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Memory saving state-sharing multi-observer for a class of multi-observer based algorithms

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***Abstract*— A multi-observer is a bank of observers which is used for state estimation in various applications. However, it has an implementation bottleneck when a large number of observers are required for the desired estimation performance. To overcome this problem, we propose the design method of a state-sharing multi-observer for a class of nonlinear systems. The state-sharing multi-observer is a single observer that integrates a bank of observers, and its state size is independent of the number of observers. We analyze the error of the state obtained from the state-sharing multi-observer, and then show its applicability to multi-observer based algorithms such as supervisory observers and in secure state estimation.**

I. Introduction

A multi-observer is a parallel implementation of observers which takes in data from the plant and whose output asymptotically converges to the output of the plant, modulo noise or disturbances. In other words, consider the plant parameterized by $p^\star \in \mathbb{P}^\star$

$$\dot{x}(t) = f(x(t), u(t), p^\star),\ y(t) = h(x(t), u(t), p^\star) + a(t), \tag{1}$$

where the state is $x \in \mathbb{R}^{n_x}$, measured output is $y \in \mathbb{R}^{n_y}$, measured input is $u \in \mathbb{R}^{n_u}$ and $a \in \mathbb{R}^{n_y}$ is an unknown signal, f and h are locally Lipschitz functions. The input u and signal a are Lebesgue measurable functions. We assume that the plant (1) is forward complete. Then, a *multi-observer* for the plant (1) is a finite bank of finite-dimensional dynamical systems, where for each $p \in \mathbb{P} \subset \mathbb{P}^\star \subset \mathbb{R}^{n_p}$ with $\mathbb{P}$ being a finite set, an observer whose input is the pair (u, y) from the plant (1), generates a state estimate $\hat{x}_p \in \mathbb{R}^{n_x}$ according to the following for all $t \in \mathbb{R}_{\geqslant 0}$

$$\dot{\hat{x}}_p(t) = \hat{f}(\hat{x}_p(t), u(t), y_p(t), p), \quad y(t) \supseteq y_p(t) \in \mathbb{R}^m. \tag{2}$$

For two vectors $y_p := (y_{p,1}, y_{p,2}, \dots, y_{p,m})^T \in \mathbb{R}^m$ for $p \in \mathbb{P}$ and $y = (y_1, y_2, \dots, y_{n_y})^T \in \mathbb{R}^{n_y}$, when we write $y_p \subseteq y$ in (2), we mean that the set of components $\{y_{p,1}, y_{p,2}, \dots, y_{p,m}\}$ of y_p is a subset of the set of components $\{y_1, y_2, \dots, y_{n_y}\}$ of y. The function $\hat{f}$ is designed such that each observer (2) is forward complete and has desirable robustness properties depending on its deployment, e.g., the estimation error $e_p := \hat{x}_p - x$ system for each $p \in \mathbb{P}$, is input-to-state stable [1] with respect to the unknown signal $a(t)$ or parameter mismatch $p - p^\star$. Notice that each observer (2) may use a subset y_p of the available measurements y which is most applicable in multi-observer based secure state estimation [2]–[7]. Moreover, the cardinality of the finite parameter set $\mathbb{P}$ is induced by the context of its use.

The multi-observer is employed in many settings, including supervisory control [8], [9], supervisory observer [10]–[12] and in secure state estimation algorithms for systems under sensor attacks [2]–[7]. A common implementation bottleneck, especially for estimation, is the need for many observers (2) either for estimation accuracy or to mitigate sensor attacks. Supervisory observers for parameter and state estimation [10], [11] require the parameter set $\mathbb{P}^\star$ to be densely sampled to achieve a desired estimation accuracy. Since an observer is designed for each parameter sample, the number of observers increases as a finer parameter estimation accuracy is desired. In the secure state estimation setting [2]–[7] where up to a positive integer M out of N sensors are allowed to be attacked, we need $\frac{N!}{M!(N-M)!} + \frac{N!}{2M!(N-2M)!}$ observers to reconstruct the state of the plant (1). One can then see that in both cases, the number of observers (2) scales unfavourably with the number of parameter samples (supervisory observer) or the number of sensors N (secure state estimator).

