2023 62nd IEEE Conference on Decision and Control (CDC)
December 13-15, 2023. Marina Bay Sands, Singapore

Adaptive Output Regulation and the Use It or Lose It Principle

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***Abstract*— It is well-known in adaptive control that when regressors are not persistently exciting (PE), then parameter adaptation is not robust. A number of adhoc modifications of parameter adaptation laws were developed to overcome this problem. In this paper we examine the PE subspace, a geometric characterization of a regressor's excitation which allows a more intrinsic modification of parameter adaptation laws. Our modular method, the μ-modification, is premised on the *Use it or Lose it Principle* of neuroplasticity, stating that parameters not excited by a regressor may be forgotten. This paper develops these ideas in the context of adaptive output regulation, with attention to the geometric properties of the PE subspace under linear filtering, such as when using augmented errors.**

I. Introduction

A longstanding issue of adaptive control is that if regressors are not persistently exciting (PE), then one can only obtain asymptotic stability, rather than exponential stability, implying adaptive control is in general not robust. This problem was intensively studied in the 1980's, resulting in a number of modular *modifications* of standard parameter adaptation laws, including the σ-modification, the e_1-modification, and projection and deadzone approaches [1, Section 8.5]. These techniques either trade off regulation to achieve robustness, or they rely on some a priori knowledge of unknown parameters. As such, they have an adhoc quality, leaving one with the unease that some theoretical structure remains to be discovered to handle non-PE regressors in an intrinsic manner.

Study of the PE requirement for robust adaptive control has recently been revived in [2], [3], [4], [5], [6], [7], [8]. This literature broadly splits into two schools of thought. Taking inspiration from the classical work [9], the papers [2], [3], [4], [5] use a so-called initial (or interval) excitation in the transient phase to extract sufficient excitation to perform parameter adaptation. The focus on extracting information from transients arises in scenarios where the PE assumption is in conflict with the control objective. For example, in adaptive stabilization, the state regressor is not PE since, by design, it tends to zero.

The second school of thought comes from the area of adaptive output regulation, where regressors arise from exogenous reference and disturbance signals that are assumed to sustain their available excitation, even if they are not PE, for the duration of tracking or disturbance rejection tasks [10], [6], [7], [8]. Our work belongs to this school and is premised on the *Use it or Lose it Principle* of neuroplasticity [11]: *parameters that are not excited by the regressor should be gradually forgotten, as they are not needed*. This principle is mathematically elaborated in a modular modification called the μ-modification [8] that does not trade off regulation for robustness. As a byproduct, we identify an intrinsic characterization of non-PE regressors to circumvent the adhoc quality of classical approaches. Similar methods have recently been proposed in [6], [7], with application to linear regression and adaptive observers, respectively. The present paper considers applications in adaptive output regulation, and we provide more details on the construction of *subspace estimators*.

The contributions as well as organization of the paper are as follows. In Section II we immediately begin by defining our notion of a PE subspace of a regressor (called PE directions in [8]), inspired by adaptive output regulation with its regressors arising from LTI exosystems. Next, we present a general error model in Section III that captures salient features of error models arising in the adaptive control literature [12], [13], [14], [15], [16]. Using the PE subspace, one can perform a coordinate transformation on the parameter estimates that illuminates the lack of robustness of parameters not excited by the regressor. Then we show how the μ-modification from [8] resolves this problem. The specific contribution of this paper relative to [8] is to address auxiliary (filtered) regressors that arise, for example, when using augmented errors; and also to show that coupling between error and parameter dynamics due to regressor estimation does not hinder the applicability of our technique.

In Section V we apply the foregoing results to the problem of linear output regulation based on an adaptive regulator from the literature [14]. This application highlights a novel perspective to account for PE subspaces in regulator design. The key property to be established is that the filtered quantities arising in regulator designs based on augmented errors preserve PE subspaces by way of a geometric invariance argument. Simulations of adaptive output regulation with and without the μ-modification are then presented in Section VI. The Appendix contains a novel supporting result on stability of perturbed systems.

II. Persistently Exciting Subspaces

Given a bounded and piecewise continuous signal $w(t) \in \mathbb{R}^q$, we say it is *persistently exciting (PE)* if there exists

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β_0, β_1, $T > 0$ such that

$$\beta_0 I \preceq \frac{1}{T}\int_t^{t+T} w(\tau)w^\intercal(\tau)\,d\tau \preceq \beta_1 I \qquad (1)$$

for all $t \geq t_0 \geq 0$, where t_0 denotes the initial time.

Given our interest in regressors arising in adaptive output regulation in the form of states of LTI exosystems, we restrict our attention to regressors that satisfy either $w = 0$ or there exists $1 \leq q_{pe} \leq q$ and $W \in \mathbb{R}^{q\times q_{pe}}$ having orthonormal columns such that $w = Ww_{pe}$ with $w_{pe}(t) \in \mathbb{R}^{q_{pe}}$ PE. Since all the excitation of $w(t)$ provided by $w_{pe}(t)$ is confined to $\mathrm{Im}(W)$, we define the *PE subspace*

$$\mathcal{W} := \mathrm{Im}(W)$$

and the *non-PE subspace* as its orthogonal complement $\mathcal{W}^\perp$.

