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Data-Driven Stability Certificate of Interconnected Homogeneous Networks via ISS Properties

Abolfazl Lavaei, *Senior Member, IEEE*, David Angeli, *Fellow, IEEE*

***Abstract*— This letter is concerned with a *compositional data-driven* approach for stability certificate of interconnected homogeneous networks with (partially) unknown dynamics while providing 100% correctness guarantees (as opposed to probabilistic confidence). The proposed framework enjoys input-to-state stability (ISS) properties of subsystems described by ISS Lyapunov functions. In our data-driven scheme, we first reformulate the corresponding conditions of ISS Lyapunov functions as a robust optimization program (ROP). Due to appearing unknown dynamics of subsystems in the constraint of ROP, we propose a scenario optimization program (SOP) by collecting data from trajectories of each unknown subsystem. We solve SOP and construct an ISS Lyapunov function for each subsystem with unknown dynamics. We accordingly leverage a compositional technique based on $\max$-*type small-gain reasoning* and construct a Lyapunov function for an unknown interconnected network based on ISS Lyapunov functions of individual subsystems. We demonstrate the efficacy of our data-driven approach over a room temperature network containing 1000 rooms with unknown dynamics. Given collected data from each unknown room, we verify that the unknown interconnected network is globally asymptotically stable (GAS) with 100% correctness guarantee.**

I. Introduction

In the past few years, data-driven optimization approaches have received remarkable attentions due to their ubiquitous applications in real-life engineering systems. In particular, given that closed-form mathematical models for many real-world systems are not available in general, two different types of *direct and indirect* data-driven techniques have been proposed in the relevant literature for formal verification and controller synthesis of complex systems with unknown dynamics. More specifically, *indirect data-driven techniques* are those which leverage system identification to learn approximate models of unknown systems, followed by model-based controller analysis approaches. On the downside, most identification techniques are mainly limited to linear or some specific classes of nonlinear systems, and accordingly, acquiring an accurate model for complex systems via those indirect techniques could be complicated, time-consuming and expensive (*e.g.,* [1, and references herein]). In comparison, *direct data-driven techniques* are those that bypass the identification stage and directly leverage measurements to provide a verification and controller design framework for unknown complex systems.

A. Lavaei is with the School of Computing, Newcastle University, United Kingdom. D. Angeli is with the Department of Electrical and Electronic Engineering, Imperial College London, United Kingdom, and with the Department of Information Engineering, University of Florence, Italy. Email: abolfazl.lavaei@newcastle.ac.uk, d.angeli@imperial.ac.uk.

State-of-the-Art. In the past two decades, there have been several works on data-driven optimization techniques. Existing results include proposing a *scenario approach* to solve a semi-infinite convex program by randomly sampling from constraints [2]; presenting a random convex program with a quantified bound on the upper-tail probability of violation [3]; establishing a probabilistic bridge between optimal values of scenario and robust convex programs [4]; and establishing an approximation bridge between an infinite-dimensional linear programming and a finite convex program [5], to name a few. There have been also several works on data-driven verification and controller synthesis of complex systems with unknown models. Existing results include stability analysis based on data for unknown linear switched systems via constructing, respectively, common and sum of squares Lyapunov functions [6], [7]; data-driven stability verification of continuous-time unknown systems [8]; a predictive control scheme for linear systems with noise-corrupted input/output data [9]; a data-driven technique for stability analysis of homogeneous systems [10]; and a data-driven scheme based on persistency of excitation for stability analysis of linear-feedback systems [11].

In comparison with our work, we propose here, for the first time, a *compositional data-driven* technique for the stability certificate of *interconnected* homogeneous networks with unknown dynamics, whereas the results in [6]- [11] are all tailored to monolithic systems and suffer severely from the, so-called, *sample complexity* when dealing with large-dimensional systems. In addition, the stability guarantee in [6]- [10] is provided over unknown systems with some probabilistic level of confidence, whereas the stability certificate in our work is offered with 100% *correctness guarantees* under some Lipschitz continuity assumptions. A similar idea of utilizing such an assumption but in the context of constructing control barrier certificates from data has been recently used in [12].

Contribution. Our main contribution is to develop a *compositional data-driven technique* for the stability certificate of interconnected homogeneous networks with unknown dynamics. The proposed scheme leverages input-to-state stability (ISS) properties of subsystems described by ISS Lyapunov functions. In our data-driven framework, we first cast conditions of ISS Lyapunov functions as a robust optimization program (ROP). Due to unknown models appearing in the constraint of ROP, we collect data from each unknown subsystem and provide a scenario optimization program (SOP) pertaining to ROP. We solve SOP and construct an ISS Lyapunov function for each unknown

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subsystem with 100% correctness guarantee. We accordingly utilize a compositional technique based on max-*type small-gain reasoning* to construct a Lyapunov function for the interconnected network via ISS Lyapunov functions of individual subsystems. We demonstrate the efficacy of our data-driven approach over an unknown room temperature network containing 1000 rooms.