The aim of this paper is to show that a class of multi-observer (2) can be implemented using a dynamical system with a dimension that is independent of the number of observers required by its use, i.e., independent of the cardinality of the parameter set $\mathbb{P}$. Moreover, this system has the same input-output behaviour from the input pair (u, y) to $\hat{x}_p$. We call such a dynamical system a *state-sharing multi-observer*, which takes the following form

$$\dot{z} = g(z, y, u), \tag{3a}$$
$$\hat{x}^a_p = \mathcal{T}_p(z), \tag{3b}$$

with appropriately designed functions $g : \mathbb{R}^{n_z} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_z}$ and $\mathcal{T}_p : \mathbb{P} \times \mathbb{R}^{n_z} \to \mathbb{R}^{n_x}$ such that it is forward complete and has state $z \in \mathbb{R}^{n_z}$ with dimension n_z that is independent of the cardinality of the parameter set $\mathbb{P}$. The state of each observer $\mathbb{O}_p$ in (2) is then extracted via (3b).

The idea of a state-sharing multi-observer was first introduced in [13, Section 8], but its memory saving construction was not developed. The form (3) is desirable and saves memory in multi-observer based algorithms because instead of implementing a bank of many observers (2), the user implements a single *state-sharing* dynamical system (3a) with dimensions that scales independently of the number of required observers. As we will see in this paper, the dimension of the state-sharing multi-observer scales linearly with the dimension of the state observer (2).

We achieve this by performing a coordinate transformation on each of the observer (2), to bring each observer (2) into a common dynamical system that takes as input (y, u) as shown in (3a). We build upon the crucial observation that for linear Luenberger observers, i.e., for each $p \in \mathbb{P}$,

$\dot{\hat{x}}_p = (A_p + L_p H_p)\hat{x}_p - L_p y$, where the pair (A_p, H_p) is observable for every $p \in \mathbb{P}$, one can choose the observer gain L_p for each $p \in \mathbb{P}$ such that the matrix $A_p + L_p H_p$ for each $p \in \mathbb{P}$ shares a common set of eigenvalues via a common characteristic polynomial such that under a linear change of coordinates, each matrix A_p can be transformed to a common matrix A that is independent of the parameter p. Therefore, if each of the linear Luenberger observer is transformed to the controllable canonical form, see [14, Section 4.3.2], then we have constructed a *state-sharing* multi-observer (3). This transformation holds whether the observer is in continuous or discrete-time. In fact, the first known state-sharing multi-observer was constructed in discrete-time by the authors in [4, Section 4].

In this paper, we consider Lur'e systems, where the presence of the state-dependent nonlinearity induces a multi-observer that is a linear Luenberger observer with a nonlinearity depending on the measured output and input. We perform a coordinate transformation to arrive at a state-sharing multi-observer (3), where its state $\hat{x}_p^a$ in (3) is exactly $\hat{x}_p$ in (2) under non-restrictive assumptions. Therefore, the contributions of this paper are:

- A state-sharing multi-observer for Lur'e systems.
- The dimension of the state-sharing multi-observer is independent of the number of observers required by the algorithm (the cardinality of $\mathbb{P}$) and scales linearly with the dimension of the observer states.
- Memory saving when deployed in settings where a large number of observers are needed by the multi-observer based algorithm, such as the supervisory observer and in secure state estimation.

II. Notation

- Let $\mathbb{C}$ be the set of complex numbers, $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_{\geqslant 0} = [0, \infty)$, $\mathbb{R}_{>0} = (0, \infty)$, $\mathbb{N}_{[i,i+k]} = \{i, i+1, i+2, \ldots, i+k\}$ and $\mathbb{N}_{\geqslant i} := \{i, i+1, \ldots, \}$.
- The cartesian product of a family $\{X_i\}_{i\in\mathbb{N}_{[1,N]}}$ of sets is denoted $\bigtimes_{i\in\mathbb{N}_{[1,N]}} X_i$. If all the sets $X_i = X$, then we use the notation X^N.
- The cardinality of a set p is denoted as $|p|$.
- The identity matrix of dimension n is denoted by $\mathbb{I}_n$ and a matrix of dimension m by n with all elements 0 is denoted by $0_{m\times n}$ or 0_m when the dimension is m by m.
- Given a set $\mathcal{A} \subseteq \mathbb{R}^{n_y}$, the set of Lebesgue measurable functions from $\mathbb{R}_{\geqslant 0}$ to $\mathcal{A}$ is denoted $\mathcal{L}_{\mathcal{A}}$.
- The Euclidean norm of a vector $x \in \mathbb{R}^n$, is denoted $\|x\|$ and for a matrix $A \in \mathbb{R}^{n\times n}$, its induced norm is $\|A\|$.
- A continuous function $\alpha : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is a class $\mathcal{K}$ function, if it is strictly increasing and $\alpha(0) = 0$.