III. General Error Model

Our goal is to demonstrate that the μ-modification is a modular technique that can be incorporated in a variety of contexts of adaptive output regulation. To do so, we identify a canonical error model for adaptive control, encompassing features arising in classical error models [12]; adaptive backstepping [13]; control affine systems [14]; uncertain Euler-Lagrange systems [15]; and contracting systems [16]. Three primary observations inform our choice of error model: (i) we consider the classical setting when the dynamics are parameterized linearly in the unknown parameters; (ii) a consistent structure that appears in the closed-loop dynamics after application of the adaptive control techniques we considered (e.g., adaptive backstepping, the MRAC matching conditions) is that unknown parameters are matched with their estimates; and (iii) the parameter adaptation law generally consists of three components: a regressor, an error signal, and an adaptation gain. Here we focus on a scalar error signal $e(t) \in \mathbb{R}$.

Taking together the preceding observations, we consider an error model $e = \mathscr{E}[\hat{\psi}, w(t), \nu]$, where $\mathscr{E}[\cdot]$ is of the form

$$\dot{\xi} = A(t,\xi) + B(t)(\hat{w}_\circ^\intercal\hat{\psi} - w_\circ^\intercal(t)\psi) \qquad (2a)$$

$$e = C(t,\xi) + D(t)(\hat{w}^\intercal\hat{\psi} - w^\intercal(t)\psi)\,. \qquad (2b)$$

Here $\xi(t) \in \mathbb{R}^n$ is the *error state*; $\psi \in \mathbb{R}^q$ and $\hat{\psi}$ are the unknown parameter and its estimate; $w_\circ(t)$ and $w(t)$ are (usually unmeasurable) regressors and $\hat{w}_\circ$, $\hat{w}$ are their estimates. The presence of both regressors $w_\circ(t)$ and $w(t)$ allows for the fact that the regressor used for adaptation may not be the same regressor interacting with the error state. For example, in Section V regressors $w_\circ$ and w are related through a filtering process, whereas for SPR linear systems, $w_\circ = w$. Finally, $\nu(t) \in \mathbb{R}^v$ is the *perturbation state* to model estimation errors, as follows:

$$\dot{\nu} = \Delta(t,\nu)\,, \qquad (3a)$$

$$\hat{w}_\circ = w_\circ(t) + \tilde{w}_\circ(t,\nu) \qquad (3b)$$

$$\hat{w} = w(t) + \tilde{w}(t,\nu)\,. \qquad (3c)$$

To complete the error model, we consider parameter adaptation laws that are structurally similar to the standard gradient algorithm:

$$\dot{\hat{\psi}} = -\gamma e\hat{w}\,, \qquad (4)$$

where $\gamma > 0$.

Next, we lay out assumptions on this error model so that the μ-modification may be seamlessly incorporated. First we have a set of assumptions on the perturbation dynamics.

Assumption 1: The perturbation (3) satisfies:

- (E1) the function $\Delta(\cdot)$ is piecewise continuous in t and globally Lipschitz in ν uniformly in t;
- (E2) the functions $\tilde{w}_\circ(\cdot)$ and $\tilde{w}(\cdot)$ are piecewise continuous in t and locally Lipschitz in ν uniformly in t. Additionally, $\tilde{w}_\circ(t,0) = \tilde{w}(t,0) = 0$;
- (E3) the equilibrium $\nu = 0$ of (3a) is globally exponentially stable (GES). ▹

Now we place assumptions on the error model itself. The crucial new assumption is (E6) stating that regressors $w_\circ(t)$ and $w(t)$ share the same PE subspace.

Assumption 2: The error model (2) satisfies:

- (E4) the functions $A(\cdot)$ and $C(\cdot)$ are piecewise continuous in t and globally Lipschitz in ξ uniformly in t. Additionally, $A(t,0) = 0$ and $C(t,0) = 0$;
- (E5) the functions $B(\cdot)$ and $D(\cdot)$ are piecewise continuous and bounded;
- (E6) the PE subspaces of $w_\circ(t)$ and $w(t)$ coincide. That is, either $w_\circ = w = 0$ or there exists $1 \leq q_{pe} \leq q$ and $W \in \mathbb{R}^{q\times q_{pe}}$ having orthonormal columns such that

$$w_\circ = Ww_{(\circ,pe)}\,, \quad w = Ww_{pe}$$

with $w_{(\circ,pe)}(t)$ and $w_{pe}(t)$ being PE. ▹

Finally, to leverage the vast literature on stability of adaptive systems, we only ask for stability properties of the *nominal unperturbed system.*

Assumption 3: Given any $q \in \mathbb{N}$ and appropriate $w_\circ(t)$ satisfying (E6), when $\nu = 0$ the system (2)-(4) satisfies:

- (E7) if $w = 0$, then $\xi = 0$ is GES;
- (E8) if $w(t)$ is PE, then $(\xi, \hat{\psi}) = (0, \psi)$ is GES. ▹

Remark 1: Global Lipschitz in (E1) and (E4) is used to guarantee global existence of appropriate converse Lyapunov functions from GES in (E3), (E7), and (E8). ◃