II. Problem Description

A. Notation and Preliminaries

Symbols $\mathbb{R}, \mathbb{R}^+$, and $\mathbb{R}_0^+$, respectively, denote sets of real, positive, and non-negative real numbers. Sets of non-negative and positive integers are represented, respectively, by $\mathbb{N} := \{0, 1, 2, \dots\}$ and $\mathbb{N}^+ = \{1, 2, \dots\}$. Considering N vectors $x_i \in \mathbb{R}^{n_i}$, we denote a column vector of dimension $\sum_i n_i$ by $x = [x_1; \dots; x_N]$. Cartesian product of sets $X_i, i \in \{1, \dots, N\}$, is represented by $\prod_{i=1}^N X_i$. Given any symmetric matrix A, its maximum eigenvalue is represented by $\lambda_{\max}(A)$. A function $\beta : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a class $\mathcal{K}$ function if it is continuous, strictly increasing, and $\beta(0) = 0$. A class $\mathcal{K}$ function $\beta(\cdot)$ is a class $\mathcal{K}_\infty$ if $\beta(s) \to \infty$ as $s \to \infty$. A function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ belongs to class $\mathcal{KL}$ if, for each fixed s, the map $\beta(r, s)$ is a class $\mathcal{K}$ with respect to r and, for each fixed $r > 0$, the map $\beta(r, s)$ is decreasing with respect to s, and $\beta(r, s) \to 0$ as $s \to \infty$.

B. Discrete-Time Nonlinear Systems

In this letter, we study interconnected systems composed of individual discrete-time nonlinear subsystems, as represented in the following definition.

Definition 2.1: A discrete-time nonlinear subsystem (dt-NS) is characterized by the tuple

$$\Upsilon_i = (X_i, W_i, f_i), \tag{1}$$

where:

- $X_i \subseteq \mathbb{R}^{n_i}$ is the state set of dt-NS;
- $W_i \subseteq \mathbb{R}^{p_i}$ is the internal input set of dt-NS;
- $f_i : X_i \times W_i \to X_i$ is a continuous function characterizing the evolution of the system. We assume f_i is a homogeneous function of degree one, *i.e.,* for any $\eta > 0$ and $x_i \in X_i, w_i \in W_i$, $f_i(\eta x_i, \eta w_i) = \eta f_i(x_i, w_i)$ [13]. We also assume that the map f_i is unknown to us.

For an initial state $x_i(0) \in X_i$ and an internal input sequence $w_i(\cdot) : \mathbb{N} \to W_i$, the evolution of dt-NS Υ_i can be described as

$$\Upsilon_i : x_i(k+1) = f_i(x_i(k), w_i(k)), \qquad k \in \mathbb{N}. \tag{2}$$

The sequence $\xi_i : \mathbb{N} \to X_i$ satisfying (2) is called the *solution process* of Υ_i under $w_i(\cdot)$ and $x_i(0)$.

Given that the ultimate goal is to ensure stability certificate of unknown interconnected dt-NS, we provide next a formal definition for interconnected dt-NS without w that can be characterized as a composition of individual dt-NS with w.

Definition 2.2: Consider $\mathcal{M} \in \mathbb{N}^+$ dt-NS $\Upsilon_i = (X_i, W_i, f_i)$, $i \in \{1, \dots, \mathcal{M}\}$, with their inputs partitioned as

$$w_i = [w_{i1}; \dots; w_{i(i-1)}; w_{i(i+1)}; \dots; w_{i\mathcal{M}}]. \tag{3}$$

The interconnection of $(\Upsilon_i)_{i=1}^{\mathcal{M}}$ is $\Upsilon = (X, f)$, denoted by $\mathcal{I}(\Upsilon_1, \dots, \Upsilon_{\mathcal{M}})$, where $X = \Pi_{i=1}^{\mathcal{M}} X_i$ and $f(x) := [f_1(x_1, w_1); \dots; f_{\mathcal{M}}(x_{\mathcal{M}}, w_{\mathcal{M}})]$, with the following interconnection constraint:

$$\forall i, j \in \{1, \dots, \mathcal{M}\}, i \neq j: \quad w_{ij} = x_j, \quad X_j \subseteq W_{ij}, \tag{4}$$

where $W_i := \prod_{j \neq i} W_{ij}$. The resulting interconnected dt-NS can be described by

$$\Upsilon : x(k+1) = f(x(k)), \quad \text{with } f : X \to X. \tag{5}$$

It is straightforward that since all $\Upsilon_i, i \in \{1, \dots, \mathcal{M}\}$, are homogeneous of degree one, the interconnected dt-NS Υ remains homogeneous with the same degree, *i.e.,* for any $\eta > 0$ and $x \in X$, $f(\eta x) = \eta f(x)$.

In the following, we define global asymptotic stability of interconnected dt-NS Υ.

Definition 2.3: An interconnected dt-NS Υ in (5) is *globally asymptotically stable (GAS)* if

$$\|x(k)\| \leq \beta(\|x(0)\|, k),$$

for any $x(0) \in \mathbb{R}^n$ and some $\beta \in \mathcal{KL}$, implying in particular that all solutions of Υ converge to the origin when $k \to \infty$.

We now present the next theorem [13] to show under which conditions the interconnected dt-NS is GAS. Notice that for homogeneous systems, GAS is equivalent to exponential stability.

