IEEE Control Systems Letters paper presented at
2023 62nd IEEE Conference on Decision and Control (CDC)
December 13-15, 2023. Marina Bay Sands, Singapore

Optimal Safety-Critical Control of Epidemics

Brooks A. Butler, *Student Member, IEEE*, and Philip E. Paré, *Member, IEEE**

***Abstract*— We present a generalized model for epidemic processes that partitions control into changes in linear and non-linear flow rates between compartments, respectively. We then define an optimal control problem that minimizes the weighted cost of rate control on the generalized model while maintaining conditions that guarantee system safety at any time using control barrier functions. Using this formulation, we prove that under homogeneous penalties the optimal controller will always favor increasing the linear flow out of an infectious process over reducing nonlinear flow in. Further, in the case of heterogeneous penalties, we provide necessary and sufficient conditions under which the optimal controller will set control of non-linear rates (i.e., the reduction of flow rate into the infection process) to zero. We then illustrate these results through the simulation of a bi-virus SEIQRS model.**

***Index Terms*— Emerging control applications, Optimal control, Biological systems**

I. Introduction

THE modeling, analysis, and control of complex epidemic processes has gained significant attention in recent years, due in large part to the emergence of the global COVID-19 pandemic and its widespread societal impact. However, despite an extensive body of literature dedicated to the understanding and control of such systems [1]–[5], key questions still remain open regarding best practices for exerting sufficient control on general epidemic models such that infection level thresholds are not exceeded at any time while also considering the cost of taking such actions. In this paper, we use principles of safety-critical control via *control barrier functions* [6], [7] to formulate an optimal control problem for a generalized compartmental epidemic model and answer questions related to the optimal control of such systems while satisfying constraints for system safety.

The use of compartments to model epidemic processes is common when the behavior of any given epidemic process can be categorized into discrete stages, where individuals or groups may transfer between stages in the process at a certain rate. A distinguishing characteristic of these models is the ability of individuals in different stages of the epidemic process to mix and interact, where interactions between infected and susceptible populations cause further infections at some given rate. This class of models allows us to make assertions about properties of convergence, stability, and equilibria for epidemic processes with respect to these flow rate parameters [8], [9]. However, making general statements about such properties for all epidemic models is non-trivial due to the nonlinearities introduced via infections through population mixing.

*Brooks A. Butler and Philip. E. Paré are with the Elmore Family School of Electrical and Computer Engineering at Purdue University. Emails: {brooksbutler, philpare}@purdue.edu. This work was partially funded by Purdue's Elmore Center for Uncrewed Aircraft Systems and the National Science Foundation, grant NSF-ECCS #2238388.

The application of control barrier functions provides a useful tool for guaranteeing population safety, according to the definition of a barrier function, as often the control of an epidemic process is focused on preventing unsafe levels of infection that may overwhelm the capacity of treatment facilities. Recent work in [10], [11] provides closed-form solutions for controllers of different classes of barrier functions for a generalized compartmental epidemic process that incorporates time delays; however, the cost of such control is not considered. In [12], the social cost with respect to the reduction of the infection rate is considered in the control of a vectorized SIR model using safety-critical conditions. Further, other work has considered the optimal control of specific epidemic processes such as influenza [13], dengue [14], information spread [15], and processes involving some quarantine compartment [16], [17]. However, in this work, we wish to consider the comparative cost of controlling either the entry to or exit from a generalized epidemic process, which offers two main advantages. First, a comparative cost analysis of control actions enhances our ability to allocate resources efficiently and maximize the impact of interventions in epidemic processes. For example, is it more effective to allocate resources toward reducing the spreading rate via preventative measures or toward increasing the rate of exit from the infectious process via random testing and quarantine? Second, analyzing a generalized model provides valuable insights into the universal properties of epidemic models, with the potential to inform numerous applications.

Therefore, we define the notation for a generalized epidemic model in the context of process rate control in Section II. We then define our conditions for safety, as well as the conditions under which safety can be guaranteed using control barrier functions in Section III. Finally, we provide an analysis of optimal control of the generalized model while satisfying the required conditions for safety and illustrate these results via simulation in Sections IV and V.

II. Model Definitions

In this section, we define the dynamics of our generalized compartmental model for epidemic processes and illustrate its use to describe a bi-virus $SEIQRS$ model.

A. Generalized Epidemic Model

Consider an n-compartment epidemic process, where each compartment represents a particular state in the epidemic process (e.g. an individual being classified as either susceptible, infected, removed, etc.). We define the state of a model with n compartments at time t as the vector

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) & \dots & x_n(t) \end{bmatrix}^\top \tag{1}$$

where $\mathbf{x}(t) \in [0,1]^n$ and $\sum_{i\in[n]} x_i(t) = 1$ with $[n] = \{1,\dots,n\}$. For convenience and conciseness of notation, we drop explicit dependence on t going forward, although the states continue to vary with time. We classify each compartment into two disjoint categories: $\mathcal{I} \subset [n]$ and $\mathcal{H} \subset [n]$, with $\mathcal{I} \cap \mathcal{H} = \emptyset$ and $\mathcal{I} \cup \mathcal{H} = [n]$, where $\mathcal{I}$ and $\mathcal{H}$ denote the sets of critical (infected, exposed, asymptomatic, etc.) and non-critical compartments (healthy, recovered, quarantined, etc.) in the epidemic process, respectively. We define critical compartments in $\mathcal{I}$ to be those which are participating (or will participate) in the infecting process of a given disease. Therefore, we wish to keep the total population in these compartments below some total safety threshold.