III. Problem formulation

Given $\mathbb{P}^\star \supset \mathbb{P}$ with $\mathbb{P}$ being a finite set, we consider plants (1) of the following form, where for $p^\star \in \mathbb{P}^\star$,

$$\begin{aligned} \dot{x} &= A_{p^\star} x + E_{p^\star}\psi(H_{p^\star} x, u), \\ y &= H_{p^\star} x + a \end{aligned} \tag{4}$$

which admits the multi-observer (a bank of observers) $\{\mathbb{O}_p : p \in \mathbb{P} \text{ with } |\mathbb{P}| = N_p\}$ with input $(y, u) \in \mathbb{R}^{n_y} \times \mathbb{R}^{n_u}$ from the plant (1), each taking the form

$$\dot{\hat{x}}_p = A_p \hat{x}_p + E_p \psi(y, u) + L_p(y_p - H_p \hat{x}_p),\ y \supseteq y_p \in \mathbb{R}^m, \tag{5}$$

where functions $A_p : \mathbb{P}^\star \to \mathbb{R}^{n_x\times n_x}$, $E_p : \mathbb{P}^\star \to \mathbb{R}^{n_x\times n_\psi}$ and H_p is a real matrix of dimension $m \times n_x$ with $m = n_y$ when $p = p^\star$. The nonlinearity $\psi : \mathbb{R}^{n_y} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_\psi}$ is locally Lipschitz in its arguments and L_p is the to-be-designed observer gain matrices of appropriate dimensions.

The class of observers (5) includes Luenberger-like observers and linear Luenberger observers (where $E_p = 0$) [14, Section 16.5]. Works such as [15] and the references therein, are devoted to (locally) transforming a nonlinear system into the linear up to an (noise-free) output injection form (4). Hence, the class of systems (4) and the Luenberger-like observer (5) that it admits is much larger than expected at the first glance. For all the aforementioned observers, a crucial assumption is that the pair (A_p, H_p) is observable for all $p \in \mathbb{P}$ such that the observer gain L_p has the following properties.

Lemma 1: For every $p \in \mathbb{P}$, the pair (A_p, H_p) is observable, so that an observer gain matrix L_p can be chosen such that

(i) the matrix $A_p - L_p H_p$ is Hurwitz, and hence there are constants $k_p \geqslant 1$ and $\lambda_p > 0$ such that $\|e^{(A_p - L_p H_p)t}\| \leqslant k_p e^{-\lambda_p t}$ for all $t \in \mathbb{R}_{\geqslant 0}$; and

(ii) there exist coefficients $q_i \in \mathbb{R}$ for $i \in \mathbb{N}_{[1,n_x]}$ independent of p, such that the characteristic polynomial of $A_p - L_p H_p$ is given by $\det(sI - (A_p - L_p H_p)) := s^{n_x} + q_1 s^{n_x - 1} + \cdots + q_{n_x}$.

Proof: Let $p \in \mathbb{P}$. We obtain items (i)-(ii) by applying Theorem 16.9 of [14] to choose L_p such that all the eigenvalues of the $\Lambda_p := A_p - L_p H_p$ have strictly negative real parts, i.e., the matrix Λ_p is Hurwitz. In other words, we can assign the eigenvalues of Λ_p for all $p \in \mathbb{P}$ to share the same eigenvalues the $c_i \in \mathbb{C}$ for $i \in \mathbb{N}_{[1,n_x]}$, which are independent of p.

To obtain (i), we write Λ_p in Jordan normal form, and see that there exist constants $k_p > 0$ and $\lambda_p > 0$ such that $\|e^{\Lambda_p t}\| \leqslant k_p e^{-\lambda_p t}$ for all $t \in \mathbb{R}_{\geqslant 0}$.