Theorem 2.4: Given an interconnected dt-NS $\Upsilon = (X, f)$, suppose there exist a homogeneous Lyapunov function $\mathcal{V} : X \to \mathbb{R}_0^+$ of degree $\kappa \in \mathbb{N}^+$, *i.e.,* for any $\eta > 0$ and $x \in X$, $\mathcal{V}(\eta x) = \eta^\kappa \mathcal{V}(x)$, and constants $\underline{\alpha}, \overline{\alpha} \in \mathbb{R}^+$, $\gamma \in (0, 1)$ such that $\forall x \in X$:

$$\underline{\alpha}\|x\|^\kappa \leq \mathcal{V}(x) \leq \overline{\alpha}\|x\|^\kappa, \tag{6a}$$

$$\mathcal{V}(f(x)) - \gamma \mathcal{V}(x) \leq 0. \tag{6b}$$

Then the interconnected dt-NS Υ is globally asymptotically stable (GAS).

Remark 2.5: Note that homogeneity of f in Theorem 2.4 implies that if the underlying system is GAS, then there exists a homogeneous GAS Lyapunov function satisfying conditions (6a)-(6b) [13].

Remark 2.6: Since the Lyapunov function $\mathcal{V}$ in Theorem 2.4 is assumed to be homogeneous of degree κ, one can readily show that this function is always radially unbounded if and only if it is positive on the unit sphere. Accordingly, conditions (6a)-(6b) may be verified only on the unit sphere rather than for all $x \in \mathbb{R}^n$; thanks to the homogeneous property of the map f together with the Lyapunov function $\mathcal{V}$.

In the next section, we analyze stability of the interconnected dt-NS via ISS properties of individual subsystems.

III. Stability Certificate via Small-Gain Reasoning

Here, we analyze stability of the interconnected dt-NS by leveraging a compositional scheme based on small-gain reasoning and constructing a Lyapunov function for the interconnected dt-NS via ISS Lyapunov functions of individual subsystems. To do so, we first define notions of ISS Lyapunov functions.

Definition 3.1: A subsystem $\Upsilon_i = (X_i, W_i, f_i)$ admits a homogeneous ISS Lyapunov function $\mathcal{S}_i : X_i \to \mathbb{R}_0^+$ of degree $\kappa \in \mathbb{N}^+$, *i.e.,* for any $\eta > 0$ and $x_i \in X_i$, $\mathcal{S}_i(\eta x_i) = \eta^\kappa \mathcal{S}_i(x_i)$, if the following inequalities hold:

$$\forall x_i \in X_i: \quad \underline{\alpha}_i \|x_i\|^\kappa \leq \mathcal{S}_i(x_i) \leq \overline{\alpha}_i \|x_i\|^\kappa, \tag{7}$$

$$\forall x_i \in X_i, \forall w_i \in W_i: \quad \mathcal{S}_i(f_i(x_i, w_i)) \leq \max\left\{\gamma_i \mathcal{S}_i(x_i), \rho_i \|w_i\|^\kappa\right\}, \tag{8}$$

for some $\underline{\alpha}_i, \overline{\alpha}_i \in \mathbb{R}^+$, $\rho_i \in \mathbb{R}_0^+$, and $\gamma_i \in (0, 1)$.

Remark 3.2: Note that, without loss of generality, we have assumed lower and upper bounds of $\mathcal{S}_i$ (*i.e.,* $\underline{\alpha}_i\|x_i\|^\kappa, \overline{\alpha}_i\|x_i\|^\kappa$) and also gain $\rho_i\|w_i\|^\kappa$ to be polynomials of degree κ, due to the homogeneity of the considered systems. Notice that this choice is consistent with a linear gain $\|w_i\| \to \|x_i\|$ which is what one expects of a homogeneous system.

In this work, we are interested in constructing ISS Lyapunov functions $\mathcal{S}_i$ to locally satisfy conditions (7)-(8) over the unit sphere $\|(x_i, w_i)\| = 1$. These conditions can then be globally transfered to $\mathbb{R}^{n_i} \times \mathbb{R}^{p_i}$; thanks to the homogeneous property of map f_i and ISS Lyapunov functions $\mathcal{S}_i$. Then under some small-gain compositionality condition, the Lyapunov function $\mathcal{V}$ for the interconnected network can be constructed via ISS Lyapunov functions $\mathcal{S}_i$ of individual subsystems. Since the interconnected network is also homogeneous, the stability certificate can then be globally guaranteed in $\mathbb{R}^n$.

Remark 3.3: Note that although the homogeneous assumption allows us to extend the stability certificate globally to $\mathbb{R}^n$, it limits the class of underlying systems. One can relax this condition to consider more classes of systems but at the cost of providing the stability certificate *locally* instead of globally.

To analyze the stability of interconnected networks via ISS properties of subsystems, we first raise the following max-type small-gain assumption that is required for the compositional reasoning.

Assumption 1: Let $\gamma_{ij} \in \mathbb{R}^+$ defined as

$$\gamma_{ij} := \frac{\rho_i}{\underline{\alpha}_j}, \qquad \text{if } i \neq j,$$

satisfy

$$\gamma_{i_1 i_2} \cdot \gamma_{i_2 i_3} \cdot \ldots \cdot \gamma_{i_{p-1} i_t} \cdot \gamma_{i_t i_1} < 1 \tag{9}$$

with $\gamma_{ij} = \gamma_i$, if $i = j$, for all sequences $(i_1, \ldots, i_t) \in \{1, \ldots, \mathcal{M}\}^t$ and $t \in \{1, \ldots, \mathcal{M}\}$. Since $\gamma_i < 1$, the circularity condition in (9) is satisfied on arbitrary sequences if and only if it is fulfilled on simple loops of length strictly bigger than one.