A. Partial Exponential Convergence

Now that we have established a suitable error model, we want to understand how the notion of a PE subspace can be used to achieve partial exponential convergence of parameters along this subspace. To that end, suppose we have $w = Ww_{pe}$ with $W \in \mathbb{R}^{q\times q_{pe}}$ and $w_{pe}(t)$ PE. By (E6), $w_\circ = Ww_{(\circ,pe)}$ with $w_{(\circ,pe)}(t)$ PE. Apply the coordinate transformation

$$\begin{bmatrix}\tilde{\psi}_{pe}\\ \hat{\psi}_\perp\end{bmatrix} = \begin{bmatrix}W^\intercal\\ W_\perp^\intercal\end{bmatrix}(\hat{\psi} - WW^\intercal\psi)\,,$$

where $W_\perp \in \mathbb{R}^{q\times(q-q_{pe})}$ is an *orthogonal completion*; namely, $\begin{bmatrix} W & W_\perp \end{bmatrix}$ is an orthogonal matrix. Then the closed-loop system can be expressed as

$$\dot{\xi} = A(t,\xi) + B(t)w^\intercal_{(\circ,pe)}(t)\tilde{\psi}_{pe} + B(t)\tilde{w}^\intercal_\circ\hat{\psi} \tag{5a}$$

$$\dot{\tilde{\psi}}_{pe} = -\gamma e_\circ w_{pe}(t) - \gamma w_{pe}(t)D(t)\tilde{w}^\intercal\hat{\psi} - \gamma e W^\intercal \tilde{w} \tag{5b}$$

$$\dot{\hat{\psi}}_\perp = 0 - \gamma e W_\perp^\intercal \tilde{w} \tag{5c}$$

with $e_\circ := C(t,\xi) + D(t)w^\intercal_{pe}(t)\tilde{\psi}_{pe}$. When $\nu = 0$, by (E2) one obtains the nominal unperturbed system

$$\begin{aligned} \dot{\xi} &= A(t,\xi) + B(t)w^\intercal_{(\circ,pe)}(t)\tilde{\psi}_{pe} \\ \dot{\tilde{\psi}}_{pe} &= -\gamma e_\circ w_{pe}(t) \\ \dot{\hat{\psi}}_\perp &= 0\,. \end{aligned}$$

It is clear from this form that the non-PE dynamics $\hat{\psi}_\perp$ are decoupled from the $(\xi, \tilde{\psi}_{pe})$ dynamics, and they are stable but not robust. Also, by (E8) we have that $(\xi, \tilde{\psi}_{pe}) = (0, 0)$ is GES. Such exponential convergence properties are in fact retained for the perturbed system (5).

Theorem 1: Consider the system (3)-(4) satisfying Assumptions 1-3. Then all states are uniformly bounded and:

1) if $w = 0$, then $\xi \to 0$ exponentially;
2) if $w(t)$ is PE, then $(\xi, \hat{\psi}) \to (0, \psi)$ exponentially;
3) otherwise, $(\xi, W^\intercal\hat{\psi}) \to (0, W^\intercal\psi)$ exponentially. ◇

Proof: Letting (5) denote the closed-loop dynamics, it suffices to apply Proposition 1 in the Appendix after making an appropriate identification. To this end, it is useful to observe that $\hat{\psi} = W\tilde{\psi}_{pe} + W_\perp\hat{\psi}_\perp + WW^\intercal\psi$.

1) Since no component of $\hat{\psi}$ is subject to PE dynamics given $w = 0 = w_\circ$, identify $\xi \mapsto \eta$, $\hat{\psi} \mapsto \eta_\perp$, and $\nu \mapsto \nu$. The nominal stability property is provided by (E7).
2) Defining $\tilde{\psi} := \hat{\psi} - \psi$ because $w(t)$ and $w_\circ(t)$ are PE, identify $(\xi, \tilde{\psi}) \mapsto \eta$ and $\nu \mapsto \nu$. The nominal stability property is provided by (E8). Note that $\eta_\perp$ does not exist but this does not affect the applicability of Proposition 1.
3) Identify $(\xi, \tilde{\psi}_{pe}) \mapsto \eta$, $\hat{\psi}_\perp \mapsto \eta_\perp$, and $\nu \mapsto \nu$. The nominal stability property is once again provided by (E8). The result then follows since $\tilde{\psi}_{pe} = W^\intercal(\hat{\psi} - \psi)$.

□

System (5) reveals that the PE component of $\hat{\psi}$ inherits the desirable exponential stability properties associated with PE regressors, as is not surprising. Any modification of the parameter adaptation law should leave the $\hat{\psi}_{pe}$ dynamics intact. Instead, a modular modification method should target the $\hat{\psi}_\perp$ dynamics which are not robust.

IV. Robust Design using the μ-modification

In this section we show how to render the closed-loop system (5) to be exponentially stable by modifying only (5c), the dynamics lacking excitation. The following regularity assumption, used in [8], is satisfied by regressors generated by LTI exosystems.