We show (ii) by observing that the characteristic polynomial of Λ_p is given by $\det(sI - \Lambda_p) := (s - c_1)(s - c_2)\ldots(s - c_{n_x})$, where we recall that c_i, for $i \in \mathbb{N}_{[1,n_x]}$ are the eigenvalues of Λ_p. Thus, we obtain the characteristic polynomial in (ii) where the coefficients $q_i \in \mathbb{R}$ for $i \in \mathbb{N}_{[1,n_x]}$ are independent of p since c_i, for $i \in \mathbb{N}_{[1,n_x]}$ are independent of p. ■

The Hurwitz stability of the matrix $A_p - L_p H_p$ ensures that the linear part of the state estimation error $\hat{x}_p - x$ is exponentially stable when $p = p^\star$. On the other hand, for all $p \in \mathbb{P}$, the matrix $A_p - L_p H_p$ share the same characteristic polynomial which will be crucial in the design of state-sharing multi-observers in Section IV.

In fact, it can be shown that for each $p \in \mathbb{P}$, the observer given by (5) can be designed to have desirable properties to be employed in the context of the supervisory observer [10], [11] and in secure state estimation [2]–[7]. In both of these uses, the multi-observer may require many observers which can be alleviated by a state-sharing multi-observer (3). In Section V, we detail the design considerations of the multi-observer (5) in both the supervisory observer and in secure state estimation to achieve the following objective.

A. Objective

Our objective is to show that by appropriate choice of the observer gain matrices $L_p : \mathbb{P} \to \mathbb{R}^{n_x \times m}$, we can construct a dynamical system with output $\hat{x}_p^a$ in the form of (3), that exhibits the same input-output behaviour as the multi-observer in (5) with state $\hat{x}_p$. We call such a dynamical system (3), a state-sharing multi-observer for the multi-observer (5), i.e.,

Definition 1: The dynamical system (3) with input (u, y) from plant (4), is a *state-sharing multi-observer* for multi-observer (5), if $\hat{x}_p^a(t) = \hat{x}_p(t)$ for all $t \in \mathbb{R}_{\geqslant 0}$.

IV. State-sharing multi-observer

We now construct a state-sharing multi-observer for the multi-observer (5). Let $\tilde{L}_p$ be an extended matrix of L_p with dimension $n_x \times n_y$ such that $\tilde{L}_p y = L_p y_p$ for all $y \in \mathbb{R}^{n_y}$ and $y_p \in \mathbb{R}^m$. Such an extended matrix can be obtained by padding L_p with zero-vectors 0_{n_x} appropriately. Then defining

$$\mathbf{A}_p := A_p - L_p H_p, \qquad \mathbf{B}_p := \left[\begin{array}{c:c} E_p & \tilde{L}_p \end{array}\right], \tag{6}$$

we rewrite (5) in the following form,

$$\dot{\hat{x}}_p^{\langle 1\rangle} = \mathbf{A}_p \hat{x}_p^{\langle 1\rangle} + \mathbf{B}_p \begin{bmatrix} \psi(y,u) \\ y \end{bmatrix}. \tag{7}$$

We will now perform a linear transformation on the state to bring the multi-observer (7) into a controllable canonical form. To this end, we impose further design requirements on the observer gain matrix L_p according to Assumption 1-(ii).

Let $\ell := n_\psi + n_y$. Define the matrices $\mathbf{A}$, $\mathbf{B}$, and T_p by

$$\mathbf{A} := \begin{bmatrix} -q_1\mathbb{I}_\ell & -q_2\mathbb{I}_\ell & \cdots & -q_{n_x-1}\mathbb{I}_\ell & -q_{n_x}\mathbb{I}_\ell \\ \mathbb{I}_\ell & 0_\ell & \cdots & 0_\ell & 0_\ell \\ 0_\ell & \mathbb{I}_\ell & \cdots & 0_\ell & 0_\ell \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_\ell & 0_\ell & \cdots & \mathbb{I}_\ell & 0_\ell \end{bmatrix}, \ \mathbf{B} := \begin{bmatrix} \mathbb{I}_\ell \\ 0_\ell \\ \vdots \\ 0_\ell \\ 0_\ell \end{bmatrix},$$

and $T_p := R_p R_q$, where

$$R_p := \begin{bmatrix} \mathbf{B}_p & \mathbf{A}_p\mathbf{B}_p & \cdots & \mathbf{A}_p^{n_x-1}\mathbf{B}_p \end{bmatrix},$$
$$R_q := \begin{bmatrix} \mathbb{I}_\ell & q_1\mathbb{I}_\ell & q_2\mathbb{I}_\ell & \cdots & q_{n_x-1}\mathbb{I}_\ell \\ 0_\ell & \mathbb{I}_\ell & q_1\mathbb{I}_\ell & \cdots & q_{n_x-2}\mathbb{I}_\ell \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_\ell & \cdots & 0_\ell & \mathbb{I}_\ell & q_1\mathbb{I}_\ell \\ 0_\ell & \cdots & 0_\ell & 0_\ell & \mathbb{I}_\ell \end{bmatrix}.$$