The circularity condition (9) indicates the existence of $\sigma_i \in \mathbb{R}^+$ [14, Theorem 5.5], fulfilling

$$\max_{i,j}\left\{\frac{\gamma_{ij}\sigma_j}{\sigma_i}\right\} < 1, \quad i, j = \{1, \ldots, \mathcal{M}\}. \tag{10}$$

Under Assumption 1, the next theorem provides the stability guarantee for the interconnected dt-NS via ISS Lyapunov functions of individual subsystems [15]–[17]. Note that this theorem requires knowing the precise models of the system to construct gains γ_{ij}, needed for the small-gain condition (9).

Theorem 3.4: Consider an interconnected dt-NS $\Upsilon = \mathcal{I}(\Upsilon_1, \ldots, \Upsilon_\mathcal{M})$ induced by $\mathcal{M} \in \mathbb{N}^+$ subsystems Υ_i with their inputs partitioned as in (3). Let each subsystem Υ_i admits an ISS Lyapunov function $\mathcal{S}_i$ as in Definition 3.1. If Assumption 1 is satisfied, then

$$\mathcal{V}(x) := \max_i\{\frac{1}{\sigma_i}\mathcal{S}_i(x_i)\}, \tag{11}$$

for σ_i as in (10), is a Lyapunov function for the interconnected dt-NS $\Upsilon = \mathcal{I}(\Upsilon_1, \ldots, \Upsilon_\mathcal{M})$, and accordingly, Υ is GAS in the sense of Definition 2.3.

In our data-driven scheme, we consider the structure of ISS Lyapunov functions as

$$\mathcal{S}_i(q_i, x_i) = \sum_{j=1}^{r} q_i^j p_i^j(x_i), \tag{12}$$

with some user-defined homogeneous basis functions $p_i^j(x_i)$ with even degree κ and unknown variables $q_i = [q_i^1; \ldots; q_i^r] \in \mathbb{R}^r$. We assume that $\mathcal{S}_i$ are continuously differentiable.

Remark 3.5: Note that basis functions p_i^j should be homogeneous of degree κ so that $\mathcal{S}_i(q_i, x_i)$ remain homogeneous of the same degree. The degree κ needs to be also even such that $\mathcal{S}_i$ be a Lyapunov function. Since p_i^j should be restricted to the unit sphere, it has to be a finite basis of this subspace (*e.g.,* a set of spline functions on the unit sphere).

Here, we first cast the required conditions for the construction of ISS Lyapunov functions in Definition 3.1 as the following robust optimization program (ROP):

$$\text{ROP}: \begin{cases} \min\limits_{[\mathcal{G}_i;\mu_i]} & \mu_i, \\ \text{s.t.} & \max\left\{\mathcal{C}_i^1(x_i, \mathcal{G}_i), \mathcal{C}_i^2(x_i, w_i, \mathcal{G}_i)\right\} \leq \mu_i, \\ & \forall (x_i, w_i) \in X_i \times W_i : \|(x_i, w_i)\| = 1, \\ & \mathcal{G}_i = [\underline{\alpha}_i; \overline{\alpha}_i; \tilde{\gamma}_i; \tilde{\rho}_i; q_i^1; \ldots; q_i^r], \\ & \underline{\alpha}_i, \overline{\alpha}_i \in [1, +\infty), \tilde{\rho}_i \in \mathbb{R}_0^+, \mu_i \in \mathbb{R}, \tilde{\gamma}_i \in (0, 1), \end{cases} \tag{13}$$

with

$$\mathcal{C}_i^1(x_i, \mathcal{G}_i) = \max\left\{\underline{\alpha}_i\|x_i\|^\kappa - \mathcal{S}_i(q_i, x_i), \mathcal{S}_i(q_i, x_i) - \overline{\alpha}_i\|x_i\|^\kappa\right\},$$
$$\mathcal{C}_i^2(x_i, w_i, \mathcal{G}_i) = \mathcal{S}_i(q_i, f_i(x_i, w_i)) - \tilde{\gamma}_i \mathcal{S}_i(q_i, x_i) - \tilde{\rho}_i\|w_i\|^\kappa. \tag{14}$$

The max term in ROP (13) implies that each $\mathcal{C}_i^1$ and $\mathcal{C}_i^2$ should be individually less than or equal to μ_i. The symbol $\mathcal{G}_i$ also denotes a decision vector that needs to be designed after solving ROP. If $\mu_i \leq 0$, any feasible solution to the ROP signifies the satisfaction of conditions (7)-(8) in Definition 3.1: $\mathcal{S}_i$ is ISS Lyapunov functions for Υ_i. We denote the optimal value of ROP by $\mu_{R_i}^*$.

Remark 3.6: Condition $\mathcal{C}_i^2$ is bilinear given that both q_i^j and $\tilde{\gamma}_i$ are decision variables. To resolve this mild bilinearity, we consider $\tilde{\gamma}_i$ in a finite set with a cardinality l, *i.e.,* $\tilde{\gamma}_i \in \{\tilde{\gamma}_i^1, \ldots, \tilde{\gamma}_i^l\}$.