Assumption 4: The autocovariance matrix of $w(t)$

$$R_w(0) := \lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{t_0+T} w(\tau)w^\intercal(\tau)\,d\tau$$

exists with convergence uniform in $t_0 \geq 0$. Furthermore, $w(t)$ is piecewise continuous. ▷

The key idea of the μ-modification is to identify the non-PE subspace of $w(t)$, such that only the non-PE dynamics are modified. Given $w(t) \in \mathbb{R}^q$, consider the *subspace estimator* $\Omega = \begin{bmatrix} v^{(1)} & \cdots & v^{(q)} \end{bmatrix}$ with dynamics

$$\dot{v} = -\varepsilon w(t)w^\intercal(t)v + \varepsilon\sigma_{tol}\left(1 - \|v\|^2\right)v\,, \tag{6}$$

where $v(t) \in \mathbb{R}^q$ denotes any column of Ω and ε, $\sigma_{tol} > 0$ are to be selected sufficiently small. The role of Ω is to asymptotically identify the non-PE subspace $\mathcal{W}^\perp = \mathrm{Im}(W_\perp)$. Selecting $\mu > 0$, we introduce the following leakage term to (4):

$$\dot{\hat{\psi}} = -\gamma e\hat{w} - \mu\Omega\Omega^\intercal\hat{\psi}\,. \tag{7}$$

The combination (6)-(7) is called the *μ-modification*.

The subspace estimator (6) is intended to be run as a slow process by choosing small ε. The leakage term (7) "nibbles away" at any dynamics not excited by the regressor (i.e., $\hat{\psi}_\perp$), in accordance with the *Use it or Lose it Principle* [11]. Slowness of the design has the desirable property of making the μ-modification less reactive to transient behaviour or to noise.

Theorem 2: Consider system (3)-(4) satisfying Assumptions 1-4. Also, let $\beta_0 > 0$ be the lower PE bound in (1) for $w_{pe}(t)$. If $w = 0$, set $\beta_0 = \infty$. Then for every $\sigma_{tol} \in (0, \beta_0)$ there exists $\varepsilon_\star > 0$ such that for every $\varepsilon \in (0, \varepsilon_\star)$ the μ-modification, which replaces (4) with (6)-(7) using an initial condition $\Omega(t_0) \in \mathbb{R}^{q\times q}$ having full rank, guarantees modulo independently and exponentially vanishing terms:

1) if $w = 0$, then $(\xi, \hat{\psi}) = (0, 0)$ is GES;
2) if $w(t)$ is PE, then $(\xi, \hat{\psi}) = (0, \psi)$ is GES;
3) otherwise, $(\xi, \hat{\psi}) = (0, WW^\intercal\psi)$ is GES. ◇

Proof Sketch: The proof is very similar to the proof of [8, Theorem 2]. After some algebra, one obtains a closed-loop system similar to (5). Of particular interest are the non-PE dynamics (5c), which are now of the form

$$\dot{\hat{\psi}}_\perp = -\mu\Lambda\hat{\psi}_\perp - \gamma e W_\perp^\intercal\tilde{w} - \mu W_\perp^\intercal\tilde{\Omega}\hat{\psi}$$

for some matrix $\Lambda \succ 0$ [8, Corollary 1] where $\tilde{\Omega} := \Omega\Omega^\intercal - W_\perp\Lambda W_\perp^\intercal$. For the purpose of establishing GES, we treat the signals ν and $\tilde{\Omega}$ as exogenous and not states. The result then follows by applying standard Lyapunov arguments using appropriate converse Lyapunov functions for GES. □

Remark 2: As discussed in [8], knowledge of some appropriate $\sigma_{tol} > 0$ can be viewed as a design tolerance setting the desired minimum excitation of the regressor $w_{pe}(t)$. ◁

We conclude that the μ-modification enables exponential stability of all states without sacrificing asymptotic regulation of the error. In cases when one only has access to an estimate $\hat{w} = w(t) + \tilde{w}(t,\nu)$, one builds the subspace estimator

$$\dot{v} = -\varepsilon\hat{w}\hat{w}^\intercal v + \varepsilon\sigma_{tol}\left(1 - \|v\|^2\right)v\,. \tag{8}$$

The results in [8, Section 6] suggest that this design works as intended provided the transient $\tilde{w}$ is sufficiently small. Smallness of the transient can be achieved by waiting before turning on the μ-modification.

V. Linear Adaptive Output Regulation

As an application of our foregoing results, we consider the problem of output regulation of a known LTI plant with unknown LTI exosystem. Consider the SISO system

$$\dot{x} = Ax + Bu + E\zeta \tag{9a}$$

$$\dot{\zeta} = S\zeta \tag{9b}$$

$$e = Cx + D\zeta\,, \tag{9c}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}$ is the control input, $\zeta(t) \in \mathbb{R}^q$ is the exosystem state, and $e(t) \in \mathbb{R}$ is the error. As is standard in regulator theory, we assume the following.

Assumption 5: The system (9) satisfies:

(A1) the triplet (C, A, B) is detectable and stabilizable;
(A2) S only has simple eigenvalues on the $j\omega$-axis;
(A3) the non-resonance condition holds:

$$\det \begin{bmatrix} A - \lambda I & B \\ C & 0 \end{bmatrix} \neq 0 \quad \forall\, \lambda \in \sigma(S)\,.$$

▷

Using (A3), let (Π, Γ) solve the regulator equations

$$\Pi S = A\Pi + B\Gamma + E\,, \quad 0 = C\Pi + D\,.$$

Consider the error state $z := x - \Pi\zeta$. Then system (9) can be expressed in matched disturbance form:

$$\dot{z} = Az + Bu - Bd \qquad \dot{\zeta} = S\zeta$$
$$e = Cz \qquad d = \Gamma\zeta\,.$$

Our goal is to regulate the error to zero while rendering the closed-loop dynamics exponentially stable.