A routine calculation (same as in the well-known single-input case) shows that the identities $T_p\mathbf{A} = \mathbf{A}_p T_p$ and $T_p\mathbf{B} = \mathbf{B}_p$ hold[1]; see also Section 4.3.2 of [14]. Therefore, using the state transformation $\hat{x}_p^{\langle 1\rangle} = T_p \hat{x}_p^{\langle 2\rangle}$, we can rewrite the state equation (7) in a controllable canonical form:

$$\dot{\hat{x}}_p^{\langle 2\rangle} = \mathbf{A}\hat{x}_p^{\langle 2\rangle} + \mathbf{B}\begin{bmatrix} \psi(y,u) \\ y \end{bmatrix}. \tag{8}$$

Since the matrices $\mathbf{A}$, $\mathbf{B}$ are parameter p independent, we may drop the subscript p. We see that (8) is then in the form of a state-sharing multi-observer (3). We summarise the results obtained thus far as follows.

Lemma 2: Consider multi-observer (5) where (A_p, H_p) is observable for every $p \in \mathbb{P}$. Then, the state estimate $\hat{x}_p^{\langle 1\rangle}$ generated by (7) is given by

$$\hat{x}_p^{\langle 1\rangle} = T_p z, \qquad \dot{z} = \mathbf{A}z + \mathbf{B}\begin{bmatrix} \psi(y,u) \\ y \end{bmatrix}. \tag{9}$$

By choosing $\hat{x}_p^a = \hat{x}_p^{\langle 1\rangle}$, we have a state-sharing multi-observer in the desired form (3). We are now ready to state our main result as follows.

Theorem 1: Consider the plant (4) and multi-observer (5) where the pair (A_p, H_p) is observable for every $p \in \mathbb{P}$. Then, the following system is a state-sharing multi-observer for multi-observer (5)

$$\dot{z} = \mathbf{A}z + \mathbf{B}\begin{bmatrix} \psi(y,u) \\ y \end{bmatrix}, \qquad \hat{x}_p^a = T_p z, \tag{10}$$

if the state-sharing multi-observer (10) and the multi-observer (5) are both initialized at zero.

Proof: Let $p \in \mathbb{P}$. Since $T_p\mathbf{A} = \mathbf{A}_p T_p$ and $T_p\mathbf{B} = \mathbf{B}_p$, the approximation error $\tilde{x}_p := \hat{x}_p - \hat{x}_p^a$ has dynamics $\dot{\tilde{x}}_p = (A_p - L_p H_p)\tilde{x}_p$, which satisfies $\tilde{x}_p(t) = e^{(A_p - L_p H_p)t}\tilde{x}_p(0) = 0$ for all $t \in \mathbb{R}_{\geqslant 0}$ since $\tilde{x}_p(0) = 0$ by choice. ∎

The state-sharing multi-observer (10) and the multi-observer (5) are both initialized at zero to ensure that we obtain $\hat{x}_p(t) = \hat{x}_p^a(t)$ for all $t \in \mathbb{R}_{\geqslant 0}$. However, even without this matching initialization, $\hat{x}_p^a(t)$ exponentially converges to $\hat{x}_p(t)$. The speed of convergence depends on the eigenvalues of the matrix $A_p - L_p H_p$ which can be assigned freely by the observer gain matrix L_p due to the observability of the pair (A_p, H_p) for every $p \in \mathbb{P}$.

The coordinate transformation we performed to obtain a state-sharing multi-observer is irrespective of whether the multi-observer (2) is in continuous or discrete time. Therefore, Theorem 1 will also hold with the appropriate modifications to the state-sharing multi-observer (10) and in the estimation of the approximation error. In fact, by doing so, we recover our result for discrete-time linear time-invariant systems in Section 4 of [4] in the context of secure state estimation. We explain how the results obtained so far can be applied to different multi-observer based algorithms in the following section.