Remark 3.7: Note that in our ROP in (13), we consider the $\max$-form condition (8) to be additive in $\mathcal{C}_i^2$. Functions γ_i, ρ_i in (8) can be then acquired based on $\tilde{\gamma}_i, \tilde{\rho}_i$ in the implication-form condition $\mathcal{C}_i^2$ as:

$$\gamma_i = 1-(1-\psi_i)(1-\tilde{\gamma}_i), \rho_i = \frac{\tilde{\rho}_i}{(1-\tilde{\gamma}_i)\psi_i}, \text{ for any } 0 < \psi_i < 1.$$

The proposed ROP in (13) is not tractable due to two main difficulties. First, there are infinitely many constraints given that the space of x_i and w_i is continuous (*i.e.,* $x_i \in X_i, w_i \in W_i$). Second and more importantly, the unknown map f_i appears in $\mathcal{C}_i^2$ in ROP (13). Given these critical challenges, we develop in the next section a data-driven approach for the construction of ISS Lyapunov functions without directly solving the ROP in (13).

IV. Data-Driven Construction of ISS Lyapunov Functions

In this section, we develop a data-driven approach to construct an ISS Lyapunov function for each unknown subsystem with 100% correctness guarantee. To do so, we assume we are given a set of two-consecutive sampled data from trajectories of unknown subsystems as the pair of $((\tilde{x}_i^z, \tilde{w}_i^z), f_i(\tilde{x}_i^z, \tilde{w}_i^z)), z \in \{1, \dots, \mathcal{N}_i\}$. In our data driven setting, we first project all data onto unit sphere by normalizing them with $\|(\tilde{x}_i^z, \tilde{w}_i^z)\|$, *i.e.,*

$$((\hat{x}_i^z, \hat{w}_i^z), f_i(\hat{x}_i^z, \hat{w}_i^z)) = \frac{((\tilde{x}_i^z, \tilde{w}_i^z), f_i(\tilde{x}_i^z, \tilde{w}_i^z))}{\|(\tilde{x}_i^z, \tilde{w}_i^z)\|}.$$

We now compute the maximum distance between any points on the unit sphere and the set of data points as

$$\delta_i = \max_{(x_i, w_i)} \min_{z} \|(x_i, w_i) - (\hat{x}_i^z, \hat{w}_i^z)\|,$$
$$(x_i, w_i) \in X_i \times W_i : \|(x_i, w_i)\| = 1. \tag{15}$$

Remark 4.1: Note that (15) is not inherently convex and the maximum distance between any points on the unit sphere and the set of data points based on (15) can be computed via gridding of the unit sphere. Since the computation based on a grid-based approach is reduced to a finite problem, the computational complexity is linear in both number of samples and number of grid points (associated to the grid size).

By considering $(\hat{x}_i^z, \hat{w}_i^z) \in X_i \times W_i, z = 1, \dots, \mathcal{N}_i$, we propose the following scenario optimization program (SOP):

$$\text{SOP:} \begin{cases} \min\limits_{[\mathcal{G}_i;\mu_i]} & \mu_i, \\ \text{s.t.} & \max\left\{\mathcal{C}_i^1(\hat{x}_i^z, \mathcal{G}_i), \mathcal{C}_i^2(\hat{x}_i^z, \hat{w}_i^z, \mathcal{G}_i)\right\} \leq \mu_i, \\ & \forall (\hat{x}_i^z, \hat{w}_i^z) \in X_i \times W_i, \forall z \in \{1, \dots, \mathcal{N}_i\}, \\ & \mathcal{G}_i = [\underline{\alpha}_i; \overline{\alpha}_i; \tilde{\gamma}_i; \tilde{\rho}_i; q_i^1; \dots; q_i^r], \\ & \underline{\alpha}_i, \overline{\alpha}_i \in [1, +\infty), \tilde{\rho}_i \in \mathbb{R}_0^+, \mu_i \in \mathbb{R}, \tilde{\gamma}_i \in (0,1), \end{cases} \tag{16}$$

where $\mathcal{C}_i^1, \mathcal{C}_i^2$ are the same functions as defined in (14). One can readily observe that the proposed SOP in (16) has finite number of constraints of the same form as in (13). We denote the optimal value of SOP by $\mu_{\mathcal{N}_i}^*$.

Remark 4.2: Notice that $\hat{w}_i$ in (16) is the state measurement of neighboring subsystems affecting an individual subsystem i. If one employs the same SOP but aims at establishing an ISS Lyapunov function with respect to an *actual disturbance*, the measurement of disturbance may become challenging, which is out of scope of this work.

Remark 4.3: Since the candidate ISS Lyapunov function in (12) is defined as a linear combination of basis functions, the SOP in (16) remains always convex with respect to decision variables. The SOP in (16) is also always feasible given that the cost function μ_i could take positive values. However, the optimal value of SOP should be non-positive so that the constructed data-driven ISS Lyapunov function becomes also valid for the original unknown subsystem (cf. Theorem 5.1).

In the next section, we employ the proposed SOP and construct ISS Lyapunov functions for unknown Υ_i by providing a GAS certificate over the unknown interconnected network.

V. Stability Certificate over Unknown Interconnected Network

Here, we aim at constructing ISS Lyapunov functions of unknown Υ_i under the following assumption.

