 satisfies Proposition 3, which implies $M_{\mathcal{F}} h_2(x) = \dot{h}_2$. Using now Corollary 1, at points where $x_1^2 - x_1 = 0$,

$$M_{\mathcal{F}}(h_1(h_2))(x) \geq \begin{cases} -\dot{h}_2(x), & \dot{h}_2(x) \geq 0 \\ \dot{h}_2(x), & \dot{h}_2(x) < 0, \end{cases} \tag{21}$$

which yields, $M_{\mathcal{F}}(h_1(h_2))(x) \geq -|\dot{h}_2(x)|$. The Mandalay derivative of h is therefore

$$M_{\mathcal{F}} h(x,u) \geq \begin{cases} -u - (2x_1 x_2 - x_2), & x_1^2 - x_1 > 0 \\ -u + (2x_1 x_2 - x_2), & x_1^2 - x_1 < 0 \\ -u - |2x_1 x_2 - x_2|, & x_1^2 - x_1 = 0. \end{cases} \tag{22}$$

It is clear that a piecewise continuous feedback controller satisfying $M_{\mathcal{F}} h(x,u) \geq -\beta h(x)$ can be derived from (22). As we have an affine system, given a Lipschitz feedback controller u_{nom}, we can choose $u(x)$ to minimize $\|u(x) - u_{nom}(x)\|^2$ subject to $u(x) \in U(x) = \{u \mid M_{\mathcal{F}} h \geq -\beta h\}$, $\beta \in \mathbb{R}_{>0}$. The results are shown in Figure 1, in which the safe region is the interior of the two intersected parabolas. The trajectory using h as a CBF successfully remains within the safe region (solid red line).

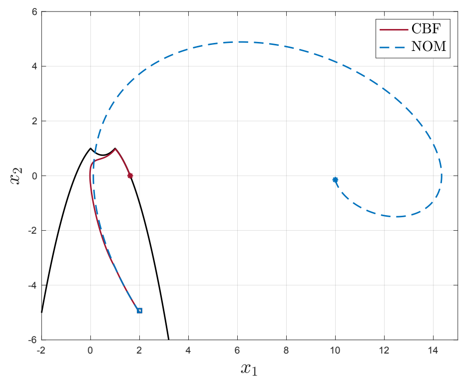

Fig. 1. The safe region corresponds to the inside area delimited by the solid black line. The initial condition of the system is represented with a square, and the final position with an asterisk. The nominal trajectory (dashed blue line), without the barrier function filter violates safety whereas the trajectory obtained with the Lipschitz CBF (solid red line) successfully avoids the unsafe region.

C. Mandalay Derivative Of Linear Combinations Of Locally Lipschitz Functions: The Superposition Rule.

Lemma 2: Let $h_i : \mathcal{D} \to \mathbb{R}$, $\mathcal{D} \subseteq \mathbb{R}^n$ be locally Lipschitz $\forall 1 \leq i \leq N$. Let $h = \sum_{i=1}^{N} s_i h_i$, $s_i > 0, \forall 1 \leq i \leq N$. If $\mathcal{F}$ is defined as in Definition 2 and takes only nonempty, compact and convex values, the following inequality holds $\forall x \in \mathcal{D}$

$$\underline{M}_{\mathcal{F}} h(x,u) \geq \sum_{i=1}^{N} s_i \underline{M}_{\mathcal{F}} h_i(x,u). \tag{23}$$

Proof: First, define the set $\mathbb{W}$

$$\begin{aligned}
\mathbb{W} &= \{\langle v, \xi \rangle \in \mathbb{R} \mid v \in \mathcal{F}(x,u), \xi \in \sum_{i=1}^{N} s_i \partial h_i(x)\} \\
&= \{\langle v, \sum_{i=1}^{N} s_i \xi_i \rangle \in \mathbb{R} \mid v \in \mathcal{F}(x,u), \xi_i \in \partial h_i(x)\} \\
&\subseteq \sum_{i=1}^{N} s_i \{\langle v, \xi_i \rangle \in \mathbb{R} \mid v \in \mathcal{F}(x,u), \xi_i \in \partial h_i(x)\} \quad (24)\\
&= \sum_{i=1}^{N} s_i L_{\mathcal{F}}^{W} h_i(x,u). \quad (25)
\end{aligned}$$

Since all $s_i > 0$, taking the infimum of both sides in (25)

$$\inf \mathbb{W} \geq \sum_{i=1}^{N} s_i \inf L_{\mathcal{F}}^{W} h_i(x,u) = \sum_{i=1}^{N} s_i \underline{M}_{\mathcal{F}} h_i(x,u).$$