V. Memory saving applications

From Theorem 1, we see that the state-sharing multi-observer (10) has dimension

$$n_x(n_\psi + n_y), \tag{11}$$

[1] Notice that this transformation does not require the invertibility of T_p when implementing the state-sharing multi-observer (10). Hence we do not need the pair $(\mathbf{A}_p, \mathbf{B}_p)$ to be controllable for every $p \in \mathbb{P}$.

which is independent of the number of the observers N_p and scales linearly with the dimension n_x of each observer in the multi-observer (5) and the sum of the dimension n_y of the output y and the dimension n_ψ of the nonlinearity $\psi(y, u)$. In comparison to multi-observer (5), its dimension is

$$n_x N_p, \tag{12}$$

where N_p is the number of observers required by the algorithm. Therefore, it is advantageous to employ a state-sharing multi-observer when $(n_\psi + n_y) \ll N_p$.

In this section, we show how the state-sharing multi-observer in Section IV can be employed in multi-observer-based algorithms. Specifically, we consider two context: the supervisory observer [10], [11] and in secure state estimation [2]–[7]. In these cases, a large number of observers is typically needed and we will show that the state-sharing multi-observer reduces the memory requirement of these algorithms.

A. Supervisory observer

The supervisory observer [10], [11] is a parameter and state estimation algorithm for the plant (1). In this context, the set $\mathbb{P}^\star$ is a compact set. The finite set $\mathbb{P}$ is the set of finite samples of $\mathbb{P}^\star$. For clarity in the supervisory observer context, we do not discuss the presence of $a(t)$, which takes the role of measurement noise. To summarise, we consider the following plant

$$\dot{x} = A_{p^\star} x + E_{p^\star} \psi(H_{p^\star} x, u), \qquad y = H_{p^\star} x, \tag{13}$$

with multi-observer

$$\dot{\hat{x}}_p = (A_p - L_p H_p)\hat{x}_p + E_p \psi(y, u) + L_p y_p, \;\; y_p = H_p x. \tag{14}$$

According to Theorem 1 of [10], along with other operating conditions, a crucial requirement is that the parameter set $\mathbb{P}^\star$ is sampled sufficiently, i.e., the cardinality of $\mathbb{P}$ is sufficiently large. Consequently, the number of observers needed in the multi-observer (2) can be many. Therefore, a state-sharing multi-observer will alleviate the computational burden of the supervisory observer.

Before showing that a state-sharing multi-observer can be constructed for the supervisory observer, we note that each of observer (2) in the multi-observer needs to posses a robustness property with respect to the parameter mismatch $\tilde{p} := p^\star - p$, see [10, Assumption 2]. To this end, we examine the state estimation error $e_p := \hat{x}_p - x$ system, for any $p \in \mathbb{P}$,

$$\begin{aligned}\dot{e}_p = (A_p - L_p H_p) e_p + \tilde{A}(p, p^\star) x \\ + \tilde{E}(p, p^\star)\psi(H_{p^\star} x, u) =: F_p(e_p, p, p^\star, u, x),\end{aligned} \tag{15}$$

where $\tilde{A}(p, p^\star) := A_p - A_{p^\star}$ and $\tilde{E}(p, p^\star) := E_p - E_{p^\star}$. We say that the multi-observer is robust with respect to the parameter mismatch if for every $p \in \mathbb{P}$, the state estimation error system (15) admits a function V_p satisfying the following.

Assumption 1: There exist scalars a_1, a_2, $\lambda_0 > 0$ and a continuous non-negative function $\tilde{\gamma} : \mathbb{R}^{n_p} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}_{\geqslant 0}$ with $\tilde{\gamma}(0, x, u) = 0$ for all $x \in \mathbb{R}^{n_x}$ and $u \in \mathbb{R}^{n_u}$, such that for any $p \in \mathbb{P}$, there exists a continuously differentiable function $V_p : \mathbb{R}^{n_x} \to \mathbb{R}_{\geqslant 0}$, which satisfies the following for all $e_p \in \mathbb{R}^{n_x}$, $x \in \mathbb{R}^{n_x}$, $u \in \mathbb{R}^{n_u}$,

$$a_1 \|e_p\|^2 \leqslant V_p(e_p) \leqslant a_2 \|e_p\|^2, \tag{16a}$$

$$\frac{\partial V_p}{\partial e_p} F_p(e_p, p, p^\star, u, x) \leqslant -\lambda_0 V_p(e_p) + \tilde{\gamma}(\tilde{p}, x, u). \tag{16b}$$

Since the class of system (13) has not been considered in the literature, we show that robustness to parameter mismatch can be achieved under mild assumptions on the plant (4).