Assumption 2: Suppose $f_i(x_i, w_i)$ is Lipschitz continuous with respect to x_i and w_i with Lipschitz constants $\mathcal{L}_{x_i}, \mathcal{L}_{w_i}$, respectively.

The Lipschitz constant of unknown models, required in Assumption 2, can be estimated via data according to the approach in [18]. Now under Assumption 2, we propose the next theorem to construct ISS Lyapunov functions $\mathcal{S}_i$ for unknown subsystems Υ_i with 100% correctness guarantees.

Theorem 5.1: Given subsystems Υ_i in (2), let Assumption 2 hold. Consider the SOP in (16) with its corresponding optimal value $\mu_{\mathcal{N}_i}^*$ and solution $\mathcal{G}_i^* = [\underline{\alpha}_i^*; \overline{\alpha}_i^*; \tilde{\rho}_i^*; q_i^{1*}; \dots; q_i^{r*}]$, with $\mathcal{N}_i$. If

$$\mu_{\mathcal{N}_i}^* + \mathscr{L}_i \delta_i \leq 0, \tag{17}$$

where $\mathscr{L}_i = \max\left\{\mathscr{L}_i^1, \mathscr{L}_i^2\right\}$, with $\mathscr{L}_i^1, \mathscr{L}_i^2$ being Lipschitz constants of $\mathcal{C}_i^1, \mathcal{C}_i^2$ with respect to x_i and (x_i, w_i), respectively, then the constructed $\mathcal{S}_i$ via solving SOP (16) are ISS Lyapunov functions for unknown subsystems Υ_i with 100% correctness guarantees.

Proof: We first show that the constructed $\mathcal{S}_i$ via solving SOP (16) satisfy $\mathcal{C}_i^2$ in (14) for the whole range of state and disturbance spaces with $\mu_i \leq 0$, *i.e.,*

$$\mathcal{C}_i^2(x_i, w_i, \mathcal{G}_i)^* \leq 0, \quad \forall (x_i, w_i) \in X_i \times W_i : \|(x_i, w_i)\| = 1.$$

Since $f_i(x_i, w_i)$ is Lipschitz continuous according to Assumption 2, one can readily show that $\mathcal{C}_i^2$ is always Lipschitz continuous with respect to (x_i, w_i) with a Lipschitz constant $\mathscr{L}_i^2 \in \mathbb{R}^+$ (cf. Lemma 5.3) given that the ISS Lyapunov function $\mathcal{S}_i$ is continuously differentiable and our analysis is on the unit sphere (bounded domain). Let $z^* :=$

$\arg\min_z \|(x_i, w_i) - (\hat{x}_i^z, \hat{w}_i^z)\|$. Then one has

$$\begin{aligned}
&\mathcal{C}_i^2(x_i, w_i, \mathcal{G}_i)^* \\
&= \mathcal{C}_i^2(x_i, w_i, \mathcal{G}_i)^* - \mathcal{C}_i^2(\hat{x}_i^{z^*}, \hat{w}_i^{z^*}, \mathcal{G}_i)^* + \mathcal{C}_i^2(\hat{x}_i^{z^*}, \hat{w}_i^{z^*}, \mathcal{G}_i)^* \\
&\leq \mathscr{L}_i^2 \min_z \|(x_i, w_i) - (\hat{x}_i^z, \hat{w}_i^z)\| + \mu_{\mathcal{N}_i}^* \\
&\leq \mathscr{L}_i^2 \max_{(x_i, w_i)} \min_z \|(x_i, w_i) - (\hat{x}_i^z, \hat{w}_i^z)\| + \mu_{\mathcal{N}_i}^* \\
&\leq \mu_{\mathcal{N}_i}^* + \mathscr{L}_i \delta_i.
\end{aligned}$$

Since $\mu_{\mathcal{N}_i}^* + \mathscr{L}_i \delta_i \leq 0$ as the main condition of the theorem, one can readily verify that

$$\mathcal{C}_i^2(x_i, w_i, \mathcal{G}_i)^* \leq 0, \quad \forall (x_i, w_i) \in X_i \times W_i \colon \|(x_i, w_i)\| = 1.$$

We now leverage a similar argument and show that the constructed $\mathcal{S}_i$ via solving SOP in (16) satisfies $\mathcal{C}_i^1$ in (14) for the whole range of the state space. Let $z^* := \arg\min_z \|x_i - \hat{x}_i^z\|$. Then one has

$$\begin{aligned}
&\mathcal{C}_i^1(x_i, \mathcal{G}_i)^* = \mathcal{C}_i^1(x_i, \mathcal{G}_i)^* - \mathcal{C}_i^1(\hat{x}_i^{z^*}, \mathcal{G}_i)^* + \mathcal{C}_i^1(\hat{x}_i^{z^*}, \mathcal{G}_i)^* \\
&\leq \mathscr{L}_i^1 \max_{(x_i, w_i)} \min_z \|x_i - \hat{x}_i^z\| + \mu_{\mathcal{N}_i}^* \leq \mu_{\mathcal{N}_i}^* + \mathscr{L}_i \delta_i \leq 0.
\end{aligned}$$

Then the constructed $\mathcal{S}_i$ via solving SOP in (16) are ISS Lyapunov functions for unknown subsystems $\Upsilon_i, i \in \{1, \ldots, \mathcal{M}\}$, with 100% correctness guarantees, which concludes the proof. ∎