We further classify $\mathcal{I}$ into three potentially overlapping subcategories: $\mathcal{X} \subset \mathcal{I}$, $\mathcal{O} \subset \mathcal{I}$, and $\mathcal{T} \subset \mathcal{I}$, where $\mathcal{X}$ is the set of infectious compartments that facilitate entry into the epidemic process via contact with compartments in $\mathcal{H}$; $\mathcal{O}$ includes compartments with an outgoing connection to any compartment in $\mathcal{H}$; and $\mathcal{T} = \mathcal{I} \setminus (\mathcal{X} \cup \mathcal{O})$.

We define the dynamics of compartment i in the model as

$$\dot{x}_i(\mathbf{x}, U, V) = f_i(\mathbf{x}, U) + g_i(\mathbf{x}, V) \tag{2}$$

where $U, V \in \mathbb{R}^{n\times n}_{\geq 0}$ are matrices of control inputs, with u_{ij} denoting the rate change of linear flow from compartment j to compartment i; v_{ij} denoting the change in the rate of infection when compartment $j \in \mathcal{H}$ comes into contact with compartment $i \in \mathcal{X}$; $g_i : \mathbb{R}^n \to \mathbb{R}$ accounts for connections of these critical multiplicative compartments to and from compartment i; and $f_i : \mathbb{R}^n \to \mathbb{R}$ accounts for connections from all other compartments. We define f_i as

$$f_i(\mathbf{x}, U) = \sum_{j\in\mathcal{J}_i} (\theta_{ij} + u_{ij})x_j - \sum_{k\in\mathcal{K}_i} (\theta_{ki} + u_{ki})x_k \tag{3}$$

where $\mathcal{J}_i, \mathcal{K}_i \subseteq \mathcal{O}\cup\mathcal{T}$ denote the sets of compartments with an incoming connection to or from compartment i, respectively, and θ_{ij} is the rate at which the compartment j flows into compartment i, where $\theta_{ij} = 0$ indicates there is no such connection. Further, let $\beta_{ij} \in \mathbb{R}_{\geq 0}$ represent the infection rate when compartment $i \in \mathcal{X}$ comes in contact with non-critical compartment $j \in \mathcal{H}$, where $\beta_{ij} = 0$ indicates no such interaction occurs. We then define

$$\mathcal{W} = \{(i,j) \in \mathcal{X} \times \mathcal{H} : \beta_{ij} > 0\}$$

to be the set of all compartment pairs that facilitate entry to the infection process $\mathcal{I}$. We can then define g_i as

$$g_i(\mathbf{x}, V) = \sum_{(j,k)\in\mathcal{Y}_i} (\beta_{jk} - v_{jk})x_j x_k - \sum_{(p,q)\in\mathcal{Z}_i} (\beta_{pq} - v_{pq})x_p x_q \tag{4}$$

where $\mathcal{Y}_i, \mathcal{Z}_i \subseteq \mathcal{W}$ denote the sets of multiplicative compartment pairs with an incoming connection to or from compartment i, respectively, and

$$v_{ij} \leq \beta_{ij}, \forall (i,j) \in \mathcal{X} \times \mathcal{H}. \tag{5}$$

Note that by our definitions of critical and non-critical compartments, if $\mathcal{Z}_i \neq \emptyset$, then $q = i$ for all $(p,q) \in Z_i$ (i.e., only compartments in $\mathcal{H}$ will have an outgoing connection due to infection). Similarly, if $\mathcal{Y}_i \neq \emptyset$, then $i \in \mathcal{I}$. Finally, we require

$$\sum_{i\in[n]} \dot{x}_i = 0 \tag{6}$$

which ensures that the population flowing out of one compartment must also be flowing into another (i.e. the total population is conserved). For a graphical illustration of our model class, see Figure 1.

B. Example Model Formulation: Bi-Virus SEIQRS

To illustrate our generalized compartmental model, we apply it to a bi-virus susceptible-exposed-infected-quarantined-recovered-susceptible ($SEIQRS$) model with state dynamics defined as

$$\begin{aligned}
\dot{S} &= \delta R - [(\beta_1 - v_{I_1S})I_1 + (\beta_2 - v_{I_2S})I_2]S \\
\dot{E}_1 &= (\beta_1 - v_{I_1S})SI_1 - \sigma_1 E_1 \\
\dot{E}_2 &= (\beta_2 - v_{I_2S})SI_2 - \sigma_1 E_2 \\
\dot{I}_1 &= \sigma_1 E_1 - [(\eta_1 + u_{QI_1}) + (\gamma_1 + u_{RI_1})]I_1 \\
\dot{I}_2 &= \sigma_2 E_2 - [(\eta_2 + u_{QI_2}) + (\gamma_2 + u_{RI_2})]I_2 \\
\dot{Q} &= (\eta_1 + u_{QI_1})I_1 + (\eta_2 + u_{QI_2})I_2 - \sigma_3 Q \\
\dot{R} &= (\gamma_1 + u_{RI_1})I_1 + (\gamma_2 + u_{RI_2})I_2 + \sigma_3 Q - \delta R
\end{aligned} \tag{7}$$

where β_1, β_2 are infection rates, σ_1, σ_2 are the rates the exposed population become infectious, γ_1, γ_2 are recovery rates, and η_1, η_2 are quarantine rates, for virus 1 and virus 2, respectively. We also define the rate at which the quarantined compartment recovers as σ_3 and a loss of immunity rate δ. In this case, we group the compartments as follows: $\mathcal{H} = \{S, Q, R\}$, $\mathcal{I} = \{E_1, E_2, I_1, I_2\}$, $\mathcal{X} = \mathcal{O} = \{I_1, I_2\}$, and $\mathcal{T} = \{E_1, E_2\}$, which is illustrated in Figure 2. Note that the control inputs v_{I_1S}, v_{I_2S} may be interpreted as social distancing policies, targeted lockdown, spread prevention measures, etc., for viruses 1 and 2, respectively. Additionally, u_{QI_1}, u_{QI_2} may be interpreted as increasing the rate of quarantine, which could be achieved via increased testing, and u_{RI_1}, u_{RI_2} as increasing the recovery rate via medical interventions, for viruses 1 and 2, respectively.