According to [12, Cor. 2, Pg. 39]:

$$\partial h(x) \subseteq \sum_{i=1}^{N} s_i \partial h_i(x). \tag{26}$$

Thus, $L_{\mathcal{F}}^{W} h(x,u) \subseteq \mathbb{W}$, which implies $\inf L_{f}^{W} h(x,u) \geq \inf \mathbb{W}$, and $\underline{M}_{\mathcal{F}} h(x,u) \geq \sum_{i=1}^{N} s_i \underline{M}_{\mathcal{F}} h_i(x,u)$. ■

Corollary 2: In Lemma 2, if $s_i < 0, \forall 1 \leq i \leq N$. Then the following inequality holds $\forall x \in \mathcal{D}$

$$\underline{M}_{\mathcal{F}} h(x,u) \geq \sum_{i=1}^{N} s_i \overline{M}_{\mathcal{F}} h_i(x,u). \tag{27}$$

Proof: The proof is exactly the same as that of Lemma 2 noting that when a nonempty compact set $S \subseteq \mathbb{R}$ is multiplied by a negative number, inf becomes sup. ■

Theorem 2: Let $h_i : \mathcal{D} \to \mathbb{R}$, $\mathcal{D} \subseteq \mathbb{R}^n$ be locally Lipschitz $\forall 1 \leq i \leq N$. Let $h = \sum_{i=1}^{N} s_i h_i$, where $s_i > 0, \forall i \in I \subseteq \{1, \dots, N\}$ and $s_i < 0, \forall i \in I^C = \{1, \cdots, N\} \setminus I$. If $\mathcal{F}$ is defined as in Definition 2 and takes only nonempty, compact and convex values, the following inequality holds $\forall x \in \mathcal{D}$

$$\underline{M}_{\mathcal{F}} h(x,u) \geq \sum_{i \in I} s_i \underline{M}_{\mathcal{F}} h_i(x,u) + \sum_{i \in I^C} s_i \overline{M}_{\mathcal{F}} h_i(x,u).$$

Moreover, equality holds if $\mathcal{F}$ is a singleton, I^C is empty and h_i are regular $\forall i \in I$.

Proof: The proof follows directly from Lemma 2 and Corollary 2. The claim of equality can be readily verified by noting that the equality holds in (24) if $\mathcal{F}$ is a singleton, and also in (26) for regular h_i and $s_i > 0$, according to [12, Cor. 3, Pg. 40]. ■

D. Application: Recursive Nonsmooth HO-CBFs

In this example we now show how the property of Mandalay derivative in Theorem 2 is particularly helpful to define HO-CBFs that are nonsmooth and Lipschitz recursively. Given the system:

$$\dot{x}_1 = -x_1^3, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = u \tag{28}$$

and the candidate nonsmooth CBF $h = L - x_2 + \mid x_1 \mid$ with $L \in \mathbb{R}_{>0}$, we can observe that the relative degree between h and u is two, and thus the system requires a HO-CBF. This new HO-CBF is built as, $\psi_1 = L - x_2 + \mid x_1 \mid$, $\psi_2 = M_f \psi_1 + \beta_1 \psi_1$, $\beta_1 \in \mathbb{R}_{>0}$. Taking Mandalay derivative of ψ_2 yields, $M_f \psi_2 = M_f(M_f \psi_1 + \beta_1 \psi_1)$. From Theorem 2 we know that

$$M_f(M_f \psi_1 + \beta_1 \psi_1) \geq M_f^2 \psi_1 + \beta_1 M_f \psi_1. \tag{29}$$

The right hand side is easier to compute and in this case yields, $\forall x \in \mathcal{D}$, $M_f \psi_1 = -x_3 - \mid x_1^3 \mid$, $M_f^2 \psi_1 = -u + \mid 3x_1^5 \mid$. We can now conclude that, if $M_f^2 \psi_1 + \beta_1 M_f \psi_1 \geq -\beta_2 \psi_2$ then the CBF inequality $M_f \psi_2 \geq -\beta_2 \psi_2$ is also satisfied. As we have an affine system, given a Lipschitz feedback controller u_{nom}, we can choose $u(x)$ to minimize $\|u(x) - u_{nom}(x)\|^2$ subject to $u(x) \in U(x) = \{u \mid M_f^2 \psi_1 + \beta_1 M_f \psi_1 \geq -\beta_2 \psi_2\}$, $\beta_1, \beta_2 \in \mathbb{R}_{>0}$. The results obtained are shown Figure 2. The safe region corresponds to the outside of the inverted triangle. The trajectory obtained with the safe controller u successfully avoids violating safety (red solid trajectory).