Assumption 6: The following information is known:

(A4) the matrices (C, A, B) are known;
(A5) wlog the pair (Γ, S) is observable;
(A6) dimension q is interpreted as a known upper bound on the exosystem order;
(A7) the measurement is e. ▷

The prototypical regulator we utilize as our starting point is that of [14, Ch. 4.1.3.2], which has the form

$$u = u_s + u_{im} \tag{11}$$

with u_s for closed-loop stability and u_{im} for disturbance rejection.

A. Invariance of Excitation under Linear Filters

A crucial step in our regulator design will be to establish that the PE subspace of some relevant regressor w is well-defined, and that any auxiliary regressors required in the design will meet the assumption (E6); namely that they share the same PE subspace as w. These properties ultimately emerge from the LTI exosystem that models the disturbance. In this section we work through the key results.

First we characterize the excitation present in disturbances generated by LTI exosystems. We say a specific $d(t) \in \mathbb{R}$ is *sufficiently rich of order* s if s is the smallest integer such that there exists $\Gamma_\circ \in \mathbb{R}^{1\times s}$ and $S_\circ \in \mathbb{R}^{s\times s}$, with $S_\circ$ only having simple eigenvalues on the $j\omega$-axis, such that

$$\dot{\zeta}_\circ = S_\circ\zeta_\circ\,, \quad d = \Gamma_\circ\zeta_\circ$$

for an appropriate initial condition $\zeta_\circ(t_0) \in \mathbb{R}^s$. If $d = 0$, we say $d(t)$ is *sufficiently rich of order* 0. Note that the pair $(\Gamma_\circ, S_\circ)$ must be observable since s is the smallest integer for which we can build an LTI exosystem for the specific $d(t)$. Moreover, the state $\zeta_\circ(t) \in \mathbb{R}^s$ must be PE. Lastly, our definition of sufficient richness is equivalent to the classical definition [17], where $d(t)$ has *exactly* s spectral lines.

Next we convert the exosystem to a canonical form [14] suitable for adaptive control. Given any controllable pair $(F, G) \in \mathbb{R}^{q\times q} \times \mathbb{R}^q$ with F Hurwitz, define the new exosystem state $w := M\zeta$ and parameter $\psi^\intercal := \Gamma M^{-1}$. Following [14], $M \in \mathbb{R}^{q\times q}$ can be selected so that the exosystem takes the form

$$\dot{w} = Fw + Gd \tag{12a}$$

$$d = \psi^\intercal w\,. \tag{12b}$$

Now we establish existence of a PE subspace of w by following the procedure in [10, Section 4.1]; see also [8].

Lemma 1: Consider system (12) with $\sigma(F + G\psi^\intercal) = \sigma(S)$ and suppose (A2) holds. For a fixed initial condition, suppose $d(t) \in \mathbb{R}$ is sufficiently rich of order $s > 0$. Then there exists a PE regressor $w_{pe}(t) \in \mathbb{R}^s$ and a matrix $W \in \mathbb{R}^{q\times s}$ having orthonormal columns such that $w = Ww_{pe}$. ⋄
Thus $\mathcal{W} := \operatorname{Im}(W)$ is the PE subspace of w.

Auxiliary regressors are required in the regulator design in [14], which arise by filtering the disturbance through a known LTI system. We must verify that such linear filtering does not destroy the PE subspace of w. A key result, stated next with a proof in the Appendix, is known in the literature [14], but it is normally presented using the Swapping Lemma, which obscures its geometric content. Here we re-interpret the result to say that if a disturbance is filtered through a linear system, then regressors arising from exosystem states associated with the unfiltered and filtered disturbances are related by a linear coordinate transformation. This prepares the groundwork for a geometric interpretation in which PE subspaces are invariant under certain types of filtering.

Lemma 2: Consider system (12) with $\sigma(F + G\psi^\intercal) = \sigma(S)$ and suppose (A2)-(A3) hold. Define $A_d := A - L_dC$ where L_d is selected so that A_d is Hurwitz. Then there exists initial conditions $z_d(t_0) \in \mathbb{R}^n$ and $w_f(t_0) \in \mathbb{R}^q$ such that the filtered disturbance d_f obtained by filtering d as

$$\dot{z}_d = A_dz_d + Bd \tag{13a}$$

$$d_f = Cz_d \tag{13b}$$

can be equivalently generated by

$$\dot{w}_f = Fw_f + Gd_f \tag{14a}$$

$$d_f = \psi^\intercal w_f\,. \tag{14b}$$

Moreover, there exists a nonsingular matrix $H_f \in \mathbb{R}^{q\times q}$ such that the *filtered regressor* satisfies $w_f = H_fw$. ⋄