III. Safety Critical Control

In order to utilize the property of safety guarantees as defined in [6], [7], we must first define a zero-superlevel set in our state space that encapsulates our notion of safety. Since our objective is to keep the total population of compartments in $\mathcal{I}$ below some threshold, which may be defined by our capacity to treat infected individuals or our threshold before operations must be shut down, we define our safe set to be

$$\mathcal{S} = \{\mathbf{x} \in [0,1]^n; h(\mathbf{x}) \geq 0\} \tag{8}$$

where

$$h(\mathbf{x}) = \mathcal{I}_{\max} - \sum_{i\in\mathcal{I}} x_i \tag{9}$$

with $\mathcal{I}_{\max} \in [0,1]$ being the chosen maximum safe threshold of the total population inside $\mathcal{I}$ at any time t. Note that in this

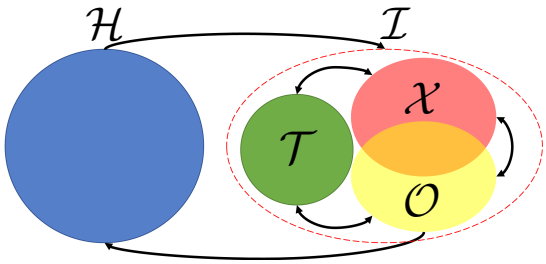


Fig. 1: A graphical illustration of the groupings for compartments in our generalized compartmental model, with $\mathcal{H}$ and $\mathcal{I}$ denoting the group of non-critical and critical compartments, respectively; $\mathcal{X}$ and $\mathcal{O}$ denoting the group of infectious and outlet compartments, respectively (note that orange denotes $\mathcal{X} \cap \mathcal{O}$); and $\mathcal{T}$ grouping all other intermediary compartments in the process $\mathcal{I}$ that neither infect compartments in $\mathcal{H}$ nor have an outlet connection.

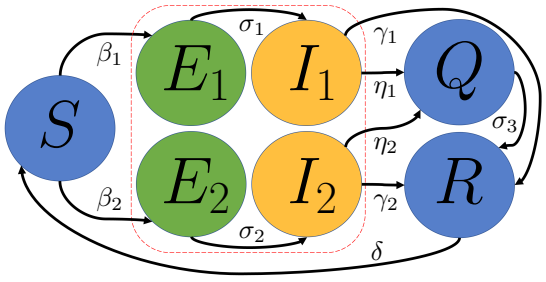


Fig. 2: A bi-virus $SEIQRS$ model, with dynamics defined by (7), where $\mathcal{H} = \{S, Q, R\}$, $\mathcal{I} = \{E_1, E_2, I_1, I_2\}$, $\mathcal{X} = \mathcal{O} = \{I_1, I_2\}$, and $\mathcal{T} = \{E_1, E_2\}$. Note that compartment colors correspond with the compartment groups in Figure 1.

paper, $\mathcal{I}_{\max}$ is a constant. Since

$$\sum_{i\in[n]} \dot{x}_i(\mathbf{x}, U, V) = \sum_{i\in\mathcal{I}} \dot{x}_i(\mathbf{x}, U, V) + \sum_{i\in\mathcal{H}} \dot{x}_i(\mathbf{x}, U, V) = 0 \tag{10}$$

by (6), we have

$$\dot{h}(\mathbf{x}, U, V) = -\sum_{i\in\mathcal{I}} \dot{x}_i(\mathbf{x}, U, V) = \sum_{i\in\mathcal{H}} \dot{x}_i(\mathbf{x}, U, V). \tag{11}$$

With our conditions for safety defined, we provide the conditions where our system is guaranteed to remain safe.

Lemma 1. *Consider the dynamics in* (2) *with control inputs* U *and* V *and the set* $\mathcal{S}$ *defined by* (8) *and* (9)*. The set* $\mathcal{S}$ *is forward invariant (safe) if* $\mathbf{x}_0 \in \mathcal{S}$ *and* $\exists U, V$ *such that*

$$\dot{h}(\mathbf{x}, U, V) \geq -\alpha(h(\mathbf{x})) \tag{12}$$

where $\alpha(\cdot)$ *is a class-*$\mathcal{K}$ *function.*

This result follows directly from the well-established literature on *control barrier functions* in [6], [7]. Note that a closed-form solution to a similar generalized compartmental epidemic model is given in [11] for different classes of barrier functions. However, the objective of our analysis in this work is to evaluate the effectiveness of different control actions, with precise reduction or increase in flow rates between individual compartments. Thus, we formulate our control problem around the notion of control cost, described in Section IV, which allows us to make statements about the optimality of different control actions implemented on a generalized compartmental epidemic model.

IV. Optimal control

In this section, we define a cost function that evaluates the relative cost of any flow rate change between any two compartments in $\mathcal{H} \cup \mathcal{I}$. We then provide an analysis of when certain control actions will be optimal with respect to the given cost function. We define the cost of exerting controls U and V at time t as

$$J(t, U(t), V(t)) = \sum_{(i,j)\in[n]\times[n]} \lambda_{ij} u_{ij}(t) + \sum_{(i,j)\in\mathcal{X}\times\mathcal{H}} \omega_{ij} v_{ij}(t) \tag{13}$$

where $\lambda_{ij}, \omega_{ij} \in \mathbb{R}_{\geq 0}$ are the associated unit cost penalties of exerting control via U and V, respectively. Further, let $\Lambda, \Omega \in \mathbb{R}_{\geq 0}^{n\times n}$ be the matrices that collect all penalty entries for λ_{ij} and ω_{ij}, respectively. Thus, the optimization problem is

$$\begin{aligned} \min_{U,V} \quad & \int_0^{t_f} J(t, U(t), V(t))\, dt \\ \text{s.t.} \quad & \dot{h}(\mathbf{x}(t), U(t), V(t)) \geq -\alpha(h(\mathbf{x}(t))), \forall t \\ & u_{ij} \geq 0,\ \beta_{ij} \geq v_{ij} \geq 0,\ \forall (i,j) \in [n] \times [n]. \end{aligned} \tag{14}$$