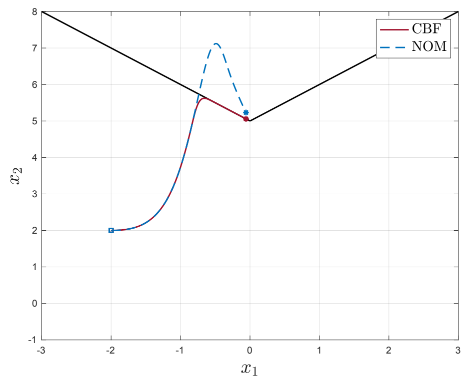

Fig. 2. The safe region corresponds to the outside of the inverted triangle. The initial condition of the system is represented with a square, and the final position with an asterisk. The nominal trajectory (dashed blue line), without the barrier function filter violates safety whereas the trajectory obtained with the Lipschitz CBF (solid red line) successfully avoids the unsafe region.

E. Product and Quotient Rules

Lastly, we derive the Mandalay derivative of products and quotients of Lipschitz functions.

Theorem 3: Suppose $\mathcal{F}$ is defined as in Definition 2 and takes only nonempty, compact and convex values. Let $h_1, h_2 : \mathbb{R}^n \to \mathbb{R}$, $\mathcal{D} \subseteq \mathbb{R}^n$ be locally Lipschitz at a point $x' \in \mathcal{D}$. Then $h = h_1 \cdot h_2$ satisfies

$$\begin{aligned}
&\underline{M}_{\mathcal{F}} h(x', u) \geq \\
&\quad \min\{h_2(x')\underline{M}_{\mathcal{F}} h_1(x', u) + h_1(x')\underline{M}_{\mathcal{F}} h_2(x', u), \\
&\quad h_2(x')\underline{M}_{\mathcal{F}} h_1(x', u) + h_1(x')\overline{M}_{\mathcal{F}} h_2(x', u), \\
&\quad h_2(x')\overline{M}_{\mathcal{F}} h_1(x', u) + h_1(x')\underline{M}_{\mathcal{F}} h_2(x', u), \\
&\quad h_2(x')\overline{M}_{\mathcal{F}} h_1(x', u) + h_1(x')\overline{M}_{\mathcal{F}} h_2(x', u)\}. \qquad (30)
\end{aligned}$$

Moreover, equality holds if $\mathcal{F}$ is a singleton, h_1, h_2 are regular at x' and $h_1(x') > 0, h_2(x') > 0$.

Proof Sketch 1: Using [14, Pag. 21 Eq. 40] given locally Lipschitz functions $h_1, h_2 : \mathbb{R}^n \to \mathbb{R}$ at a point $x' \in \mathbb{R}^n$, the composite function $h_3 = h_1 \cdot h_2$ satisfies $L^W_{\mathcal{F}} h_3(x', u) \subseteq \mathbb{S}(x', u)$, where $\mathbb{S}(x', u) = \{\langle v, \xi \rangle \in \mathbb{R} \mid v \in \mathcal{F}(x'),\ \xi \in (h_2(x')\partial h_1(x') + h_1(x')\partial h_2(x'))\}$. Therefore $\underline{M}_{\mathcal{F}} h_3(x', u) \geq \inf \mathbb{S}(x', u)$.

Theorem 4: Suppose $\mathcal{F}$ is defined as in Definition 2 and takes only nonempty, compact and convex values. Let $h_1, h_2 : \mathbb{R}^n \to \mathbb{R}$, $\mathcal{D} \subseteq \mathbb{R}^n$ be locally Lipschitz at a point $x' \in \mathcal{D}$ and $h_2(x') \neq 0$. Then $h = \frac{h_1}{h_2}$ satisfies

$$\begin{aligned}
&\underline{M}_{\mathcal{F}} h(x', u) \geq \qquad (31) \\
&\frac{1}{h_2(x')^2} \min\{h_2(x')\underline{M}_{\mathcal{F}} h_1(x', u) - h_1(x')\underline{M}_{\mathcal{F}} h_2(x', u), \\
&\qquad h_2(x')\underline{M}_{\mathcal{F}} h_1(x', u) - h_1(x')\overline{M}_{\mathcal{F}} h_2(x', u), \\
&\qquad h_2(x')\overline{M}_{\mathcal{F}} h_1(x', u) - h_1(x')\underline{M}_{\mathcal{F}} h_2(x', u), \\
&\qquad h_2(x')\overline{M}_{\mathcal{F}} h_1(x', u) - h_1(x')\overline{M}_{\mathcal{F}} h_2(x', u)\}.
\end{aligned}$$

Moreover, equality holds when $\mathcal{F}$ is a singleton, h_1 and $-h_2$ are regular at x' and $h_1(x') > 0$ and $h_2(x') > 0$.