As a result of Lemma 2, we can write $d = \psi_f^\intercal w_f$ where $\psi_f^\intercal := \psi^\intercal H_f^{-1}$. Then, provided we build an observer for w_f, it is clear that we need to derive an adaptation law to estimate $\psi_f \in \mathbb{R}^q$ if we want to reject the disturbance d. Noting that the filtered disturbance satisfies

$$d_f = \psi^\intercal w_f = \psi_f^\intercal H_f w_f =: \psi_f^\intercal \bar{w}\,,$$

it suggests that we can construct the *augmented error*

$$\bar{e} := \hat{\psi}_f^\intercal \bar{w} - d_f = (\hat{\psi}_f - \psi_f)^\intercal \bar{w}$$

using an estimate $\hat{\psi}_f$ of ψ_f for the purpose of parameter adaptation. Given that the matrix H_f is not directly available, the following lemma shows how one can generate the *augmented regressor* $\bar{w}$. Its proof is similar to Lemma 2 and is thus omitted.

Lemma 3: Consider H_f and w_f as defined in Lemma 2, and let $\bar{w} := H_f w_f$. Then there exists a filter satisfying

$$\dot{Z} = A_d Z + B w_f^\intercal \tag{15a}$$

$$\bar{w}^\intercal = CZ \tag{15b}$$

for some appropriate $Z(t_0) \in \mathbb{R}^{n\times q}$. ◇

The key theme underlying Lemmas 2-3 is that filtering through the stable transfer function $H_d(s) := C(sI - A_d)^{-1}B$ is equivalent to the application of the linear transformation H_f. With this insight in mind, we can now explain the effect of filtering on PE subspaces.

Suppose we are given a disturbance $d(t)$ for some initial condition $\zeta(t_0)$ and let s denote its sufficient richness. If $s = 0$ then $d = 0$, implying that $w = w_f = \bar{w} = 0$. Otherwise, we have $1 \le s \le q$. If $s < q$, then the exosystem is overmodeled, and it is known that the regressor $w(t)$ (and $w_f(t)$, $\bar{w}(t)$) is not PE. The following is the key statement when $s \ge 1$: the PE subspaces of w, w_f, and $\bar{w}$ are well defined, and they coincide. The proof relies on the geometrically appealing fact that $\mathcal{W}$ is invariant under the map H_f.

Lemma 4: Consider $w_f(t)$ and $\bar{w}(t)$ as defined in Section V-A and let W be as defined in Lemma 1. Then there exist PE regressors $w_{(f,pe)}(t)$, $\bar{w}_{pe}(t) \in \mathbb{R}^s$ such that $w_f = W w_{(f,pe)}$ and $\bar{w} = W\bar{w}_{pe}$. ◇

Proof: Since $w_{pe}(t) \in \mathbb{R}^s$ is PE, there exists times $\{\,t_i\,\}_{i=1}^s$ such that $Y := \begin{bmatrix} w_{pe}(t_1) & \cdots & w_{pe}(t_s)\end{bmatrix} \in \mathbb{R}^{s\times s}$ is invertible. Noting $\dot{w} = (F + G\psi^\intercal)W w_{pe} = W\dot{w}_{pe}$ and letting $\dot{Y} := \begin{bmatrix} \dot{w}_{pe}(t_1) & \cdots & \dot{w}_{pe}(t_s)\end{bmatrix}$, we have that $(F + G\psi^\intercal)W = W\bar{Y}$ with $\bar{Y} := \dot{Y}Y^{-1}$. Let the polynomials $N_d(\cdot)$, $D_d(\cdot)$, and the matrix H_f be as in the proof of Lemma 2. Then $N_d(F + G\psi^\intercal)W = WN_d(\bar{Y})$ and $D_d(F + G\psi^\intercal)W = WD_d(\bar{Y})$, where $N_d(\bar{Y})$ and $D_d(\bar{Y})$ have full rank (see related statements in the proof of Lemma 2) because all other matrices do. Consequently $D_d(F + G\psi^\intercal)^{-1}W = WD_d(\bar{Y})^{-1}$ and so $H_f W = W\bar{H}_f$ with $\bar{H}_f := D_d(\bar{Y})^{-1}N_d(\bar{Y})$ invertible. As a result, one has

$$w_f = H_f w = H_f W w_{pe} = W\bar{H}_f w_{pe}$$

$$\bar{w} = H_f w_f = H_f W \bar{H}_f w_{pe} = W\bar{H}_f^2 w_{pe}\,.$$

Defining $w_{(f,pe)} := \bar{H}_f w_{pe}$ and $\bar{w}_{pe} := \bar{H}_f^2 w_{pe}$, which are PE by [12, Lemma 6.1], proves the result. □

Remark 3: Lemmas 1-4 make it clear that PE subspaces depend on the initial conditions of the exosystem. Since initial conditions are generally assumed unknown, one should not expect to know the relevant PE subspaces and thus a subspace estimator must be employed, as done in the μ-modification. ◁