Note that (14) is convex since, (3), (4), (11), and (13) are linear with respect to $U(t)$ and $V(t)$. We define $U^*(t), V^*(t)$ to be the set of control inputs that solve $\arg\min J(t, U(t), V(t))$ subject to (14) at time t. Note that the goal of (14) is to minimize the total cost of taking actions to keep the population in the infecting process ($\mathcal{I}$) below a defined threshold at any time t, where this threshold may represent the capacity of a given population to treat infected individuals. Again, for conciseness of notation, we omit the notation of time dependence of J for the remainder of this section.

For our analysis, we impose the following assumption.

Assumption 1. *Let* $\mathcal{X} = \mathcal{O} \neq \emptyset$.

In other words, we assume that for each infectious compartment, there exists at least one outlet connection to a non-critical compartment in $\mathcal{H}$. Note that this assumption still captures many, if not most, commonly used epidemic models, since after being infectious it is common to transition to a removed or susceptible state; however, this assumption excludes some multi-stage infection models, such as $SAIR$, where A is an asymptomatic stage that can cause infection and only transitions to the infection compartment I.

We now consider properties of the optimal solution for (14) with respect to the penalties Λ and Ω. To aid in the proof of our analytical results, we derive the following reduced expressions under Assumption 1. Consider the expression for $\dot{h}(\mathbf{x}, U, V)$, given by (11),

$$\begin{aligned} \dot{h} &= \sum_{i\in\mathcal{H}} f_i(\mathbf{x}, U) + g_i(\mathbf{x}, V) \\ &= \sum_{i\in\mathcal{H}} \Bigg[\sum_{j\in\mathcal{J}_i} (\theta_{ij} + u_{ij}) x_j - \sum_{k\in\mathcal{K}_i} (\theta_{ki} + u_{ki}) x_k \\ &\quad + \sum_{(l,m)\in\mathcal{Y}_i} (\beta_{lm} - v_{lm}) x_l x_m - \sum_{(p,q)\in\mathcal{Z}_i} (\beta_{pq} - v_{pq}) x_p x_q \Bigg]. \end{aligned} \tag{15}$$

By (6), we have

$$\sum_{k\in\mathcal{K}_i}(\theta_{ki}+u_{ki})x_k=\sum_{j\in\mathcal{J}_i\cap\mathcal{H}}(\theta_{ij}+u_{ij})x_j. \tag{16}$$

Thus, by (16) and since $\mathcal{Y}_i=\emptyset,\forall i\in\mathcal{H}$, (15) reduces to

$$\dot{h}=\sum_{i\in\mathcal{H}}\left[\sum_{j\in\mathcal{J}_i^{\mathcal{X}}}(\theta_{ij}+u_{ij})x_j-\sum_{(p,q)\in\mathcal{Z}_i}(\beta_{pq}-v_{pq})x_px_q\right] \tag{17}$$

where, by Assumption 1, $\mathcal{J}_i^{\mathcal{X}}=\mathcal{J}_i\setminus\mathcal{H}\subseteq\mathcal{X}$ is the set of remaining compartments in $\mathcal{I}$ with an outgoing connection to compartment $i\in\mathcal{H}$. We collect the terms of (17) that are independent of control inputs as

$$\begin{aligned}d(\mathbf{x})&=\sum_{i\in\mathcal{H}}\left[\sum_{j\in\mathcal{J}_i^{\mathcal{X}}}\theta_{ij}x_j-\sum_{(p,q)\in\mathcal{Z}_i}\beta_{pq}x_px_q\right]\\&=\sum_{(j,i)\in\mathcal{H}\times\mathcal{X}}\theta_{ji}x_i-\sum_{(i,j)\in\mathcal{X}\times\mathcal{H}}\beta_{ij}x_ix_j\end{aligned} \tag{18}$$

where the indices i and j are swapped for convenience such that in both terms of (18) we have $i\in\mathcal{X}$ and $j\in\mathcal{H}$. Similarly, we collect the terms of (17) that are dependent on control inputs U,V as

$$\begin{aligned}H(\mathbf{x},U,V)&=\sum_{i\in\mathcal{H}}\left[\sum_{j\in\mathcal{J}_i^{\mathcal{X}}}u_{ij}x_j+\sum_{(p,q)\in\mathcal{Z}_i}v_{pq}x_px_q\right]\\&=\sum_{(j,i)\in\mathcal{H}\times\mathcal{X}}u_{ji}x_i+\sum_{(i,j)\in\mathcal{X}\times\mathcal{H}}v_{ij}x_ix_j\end{aligned} \tag{19}$$

where H may be considered the resulting control force exerted on the system given U,V with respect to $\dot{h}$. Thus, by Lemma 1, we have that if $\mathbf{x}\in\mathcal{S}$ and $\exists U,V$ such that

$$d(\mathbf{x})+H(\mathbf{x},U,V)+\alpha(h(\mathbf{x}))\geq 0 \tag{20}$$

then the system state is guaranteed to remain within the safe set $\mathcal{S}$, i.e., $\mathcal{S}$ is forward invariant. We can evaluate the cost of exerting the control U,V at time t according to (13) and (20) as

$$J(U,V)=\sum_{(j,i)\in\mathcal{H}\times\mathcal{X}}\lambda_{ji}u_{ji}+\sum_{(i,j)\in\mathcal{X}\times\mathcal{H}}\omega_{ij}v_{ij}. \tag{21}$$