The data-driven results of Theorem 5.1 provide an ISS Lyapunov function for each individual subsystem within the network. Nevertheless, the constructed ISS certificate of each subsystem is of limited appeal on its own, without utilizing the small-gain reasoning and transferring the stability result to the network. In particular, as the final step, if the circularity condition in (9) is fulfilled, then $\mathcal{V}$ in (11), composed of data-driven ISS Lyapunov functions of individual subsystems, is a Lyapunov function for the unknown interconnected dt-NS $\Upsilon = \mathcal{I}(\Upsilon_1, \ldots, \Upsilon_{\mathcal{M}})$, and accordingly, Υ is GAS in the sense of Definition 2.3. It is worth mentioning that if one synthesizes $\underline{\alpha}_j$ and ρ_i during solving the SOP in (16) such that $\frac{\rho_i}{\underline{\alpha}_j} < 1$, the small-gain condition (9) is automatically fulfilled without requiring any posteriori check. In this case, one can readily conclude that $\mathcal{V}$ is a Lyapunov function for unknown interconnected dt-NS.

Remark 5.2: Note that one can establish the results of Theorem 5.1 directly for an interconnected network; however, the proposed SOP in this case suffers severely from the, so-called, *sample complexity* due to dealing with large-dimensional networks: the required number of data for providing stability guarantee exponentially increases with the size of underlying networks. This is the main motivation that we use a *divide and conquer strategy* and reduce our complex problem to search for ISS Lyapunov functions of unknown individual subsystems via the proposed data-driven results in Theorem 5.1. We then use small-gain reasoning and certify the GAS property of unknown interconnected networks based on those data-driven ISS properties of underlying subsystems.

We propose Algorithm 1 to describe the required steps for the construction of ISS Lyapunov functions of unknown Υ_i from data according to Theorem 5.1. It is worth mentioning that if $\mu_{\mathcal{N}_i}^* + \mathscr{L}_i \delta_i \nleq 0$, one can either change basis functions of $\mathcal{S}_i$ (cf. Remark 3.5) or project more data onto the unit sphere to potentially decrease the distance δ_i, and repeat Steps 1-4 of Algorithm 1. When $\mu_{\mathcal{N}_i}^* + \mathscr{L}_i \delta_i \leq 0$, the small-gain compositionality condition in (9) needs to be checked with the obtained γ_{ij}. If circularity condition (9) is fulfilled, then unknown dt-NS is GAS with the Lyapunov function $\mathcal{V}(q, x) := \max_i\{\frac{1}{\sigma_i}\mathcal{S}_i(q_i, x_i)\}$. If circularity condition (9) is not satisfied, one needs to repeat Steps 1-4 to design other γ_{ij} potentially satisfying condition (9).

Algorithm 1 Data-driven construction of ISS Lyapunov functions of unknown individual subsystem Υ_i

Require: Sampled data-points $((\tilde{x}_i^z, \tilde{w}_i^z), f_i(\tilde{x}_i^z, \tilde{w}_i^z)), z \in \{1, \ldots, \mathcal{N}_i\}$

1: Project all data-points onto unit sphere by normalizing them as $((\hat{x}_i^z, \hat{w}_i^z), f_i(\hat{x}_i^z, \hat{w}_i^z)) = \frac{((\tilde{x}_i^z, \tilde{w}_i^z), f_i(\tilde{x}_i^z, \tilde{w}_i^z))}{\|(\tilde{x}_i^z, \tilde{w}_i^z)\|}$

2: Solve SOP in (16) with the normalized data and obtain $\mu_{\mathcal{N}_i}^*$

3: Compute δ_i as the maximum distance between any points in the unit sphere and the set of data points, *i.e.,* $\delta_i = \max_{(x_i, w_i)} \min_z \|(x_i, w_i) - (\hat{x}_i^z, \hat{w}_i^z)\|$, $(x_i, w_i) \in X_i \times W_i : \|(x_i, w_i)\| = 1$

4: Compute Lipschitz constant $\mathscr{L}_i$ according to Lemma (5.3)

5: If $\mu_{\mathcal{N}_i}^* + \mathscr{L}_i \delta_i \leq 0$, then the constructed $\mathcal{S}_i$ via solving SOP (16) are ISS Lyapunov functions for unknown subsystems Υ_i

6: Otherwise, inconclusive given the choice of $\mathcal{S}_i$ and δ_i

In order to check condition (17) in Theorem 5.1, one needs to first compute $\mathscr{L}_i$. We propose in the next lemma an explicit way to compute $\mathscr{L}_i$ for dt-NS.

Lemma 5.3: Consider unknown dt-NS $x_i(k+1) = f_i(x_i(k), w_i(k))$. Let $f_i(x_i, w_i)$ be Lipschitz continuous with respect to x_i and w_i with Lipschitz constants $\mathcal{L}_{x_i}, \mathcal{L}_{w_i}$ according to Assumption 2. Then $\mathscr{L}_i$ for a quadratic ISS Lyapunov function of the form $x_i^\top P_i x_i$, with a positive-definite matrix $P_i \in \mathbb{R}^{n_i \times n_i}$ and $\kappa = 2$, is quantified as $\mathscr{L}_i = \max\{\mathscr{L}_i^1, \mathscr{L}_i^2\} = \max\{2\lambda_{\max}(P_i)(\mathcal{L}_{x_i}\mathcal{L}_i + \mathcal{L}_{w_i}\mathcal{L}_i + 1) + 2\tilde{\rho}_i, 2\,\lambda_{\max}(P) + 2\overline{\alpha}_i\}$, where $\|f_i(x_i, w_i)\| \leq \mathcal{L}_i, \forall (x_i, w_i) \in X_i \times W_i$.