B. Regulator Design

The main obstacles in arriving at an error model amenable to application of the μ-modification have been surpassed in the previous section. The rest of the design steps now follow the literature [14]. The adaptive internal model for the regulator is

$$\dot{\hat{z}}_d = A\hat{z}_d + Bu + L_d(e - C\hat{z}_d) \tag{16a}$$

$$\dot{\hat{w}}_f = F\hat{w}_f + G(C\hat{z}_d - e) \tag{16b}$$

$$u_{im} = \hat{\psi}_f^\intercal \hat{w}_f\,, \tag{16c}$$

where $L_d \in \mathbb{R}^n$ is selected such that $A_d := A - L_d C$ is Hurwitz, by (A1). The parameter adaptation law is

$$\dot{\hat{Z}} = A_d\hat{Z} + B\hat{w}_f^\intercal \tag{17a}$$

$$\hat{\bar{w}}^\intercal = C\hat{Z} \tag{17b}$$

$$\hat{\bar{e}} = \hat{\psi}_f^\intercal\hat{\bar{w}} - (C\hat{z}_d - e) \tag{17c}$$

$$\dot{\hat{\psi}}_f = -\gamma\hat{\bar{e}}\hat{\bar{w}}\,, \tag{17d}$$

where $\gamma > 0$. Finally, the dynamic stabilizer is

$$\dot{\hat{z}}_s = A\hat{z}_s + Bu_s + L_s(e - C\hat{z}_s) \tag{18a}$$

$$u_s = K\hat{z}_s\,, \tag{18b}$$

where L_s, $K^\intercal \in \mathbb{R}^n$ are selected such that $A_s := A - L_s C$ and $A_{cl} := A + BK$ are Hurwitz, by (A1).

C. Closed-Loop System

To apply our prior developments in Sections III-IV, including partial exponential convergence and robustness with the addition of the μ-modification, we need to show that the resulting closed-loop system for the considered output regulation problem adheres to the presented error model.

First, we start with the perturbation state representing estimation errors. Defining $\tilde{z}_d := \hat{z}_d - z$, $\check{z}_d := \tilde{z}_d - z_d$, $\tilde{w}_f := \hat{w}_f - w_f$, and $\tilde{Z} := \hat{Z} - Z$, it is easily verified that

$$\begin{bmatrix}\dot{\check{z}}_d \\ \dot{\tilde{w}}_f\end{bmatrix} = \begin{bmatrix} A_d & 0 \\ GC & F\end{bmatrix}\begin{bmatrix}\check{z}_d \\ \tilde{w}_f\end{bmatrix}, \qquad \dot{\tilde{Z}} = A_d\tilde{Z} + B\tilde{w}_f^\intercal\,.$$

Additionally, we have

$$\tilde{\bar{w}} := \hat{\bar{w}} - \bar{w} = (C\tilde{Z})^\intercal\,.$$

Making the identifications $(\check{z}_d, \tilde{w}_f, \tilde{Z}) \mapsto \nu$, $\tilde{w}_f \mapsto \tilde{w}_\circ$, and $\tilde{\bar{w}} \mapsto \tilde{w}$, we can verify Assumption 1.

(E1): holds because the $(\check{z}_d, \tilde{w}_f, \tilde{Z})$ dynamics are LTI;
(E2): holds because $\tilde{w}_f$ and $\tilde{\bar{w}}$ are linear in $(\check{z}_d, \tilde{w}_f, \tilde{Z})$;
(E3): holds because A_d and F are Hurwitz.

Second, we write out the error dynamics. Defining the state $\tilde{z}_s := \hat{z}_s - z$, one obtains

$$\begin{bmatrix} \dot{z} \\ \dot{\tilde{z}}_s \\ \dot{\check{z}}_d \end{bmatrix} = \begin{bmatrix} A_{cl} & BK & 0 \\ 0 & A_s & 0 \\ 0 & 0 & A_d \end{bmatrix} \begin{bmatrix} z \\ \tilde{z}_s \\ \check{z}_d \end{bmatrix} + \begin{bmatrix} B \\ B \\ 0 \end{bmatrix} (\hat{w}_f^\intercal \hat{\psi}_f - w_f^\intercal \psi_f)$$
$$\hat{\hat{e}} = -C\check{z}_d + (\hat{\bar{w}}^\intercal \hat{\psi}_f - \bar{w}^\intercal \psi_f)\,.$$

A small technicality that appears is that now $\check{z}_d$ must be a component of both the error state ξ and the perturbation state ν. This can be circumvented by defining an additional state $\check{z}_d'$ having the same dynamics as $\check{z}_d$, and using it for the above error dynamics. In doing so, we are considering a slightly more general system, with the original closed-loop system recovered when $\check{z}_d(t_0) = \check{z}_d'(t_0)$. Making the identifications $(z, \tilde{z}_s, \check{z}_d') \mapsto \xi$, $\hat{\psi}_f \mapsto \hat{\psi}$, $w_f \mapsto w_\circ$, $\bar{w} \mapsto w$, and $\hat{\hat{e}} \mapsto e$, we can verify Assumption 2.

(E4): holds by linearity in $(z, \tilde{z}_s, \check{z}_d')$;
(E5): holds because the associated functions are constant;
(E6): holds by Lemma 4.