Lemma 3: Consider the plant (13) with a compact $\mathbb{P}^\star$ and the multi-observer (14) under Assumption 1. Suppose the matrices A_p, E_p, H_p are continuous in $p \in \mathbb{P}^\star$. Then, the multi-observer (14) satisfies Assumption 1, i.e., robust to parameter mismatch.

Proof: [Sketch] For any $p \in \mathbb{P}$, choose a quadratic candidate Lyapunov function $V_p(e_p) = e_p^T P_p e_p$ with $P_p = P_p^T > 0$ and following the same procedure as in the proof of Proposition 2 in [10] for linear systems, we arrive at $\tilde{\gamma}(\tilde{p}, x, u) := \frac{2}{\nu} |P_p|^2 \gamma(\tilde{p}, x, u)^2$, where $\nu > 0$ and $\gamma(\tilde{p}, x, u) := \max_{p \in \mathbb{P}^\star} \left| \tilde{A}(p, p^\star)x + \tilde{E}(p, p^\star)\psi(H_{p^\star} x, u) \right|$. The function γ is continuous since A_p, E_p, H_p are continuous in its arguments, the function ψ is locally Lipschitz in both its arguments (which implies continuity) and $\mathbb{P}^\star$ is a compact set. ∎

Now that we have verified that the multi-observer (14) is robust to parameter mismatch, we are ready to provide conditions such that a state-sharing multi-observer can be used in the supervisory observer. The proof of the following result is done by application of Theorem 1 and Lemma 3.

Proposition 1: Consider the plant (13) and the multi-observer (14) under Assumption 1, where A_p, E_p and H_p are continuous in $p \in \mathbb{P}^\star$. Suppose for every $p \in \mathbb{P}$, there exist observer gains L_p such that Assumption 1 hold. If the state-sharing multi-observer (10) and the multi-observer (19) are both initialized at zero, then the state-sharing multi-observer (10) also satisfies the robustness property in Assumption 1.

The class of multi-observers (5) considered in this paper is applicable to the class of linear systems presented in Section VI of [10].

B. Secure state observer

The problem of estimating the states of a plant where a subset of the sensors can be maliciously manipulated is known in the literature as secure state estimation [2]–[7]. Here, the signal $a(t)$ in the plant (1) models additive sensor attacks. The multi-observer has been employed in secure state estimation for consistency checking, such as in [3], [5], [6], or in a satisfiability checking framework [4]. A common requirement in all these works where up to a positive integer M out of N sensors are allowed to be attacked, is $N > 2M$ and we need

$$N_p := \frac{N!}{M!(N-M)!} + \frac{N!}{2M!(N-2M)!} \tag{17}$$

observers to reconstruct the states of the plant. The number of observers N_p is large when N is large (many sensors). For example, a system with $N = 100$ sensors where $M = 40$ can be corrupted means that a secure state observer needs to employ a multi-observer with $N_p = 1.3746 \times 10^{28}$ observers! Since the dimension of a state-sharing multi-observer is given by (11) which is independent of N_p, the memory burden of secure state observers is alleviated.

The class of plant (4) and multi-observer (5) capture the linear systems in [3], [4]. To this end, the plant with N sensors takes the form of (4) with the parameter set $\mathbb{P}^\star := \mathbb{N}_{[1,N]}^{N_p}$, with $A_{p^\star} = A$, $E_{p^\star} = E$ being parameter independent matrices and the nonlinearity ψ being input-dependent only[2], i.e., for $i \in \mathbb{N}_{[1,N]}$,

$$\dot{x} = Ax + E\psi(0, u), \quad y_i = H_i x + a_i \in \mathbb{R}^{n_i}, \tag{18}$$

where $\sum_{i \in \mathbb{N}_{[1,N]}} n_i =: n_y$, and for a set $\mathsf{I} \subset \mathbb{N}_{[1,N]}$ with cardinality $|\mathsf{I}| \leqslant M$, the potentially unbounded attack signal a_i satisfies $a_i(t) = 0$ for all $t \in \mathbb{R}_{\geqslant 0}$ and $i \notin \mathsf{I}$.