We may further partition the control and cost terms of (19) and (21), respectively, relative to the pairs of interacting compartments $(i,j)\in\mathcal{W}\subseteq\mathcal{X}\times\mathcal{H}$ where

$$H(\mathbf{x},U,V)=\sum_{(i,j)\in\mathcal{W}}H_{(i,j)},\quad J(U,V)=\sum_{(i,j)\in\mathcal{W}}J_{(i,j)}$$

with $H_{(i,j)}$ and $J_{(i,j)}$ defined as

$$H_{(i,j)}(\mathbf{x},U,V)=\sum_{k\in\mathcal{K}_i^{\mathcal{H}}}u_{ki}x_i+v_{ij}x_ix_j \tag{22}$$

$$J_{(i,j)}(U,V)=\sum_{k\in\mathcal{K}_i^{\mathcal{H}}}\lambda_{ki}u_{ki}+\omega_{ij}v_{ij} \tag{23}$$

where $\mathcal{K}_i^{\mathcal{H}}=\mathcal{K}_i\setminus\mathcal{I}$. In other words, we can encapsulate the total control cost and relative effectiveness by considering separately each interaction of compartments $(i,j)\in\mathcal{W}$ which includes the flow out of $j\in\mathcal{H}$ due to infections from $i\in\mathcal{X}$ and the flow into $\mathcal{H}$ from all outlet connections of i.

Given the above notation, we consider properties of the optimal solution to (14) for the case defined as follows.

Assumption 2. *Let $\Lambda=a\cdot\mathbf{1}^{n\times n}$, $\Omega=a\cdot\mathbf{1}^{n\times n}$, where $a\in\mathbb{R}_{\geq 0}$.*

Assumption 2 implies that all control variables have equal weighting with respect to the control cost (i.e., the cost of reducing the rate of entry is equal to the cost of increasing the rate of exit from the infecting process, respectively), which yields the following result.

Theorem 1. *Given Assumptions 1-2, $V^*(t)=\mathbf{0},\forall t\in\mathbb{R}_{\geq 0}$.*

Proof. If $x_i=0,\forall i\in\mathcal{X}$, then $V^*(t)=\mathbf{0}$ is trivially true.

If $\exists i\in\mathcal{X}$ such that $x_i>0$ and we assume, by way of contradiction, that $\exists(i,j)\in\mathcal{W}$ such that $v_{ij}^*>0$. By (22), (23), and Assumption 2, we must have

$$v_{ij}x_ix_j\geq u_{ki}x_i\ \text{ and }\ v_{ij}<u_{ki},\ \forall k\in\mathcal{K}_i^{\mathcal{H}}$$

which is a contradiction since $\frac{u_{ki}}{v_{ij}}\leq x_j\leq 1$ and $\frac{u_{ki}}{v_{ij}}>1$ cannot both be true. □

This result implies that if the cost of all control is equal then an unconstrained optimal controller will always choose to increase the flow out of the infectious process rather than decrease the rate of entry. Note, however, that (14) does not account for the cost of each state, rather it only considers the cost of allocating resources to change flow rates between compartments to satisfy the safety condition. For example, our cost function only considers the instantaneous cost of increasing the rate of quarantine and not the cost of a certain proportion of the population being in quarantine.

We now consider the following more general case of non-uniform penalties.

Assumption 3. *Let $\Lambda\neq a\cdot\mathbf{1}^{n\times n}$, $\Omega\neq a\cdot\mathbf{1}^{n\times n}$, where $a\in\mathbb{R}_{\geq 0}$ and $d(\mathbf{x})+\alpha(h(\mathbf{x}))<0$.*

In this case, we require the additional condition of $d(\mathbf{x})+\alpha(h(\mathbf{x}))<0$ since otherwise the minimal control necessary to keep the system safe is trivially $U^*=\mathbf{0},V^*=\mathbf{0}$, where $d(\mathbf{x})$ may be interpreted as the control-free dynamics of the system and $\alpha(h(\mathbf{x}))$ as a relaxation term. Thus, given Assumption 3, we have the following result.

Theorem 2. *Given Assumptions 1 and 3, $V^*(t)=\mathbf{0}$ if and only if $\forall(i,j)\in\mathcal{W},\exists k\in\mathcal{K}_i^{\mathcal{H}}$ such that $x_j<\frac{\omega_{ij}}{\lambda_{ki}}$ or $\exists p\in\mathcal{X}\setminus\{i\}$ and $\exists q\in\mathcal{K}_p^{\mathcal{H}}$ such that $\frac{x_ix_j}{x_p}<\frac{\omega_{ij}}{\lambda_{qp}}$.*

Proof. For proof of necessity, consider the contrapositive condition: assume $\exists(i,j)\in\mathcal{W}$ such that $v_{ij}^*>0$. Thus, for v_{ij} to be optimal, when compared to the alternative control terms of $u_{ki},\forall i\in\mathcal{X},\forall k\in\mathcal{K}_i^{\mathcal{H}}$, by (22), (23), we must have that both

$$v_{ij}x_ix_j\geq u_{ki}x_i\ \text{ and }\ \omega_{ij}v_{ij}<\lambda_{ki}u_{ki},\ \forall k\in\mathcal{K}_i^{\mathcal{H}} \tag{24}$$

and $\forall p\in\mathcal{X}\setminus\{i\}$,

$$v_{ij}x_ix_j\geq u_{qp}x_p\ \text{ and }\ \omega_{ij}v_{ij}<\lambda_{qp}u_{qp},\ \forall q\in\mathcal{K}_p^{\mathcal{H}} \tag{25}$$