Note that the upper bound of unknown models, *i.e.,* $\mathcal{L}_i$, can be quantified via the range of the state space since unknown dynamics evolve in discrete time.

VI. Case Study: Room Temperature Network

We verify our data-driven results over a room temperature network containing 1000 rooms in a circular topology. The evolution of the temperature $T(\cdot)$ can be described by the following interconnected network [19]:

$$\Upsilon \colon T(k+1) = AT(k),$$

where A is a matrix with diagonal elements $a_{ii} = 1 - 2\varphi - \theta$, $i \in \{1, \ldots, \mathcal{M}\}$, off-diagonal elements $a_{i,i+1} = a_{i+1,i} =$

$a_{1,\mathcal{M}} = a_{\mathcal{M},1} = \varphi$, $i \in \{1, \ldots, \mathcal{M}-1\}$, and zero for other elements. In addition, $T(k) = [T_1(k); \ldots; T_{\mathcal{M}}(k)]$, and φ, θ are thermal factors between rooms $i \pm 1$ and i, and the external environment and the room i, respectively. Now by characterizing each individual room as

$$\Upsilon_i \colon T_i(k+1) = a_{ii} T_i(k) + \varphi(w_{i-1}(k) + w_{i+1}(k)), \quad (18)$$

with $w_0 = w_{\mathcal{M}}, w_{\mathcal{M}+1} = w_1$, one has $\Upsilon = \mathcal{I}(\Upsilon_1, \ldots, \Upsilon_{\mathcal{M}})$. We assume that the underlying model is unknown. The main target is to compositionally construct a Lyapunov function for the interconnected dt-NS based on ISS Lyapunov functions of individual subsystems via solving SOP (16). Accordingly, we verify that the interconnected network is GAS with respect to its equilibrium point $x = 0$ with 100% correctness guarantee.

We first fix the structure of our ISS Lyapunov functions as $\mathcal{S}_i(q_i, T_i) = q_i T_i^2$ for all $i \in \{1, \ldots, 1000\}$. We employ Algorithm 1 as our proposed data-driven scheme. We collect 730 samples from trajectories of each unknown room (*i.e.,* $\mathcal{N}_i = 730$) and normalize them to be projected onto unit sphere. We now solve the SOP (16) with $\mathcal{N}_i = 730$ and compute coefficients of ISS Lyapunov functions together with other decision variables as

$$\begin{aligned} \mathcal{S}_i(q_i, T_i) = 1.25 T_i^2, \underline{\alpha}_i = 1, \overline{\alpha}_i = 1.5, \rho_i = 0.03,\\ \mu^*_{\mathcal{N}_i} = -0.0644, \end{aligned} \quad (19)$$

with a fixed $\gamma_i = 0.9$. We compute $\delta_i = 0.0109$ and $\mathscr{L}_i = 5.5$ according to Steps 3, 4 in Algorithm 1. Since $\mu^*_{\mathcal{N}_i} + \mathscr{L}_i \delta_i = -47 \times 10^{-4} \leq 0$, according to Theorem 5.1, one can verify that the constructed ISS Lyapunov functions via collected data are valid for the original ROP (13) with 100% correctness guarantees.

We now proceed with Theorem 3.4 to construct a Lyapunov function for the interconnected dt-NS using ISS Lyapunov functions of individual subsystems, constructed from data. By taking $\sigma_i = 1, \forall i \in \{1, \ldots, \mathcal{M}\}$, the circularity condition in (9) is fulfilled. Then by employing the results of Theorem 3.4, one can certify that the interconnected dt-NS $\Upsilon = \mathcal{I}(\Upsilon_1, \ldots, \Upsilon_{\mathcal{M}})$ is GAS with respect to $x = 0$, and $\mathcal{V}(q, T) = \max_i\{\mathcal{S}_i(q_i, T_i)\} = \max_i\{1.25 T_i^2\}$ is a Lyapunov function for the interconnected dt-NS with 100% correctness guarantee.

The computation of ISS Lyapunov functions took 1.13 seconds for each room (1130 seconds for all 1000 rooms in a serial computation) on a machine with Windows operating system (Intel i7-8665U CPU with 16 GB of RAM).

VII. Discussion

In this letter, we proposed a compositional data-driven technique to ensure stability certificate of interconnected homogeneous systems with unknown models. In our data-driven scheme, we collected data from trajectories of each unknown subsystem to propose a scenario optimization program (SOP). We solved the resulting SOP and constructed an ISS Lyapunov function for each unknown subsystem with 100% correctness guarantee. We accordingly utilized a compositional technique based on *small-gain reasoning* and constructed a Lyapunov function for the interconnected network based on ISS Lyapunov functions of individual subsystems. We demonstrated the efficacy of our data-driven results over an unknown room temperature network containing 1000 rooms. Developing a compositional data-driven technique for ensuring *incremental ISS property* of *general nonlinear class* of interconnected systems is under investigation as a future work.

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