According to Theorem 1 of [5], the multi-observer takes the form (5) considered in this paper, with $\mathbb{P}$ being the collection of all sets $\mathsf{J} \subset \mathbb{N}_{[1,N]}$ and $\mathsf{S} \subset \mathbb{N}_{[1,N]}$ with cardinality $|\mathsf{J}| = N - 2M$ and $|\mathsf{S}| = N - M$, respectively. Hence, the parameter set $\mathbb{P}$ has cardinality N_p given by (17). Further, the matrices $A_p = A$, and $E_p = E$ are parameter independent and $y_p = H_p \hat{x}_p$, where H_p is the stacking of the j-th row of the matrix H for all $j \in p$. To summarise, the multi-observer in the secure state observer [5] for plant (18) takes the following form for $p \in \mathbb{P}$

$$\dot{\hat{x}}_p = A\hat{x}_p + E\psi(0, u) + L_p(y_p - H_p \hat{x}_p), \tag{19}$$

where $y_p = (y_i)_{i \in p}$ (stacking of all y_i indexed by the index set p). Crucially, for every $p \in \mathbb{P}$, each observer (19) is designed to be input-to-state stable (ISS) with respect to the attack vector a_p for $p \in \mathbb{P}$, i.e.,

Assumption 2: For every $p \in \mathbb{P}$, each observer (19) is designed such that its state estimation error $e_p := \hat{x}_p - x$ satisfies

$$\|e_p(t)\| \leqslant k_p e^{-\lambda_p t} \|e_p(0)\| + \gamma_p \left(\underset{s \in [0,t]}{\text{esssup}} \|a_p(s)\| \right), \ t \in \mathbb{R}_{\geqslant 0}, \tag{20}$$

for all $e_p(0) \in \mathbb{R}^{n_x}$, where $(k_p, \lambda_p) \in \mathbb{R}^2_{>0}$ and $\gamma_p \in \mathcal{K}$ and $a \in \mathcal{L}_{\mathcal{A}}$.

Remark 1: The multi-observer based solution in [3], [5] to the secure state estimation problem is dependent on the fact that each observer (19) receives only a subset y_p of the full measurement vector y such that the desired ISS property with respect to the attack vector a_p stated in Assumption 2 can be fulfilled.

We show that the ISS property can be attained by choosing the observer matrices L_p for every $p \in \mathbb{P}$ according to Assumption 1.

[2]A plant of this form induces a multi-observer that receives only a subset y_p of the full measurement vector y. We elaborate on this crucial point that is particular to the secure state estimation problem in Remark 1.

Lemma 4: Consider the plant (18) and multi-observer (19). Suppose that for every $p \in \mathbb{P}$, the observer gain L_p are chosen such that Assumption 1 hold. Then for every $p \in \mathbb{P}$, observer (19) satisfies Assumption 2.

Proof: Let $p \in \mathbb{P}$. The state estimation error e_p system satisfies

$$\dot{e}_p = \Lambda_p e_p - L_p a_p, \tag{21}$$

where $\Lambda_p := A - L_p H_p$. The solution e_p to (21) is

$$e_p(t) = e^{\Lambda_p t} e_p(0) - \int_0^t e^{\Lambda_p (t-s)} L_p a_p(s) ds. \tag{22}$$

Since Λ_p satisfies Assumption 1, we obtain (20) in Assumption 2 with $\gamma_p(r) := \frac{k_p}{\lambda_p} \|L_p\| r$ where k_p and λ_p come from Assumption 1. ■

By straightforward application of Theorem 1 and Lemma 4, we obtain the following sufficient conditions for constructing a state-sharing multi-observer for (19).

Proposition 2: Consider the plant (18) and multi-observer (19) under Assumption 2. Suppose for every $p \in \mathbb{P}$, the observer gain L_p is chosen such that Assumption 1 holds. If the state-sharing multi-observer (10) and the multi-observer (19) are both initialized at zero, then the state-sharing multi-observer (10) also satisfies Assumption 2.

Proposition 2 is applicable to the linear systems considered in [3]. Although the results here were developed for continuous-time systems, similar developments can be done for discrete-time systems, which has been done for discrete-time linear systems in [4].

VI. Conclusion and Future Work

We have shown that a state-sharing multi-observer can be constructed for a class of multi-observers, which is advantageous in applications such as the supervisory observer and the secure state observer where a large number of observers need to be employed in parallel. Future work will focus on the design of a state-sharing multi-observer for multi-observers with a state-dependent nonlinearity.

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