Therefore, (24) and (25) are satisfied if

$$x_j \geq \frac{u_{ki}}{v_{ij}} \text{ and } \frac{\omega_{ij}}{\lambda_{ki}} < \frac{u_{ki}}{v_{ij}}, \ \forall k \in \mathcal{K}_i^{\mathcal{H}}$$

and $\forall p \in \mathcal{X} \setminus \{i\}$

$$\frac{x_i x_j}{x_p} \geq \frac{u_{qp}}{v_{ij}} \text{ and } \frac{\omega_{ij}}{\lambda_{qp}} < \frac{u_{qp}}{v_{ij}}, \ \forall q \in \mathcal{K}_p^{\mathcal{H}}$$

which implies $\exists (i,j) \in \mathcal{W}$ such that $\forall p \in \mathcal{X} \setminus \{i\}$

$$x_j \geq \frac{\omega_{ij}}{\lambda_{ki}}, \forall k \in \mathcal{K}_i^{\mathcal{H}} \text{ and } \frac{x_i x_j}{x_p} \geq \frac{\omega_{ij}}{\lambda_{qp}}, \ \forall q \in \mathcal{K}_p^{\mathcal{H}}$$

yielding the desired condition.

For proof of sufficiency, we again consider the contrapositive. Let $\exists (i,j) \in \mathcal{W}$ such that $x_j \geq \frac{\omega_{ij}}{\lambda_{ki}}, \forall k \in \mathcal{K}_i^{\mathcal{H}}$ and $\frac{x_i x_j}{x_p} \geq \frac{\omega_{ij}}{\lambda_{qp}}, \forall p \in \mathcal{X} \setminus \{i\}, \forall q \in \mathcal{K}_p^{\mathcal{H}}$. Assume, by way of contradiction, that $V^* = \mathbf{0}$. We can then express (20) in terms of the optimal controller as

$$d_{(i,j)}(\mathbf{x}) + \sum_{(p,l) \in \overline{\mathcal{W}}} \left[d_{(p,l)}(\mathbf{x}) + H_{(p,l)}(\mathbf{x}, U^*, \mathbf{0}) \right] + \alpha(h(\mathbf{x})) \geq 0 \tag{26}$$

where

$$d_{(i,j)}(\mathbf{x}) = \sum_{k \in \mathcal{K}_i} \theta_{ki} x_i + \beta_{ij} x_i x_j$$

and $\overline{\mathcal{W}} = \mathcal{W} \setminus \{(i,j)\}$. Without loss of generality, we may choose for each $p \in \mathcal{X}$ a $q \in \mathcal{K}_p^{\mathcal{H}}$ such that $\lambda_{qp} \leq \lambda_{lp}, \forall l \in \mathcal{K}_p^{\mathcal{H}} \setminus \{q\}$, making our optimal control, in terms of (26),

$$d_{(i,j)}(\mathbf{x}) + \sum_{(p,l) \in \overline{\mathcal{W}}} \left[d_{(p,l)}(\mathbf{x}) + u_{qp}^* x_p \right] + \alpha(h(\mathbf{x})) \geq 0.$$

Thus, in order for U^* to be optimal with $V^* = \mathbf{0}$, there must exist a $p \in \mathcal{X}$ such that

$$v_{ij} x_i x_j \leq u_{qp} x_p \text{ and } \lambda_{qp} u_{qp} < \omega_{ij} v_{ij}$$

which implies $\frac{x_i x_j}{x_p} < \frac{\omega_{ij}}{\lambda_{qp}}$, which is a contradiction. Thus, $V^* \neq \mathbf{0}$, completing the proof. □

Note that when Assumption 2 holds, as is the case for Theorem 1, the condition for Theorem 2 is trivially true. One key insight gained from Theorem 2 is that, in the case of a single virus, a greater proportion of x_j that becomes infected by population $i \in \mathcal{X}$ requires a greater magnitude of control v_{ij} to prevent further infections and thus the ratio of costs must be small enough to justify continued exertion of control via V. Further, we see in the case of a multi-virus model (i.e., $|\mathcal{X}| > 1$) that we must also consider the condition where it may still be optimal to meet the requirements for safety according to (20) by compensating with control of other infectious compartments with potentially cheaper linear control terms in Λ, which is otherwise not considered in the single-virus case as illustrated by the following corollary.

Corollary 1. *Given Assumptions 1 and 3 with* $|\mathcal{X}| = 1$, $V^*(t) = \mathbf{0}$ *if and only if* $\forall (i,j) \in \mathcal{W}, \exists k \in \mathcal{K}_i^{\mathcal{H}}$ *such that* $x_j(t) < \frac{\omega_{ij}}{\lambda_{ki}}$.

Proof. This follows from the fact that when $|\mathcal{X}| = 1$ we have $\mathcal{X} \setminus \{i\} = \emptyset$, where $\mathcal{X} = \{i\}$. Thus, $\exists (i,j) \in \mathcal{W}$ such that $\frac{x_i x_j}{x_p} \geq \frac{\omega_{ij}}{\lambda_{qp}}, \forall p \in \mathcal{X} \setminus \{i\}, \forall q \in \mathcal{K}_p^{\mathcal{H}}$ is trivially true. □

Given the results of this section, we see that, in general, the conditions that cause $V^* = \mathbf{0}$ are less strict than those that enable $V^* \neq \mathbf{0}$, which rely on both favorable cost incentives and compartment state conditions. We now illustrate these results in the following section via simulation examples.

V. Simulations

We simulate a bi-virus $SEIQRS$ with dynamics defined in (7) in discrete time via Euler's method with sampling parameter $\tau = 0.01$ and model parameters $\beta_1 = \beta_2 = 1$, $\sigma_1 = \sigma_2 = \sigma_3 = \gamma_1 = \gamma_2 = \delta = 0.25$, and $\eta_1 = \eta_2 = 0.025$, which are selected as such to induce an endemic state in the system. To simulate the optimal control of this system, we utilize a linear program solver at each time step which minimizes the cost according to (13) subject to the safety constraint defined in (12).