Proof Sketch 2: The proof can be derived similarly to the proof of Theorem 3 but using [14, Pag. 21 Eq. 41] instead.

IV. CONCLUSIONS

In this paper we presented the Mandalay derivative, a generalized directional derivative, based on set-valued Lie derivatives to handle nondifferentiability in nonsmooth analysis. We derived the analogues to the classical chain rule, superposition rule, product rule, and quotient rule in the form of inequalities. Having a new set of rules to compute the Mandalay derivative of more complex functions facilitates using nonsmooth functions for system analysis and design, as demonstrated in our applications with nonsmooth CBFs and HO-CBFs. Additionally, our results are derived from well-established interval operations and connect the Mandalay derivative to the emerging field of interval arithmetic.

REFERENCES

[1] A. Bacciotti and F. Ceragioli, "Stability and stabilization of discontinuous systems and nonsmooth lyapunov functions," *ESAIM: Control, Optimisation and Calculus of Variations*, vol. 4, pp. 361–376, 1999.

[2] P. Glotfelter, J. Cortés, and M. Egerstedt, "Nonsmooth barrier functions with applications to multi-robot systems," *IEEE control systems letters*, vol. 1, no. 2, pp. 310–315, 2017.

[3] ——, "A nonsmooth approach to controller synthesis for boolean specifications," *IEEE Transactions on Automatic Control*, vol. 66, no. 11, pp. 5160–5174, 2020.

[4] P. Glotfelter, I. Buckley, and M. Egerstedt, "Hybrid nonsmooth barrier functions with applications to provably safe and composable collision avoidance for robotic systems," *IEEE Robotics and Automation Letters*, vol. 4, no. 2, pp. 1303–1310, 2019.

[5] M. Maghenem and R. G. Sanfelice, "Sufficient conditions for forward invariance and contractivity in hybrid inclusions using barrier functions," *Automatica*, vol. 124, p. 109328, 2021.

[6] ——, "Characterizations of safety in hybrid inclusions via barrier functions," in *Proceedings of the 22nd ACM International Conference on Hybrid Systems: Computation and Control*, 2019, pp. 109–118.

[7] A. Bacciotti and F. Ceragioli, "Nonpathological lyapunov functions and discontinuous carathéodory systems," *Automatica*, vol. 42, no. 3, pp. 453–458, 2006.

[8] A. D. Ames, J. W. Grizzle, and P. Tabuada, "Control barrier function based quadratic programs with application to adaptive cruise control," in *53rd IEEE Conference on Decision and Control*. IEEE, 2014, pp. 6271–6278.

[9] A. D. Ames, S. Coogan, M. Egerstedt, G. Notomista, K. Sreenath, and P. Tabuada, "Control barrier functions: Theory and applications," in *2019 18th European control conference (ECC)*. IEEE, 2019, pp. 3420–3431.

[10] S. Prajna, A. Jadbabaie, and G. J. Pappas, "A framework for worst-case and stochastic safety verification using barrier certificates," *IEEE Transactions on Automatic Control*, vol. 52, no. 8, pp. 1415–1428, 2007.

[11] W. Xiao, C. G. Cassandras, and C. Belta, *Safe Autonomy with Control Barrier Functions: Theory and Applications*. Springer Nature, 2023.

[12] F. H. Clarke, *Optimization and nonsmooth analysis*. SIAM, 1990.

[13] G. Alefeld and G. Mayer, "Interval analysis:theory and applications," *journal of computational and applied mathematics*, vol. 121, p. 421–464, 1999.

[14] J. Cortes, "Discontinuous dynamical systems," *IEEE Control systems magazine*, vol. 28, no. 3, pp. 36–73, 2008.

[15] W. Xiao and C. Belta, "Control barrier functions for systems with high relative degree," in *2019 IEEE 58th conference on decision and control (CDC)*. IEEE, 2019, pp. 474–479.

[16] X. Tan, W. S. Cortez, and D. V. Dimarogonas, "High-order barrier functions: Robustness, safety, and performance-critical control," *IEEE Transactions on Automatic Control*, vol. 67, no. 6, pp. 3021–3028, 2021.

[17] W. Xiao and C. Belta, "High-order control barrier functions," *IEEE Transactions on Automatic Control*, vol. 67, no. 7, pp. 3655–3662, 2021.