Given this model setup, we simulate the optimal controller with the objective $\mathcal{I}_{\max} = 0.2$ for three different sets of penalties, (Λ_1, Ω_1), (Λ_1, Ω_2), and (Λ_2, Ω_2), where, for both Λ_1 and Λ_2, we set $\lambda_{RI_1} = \lambda_{RI_2} = 2$; in Λ_1, we set $\lambda_{QI_1} = \lambda_{QI_2} = 1$; in Ω_1, we set $\omega_{I_1 S} = \omega_{I_2 S} = 1$; in Λ_2, we set $\lambda_{QI_1} = 1, \lambda_{QI_2} = 0.1$; and in Ω_2, we set $\omega_{I_1 S} = \omega_{I_2 S} = 0.8$. In other words, (Λ_1, Ω_1) simulates equal penalties across all relevant control inputs, (Λ_1, Ω_2) simulates the cost of reducing the infection rate being twenty percent cheaper than the cost of increasing the rate of quarantine, and (Λ_2, Ω_2) simulates when the cost of increasing the rate of quarantine for I_2 is ten times cheaper than it is for I_1.

We simulate the system for the above penalties with initial conditions $(S, I_1, I_2) = (0.98, 0.1, 0.1)$, where all other states are set to zero. In Figure 3, we see the results of the optimal controller for each set of penalties compared with the system under no control. Note that the controller maintains the condition for safety, $E_1 + E_2 + I_1 + I_2 \leq \mathcal{I}_{\max} = 0.2$, in all cases. In the case of uniform penalties (Λ_1, Ω_1), we see in Figure 4a that the controller at each time step sets $v_{I_1 S} = v_{I_2 S} = 0$, which is consistent with the results of Theorem 1. In the case of (Λ_1, Ω_2), we see in Figure 4b that the controller sets $u_{QI_1} = u_{QI_2} = 0$ while $S \geq 0.8$ where after $S < 0.8$ the controller immediately switches to setting $v_{I_1 S} = v_{I_2 S} = 0$, which is consistent with Theorem 2 where, if $S < \frac{\omega_{I_1 S}}{\lambda_{QI_1}}$ or $S < \frac{\omega_{I_2 S}}{\lambda_{QI_2}}$, then $V^* = \mathbf{0}$.

In the final case of (Λ_2, Ω_2), we see, both in the simulations of the state dynamics in Figure 3 and the implemented control policy in Figure 4c, three distinct phases. First, we see that the controller sets $v_{I_1 S} = v_{I_2 S} = 0$ while is it able to compensate by exerting control via u_{QI_2} since $\frac{SI_1}{I_2} < \frac{\omega_{I_1 S}}{\lambda_{QI_2}}$, consistent with the second condition of Theorem 2. Second, we see a period when $\frac{SI_1}{I_2} \geq \frac{\omega_{I_1 S}}{\lambda_{QI_2}}$ and $S \geq \frac{\omega_{I_1 S}}{\lambda_{QI_2}}$ causing the controller to switch between $v_{I_1 S}$ and u_{QI_2}. Finally, when $S < \frac{\omega_{I_1 S}}{\lambda_{QI_2}}$ we see again that $v_{I_1 S} = v_{I_2 S} = 0$, where control continues to switch between u_{QI_1} and u_{QI_2}. This switching behavior in phases two and three occurs since our controller is not penalized for switching between control variables. Therefore, the controller maintains the safety condition for the minimum cost at each time step according to the system states, which may be unrealistic in practice. This impracticality can be overcome by incorporating the cost of changing the selection of control variables into the cost function; however, we leave

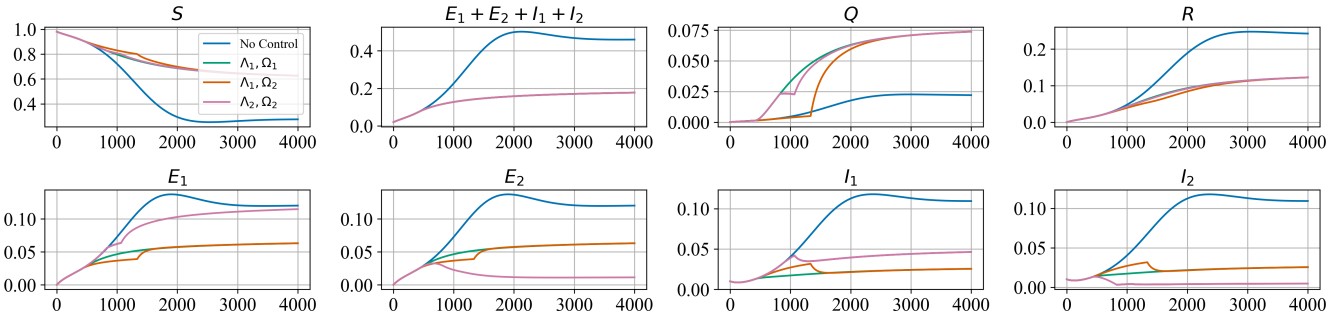
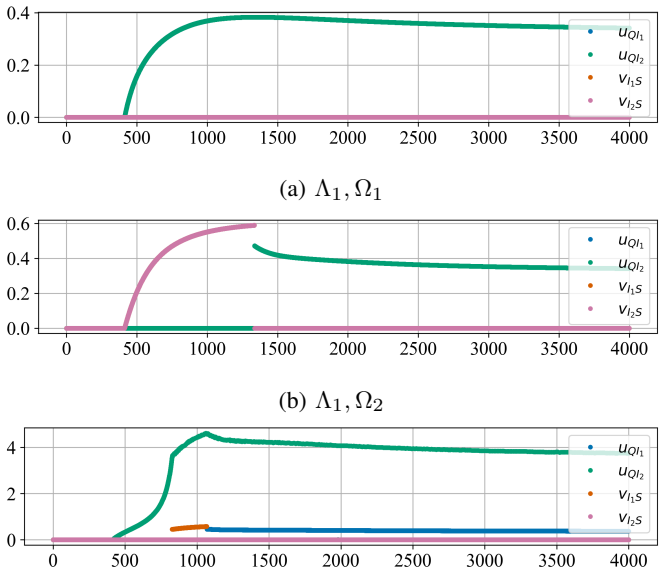


Fig. 3: Simulations of the multi-virus $SEIQRS$ model, with dynamics defined in (7), where the optimal control U^*, V^* is computed at each time step using a linear program solver according to the penalties (Λ_1, Ω_1) (green), (Λ_1, Ω_2) (orange), and (Λ_2, Ω_2) (pink) with the objective of $\mathcal{I}_{\max} = 0.2$ compared with the system under no control (blue). Control values for each penalty configuration are shown in Figure 4.



(a) Λ_1, Ω_1

(b) Λ_1, Ω_2

(c) Λ_2, Ω_2

Fig. 4: Corresponding control values at each time step for the simulations in Figure 3 for each penalty configuration (Λ_1, Ω_1), (Λ_1, Ω_2), and (Λ_2, Ω_2), respectively.

this extension as future work. Further, regardless of the speed at which the controller is allowed to switch between control inputs if the controller is capable of satisfying (12), the system is guaranteed to remain safe.

VI. Conclusion

In this paper, we have constructed a generalized model for epidemic processes that allows for control of rates between any compartment. Using this model class and the objective of maintaining system safety, we have provided conditions under which it is more cost-effective to increase the rates of exit from an epidemic process rather than decrease the rates of infection. While this work offers insight into optimal resource allocation for controlling the spread of an idealized process with known system structure, states, parameters, and unconstrained resources, its results can be applied broadly to many compartmental epidemic processes.

References

[1] R. M. Anderson and R. M. May, *Infectious Diseases of Humans: Dynamics and Control*. Oxford University Press, 1992.

[2] H. Andersson and T. Britton, *Stochastic Epidemic Models and their Statistical Analysis*. Springer Science & Business Media, 2012, vol. 151.

[3] M. J. Keeling and P. Rohani, *Modeling Infectious Diseases in Humans and Animals*. NJ: Princeton University Press, 2008.

[4] M. De Domenico, C. Granell, M. A. Porter, and A. Arenas, "The physics of spreading processes in multilayer networks," *Nature Physics*, vol. 12, no. 10, pp. 901–906, 2016.

[5] J. Sereno, A. Anderson, A. Ferramosca, E. A. Hernandez-Vargas, and A. H. González, "Minimizing the epidemic final size while containing the infected peak prevalence in SIR systems," *Automatica*, vol. 144, p. 110496, 2022.

[6] A. D. Ames, X. Xu, J. W. Grizzle, and P. Tabuada, "Control barrier function based quadratic programs for safety critical systems," *IEEE Transactions on Automatic Control*, vol. 62, no. 8, pp. 3861–3876, 2016.

[7] A. D. Ames, S. Coogan, M. Egerstedt, G. Notomista, K. Sreenath, and P. Tabuada, "Control barrier functions: Theory and applications," in *Proceedings of the 2019 18th European Control Conference (ECC)*. IEEE, 2019, pp. 3420–3431.

[8] R. Pastor-Satorras, C. Castellano, P. Van Mieghem, and A. Vespignani, "Epidemic processes in complex networks," *Reviews of Modern Physics*, vol. 87, no. 3, p. 925, 2015.

[9] P. E. Paré, C. L. Beck, and T. Başar, "Modeling, estimation, and analysis of epidemics over networks: An overview," *Annual Reviews in Control*, vol. 50, pp. 345–360, 2020.

[10] A. D. Ames, T. G. Molnár, A. W. Singletary, and G. Orosz, "Safety-critical control of active interventions for COVID-19 mitigation," *IEEE Access*, vol. 8, pp. 188 454–188 474, 2020.

[11] T. G. Molnár, A. W. Singletary, G. Orosz, and A. D. Ames, "Safety-critical control of compartmental epidemiological models with measurement delays," *IEEE Control Systems Letters*, vol. 5, no. 5, pp. 1537–1542, 2020.

[12] P. Mestres and J. Cortés, "Safe policy design for controlling epidemic spreading under heterogeneous testing capabilities," in *Proceedings of the 2022 American Control Conference (ACC)*. IEEE, 2022, pp. 697–702.

[13] E. Gubar and Q. Zhu, "Optimal control of influenza epidemic model with virus mutations," in *Proceedings of the 2013 European Control Conference (ECC)*. IEEE, 2013, pp. 3125–3130.

[14] H. S. Rodrigues, M. T. T. Monteiro, and D. F. Torres, "Vaccination models and optimal control strategies to dengue," *Mathematical Biosciences*, vol. 247, pp. 1–12, 2014.

[15] K. Kandhway and J. Kuri, "How to run a campaign: Optimal control of SIS and SIR information epidemics," *Applied Mathematics and Computation*, vol. 231, pp. 79–92, 2014.

[16] K. Li, G. Zhu, Z. Ma, and L. Chen, "Dynamic stability of an SIQS epidemic network and its optimal control," *Communications in Nonlinear Science and Numerical Simulation*, vol. 66, pp. 84–95, 2019.

[17] X. Lü, H.-w. Hui, F.-f. Liu, and Y.-l. Bai, "Stability and optimal control strategies for a novel epidemic model of COVID-19," *Nonlinear Dynamics*, vol. 106, no. 2, pp. 1491–1507, 2021.