Third, we recall that the adaptation law is simply

$$\dot{\hat{\psi}}_f = -\gamma \hat{\hat{e}} \hat{\bar{w}}$$

with $\gamma > 0$. By setting $(\check{z}_d, \tilde{w}_f, \tilde{Z}) = (0, 0, 0)$ we have the nominal unperturbed system

$$\begin{bmatrix} \dot{z} \\ \dot{\tilde{z}}_s \\ \dot{\check{z}}_d' \end{bmatrix} = \begin{bmatrix} A_{cl} & BK & 0 \\ 0 & A_s & 0 \\ 0 & 0 & A_d \end{bmatrix} \begin{bmatrix} z \\ \tilde{z}_s \\ \check{z}_d' \end{bmatrix} + \begin{bmatrix} B \\ B \\ 0 \end{bmatrix} w_f^\intercal(t)(\hat{\psi}_f - \psi_f)$$
$$\dot{\hat{\psi}}_f = -\gamma \bar{w}(t)\bar{w}^\intercal(t)(\hat{\psi}_f - \psi_f) + \gamma \bar{w}(t) C \check{z}_d'$$

for which we can verify Assumption 3.

(E7): holds because A_{cl}, A_s, and A_d are Hurwitz;
(E8): holds by the classical result [12, Theorem 2.16] and because we have a cascade interconnection of GES systems.

Lastly, since w_f and $\bar{w}$ are states of LTI exosystems, they are piecewise continuous and almost periodic. Thus by [18, Appendix, Theorem 6] their autocovariance matrices exists, satisfying Assumption 4.

VI. Simulation

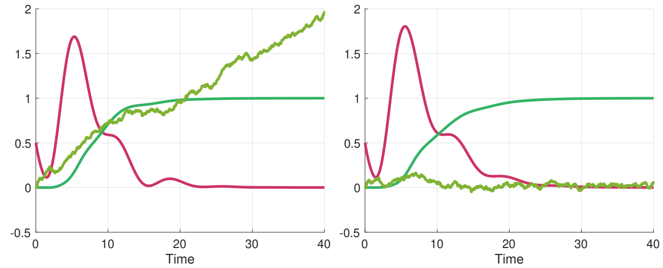
We simulate system (9) with matrices

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad E = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad D = 1, \qquad S = 0,$$

and initial conditions $x(t_0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^\intercal$ and $\zeta(t_0) = 0.5$. The regulator (11), (16)-(18) is built with values

$$F = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad L_d = L_s = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$
$$K = \begin{bmatrix} -1 & -1 \end{bmatrix}, \qquad \gamma = 1,$$

and all initial conditions set to zero. When including the μ-modification (7)-(8), we use $\sigma_{tol} = 0.1$, $\varepsilon = 1$, and $\mu = 1$ with initial condition $\Omega(t_0)$ selected randomly as an orthogonal matrix. The subspace estimator (8) is built using $\hat{w}_f$ rather than $\hat{\bar{w}}$ to reduce the effects of transients due to filtering since their PE subspaces coincide (see Lemma 4).



(a) μ-modification off (b) μ-modification on

Fig. 1: Comparison of the error e (red) and parameter $\hat{\psi}_f$ adaptation dynamics (green, olive green) with noise injected.

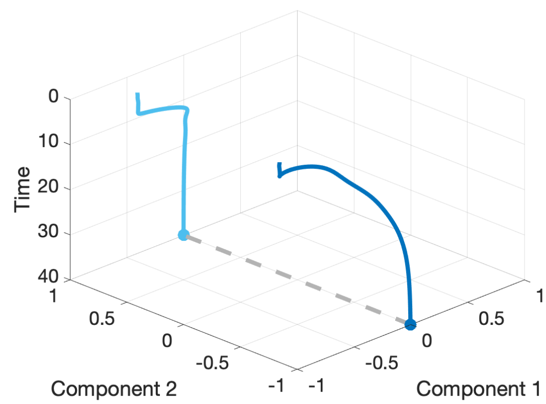


Fig. 2: Columns of the subspace estimator (blue, light blue) and a plot of the non-PE subspace (grey dashed lines).

Since the exosystem can only generate constants, the effective disturbance d is sufficiently rich of order at most $s = 1$. Consequently, our internal model of dimension $q = 2$ is overparameterized. It can then be shown that the non-PE dynamics consists of $\hat{\psi}_{(f,\perp)} = (\hat{\psi}_f)_2$ (olive green). To demonstrate that we have robustified the non-PE parameter dynamics, we additively inject white Gaussian noise with 0.05 mean and 0.2 variance into the $(\hat{\psi}_f)_2$ dynamics. Figure 1 shows how the parameter adaptation dynamics are rendered robust using the μ-modification in the presence of noise. In particular, we retain asymptotic error regulation while the μ-modification keeps all states bounded. A visualization of the non-PE subspace $\mathcal{W}^\perp = \{0\} \times \mathbb{R}$ recovered by our subspace estimator is found in Figure 2.

VII. Conclusion

The paper further develops the idea of PE subspaces, introduced in [8], in the context of adaptive output regulation where regressors generally arise from LTI exosystems. We show by way of a general error model how to robustify parameter adaptation laws by *forgetting* the parameters that are not excited by associated regressors, in line with the *Use It or Lose It Principle* of neuroplasticity. That is, parameters that lie in the subspace orthogonal to the PE subspace should be gradually forgotten. An application of adaptive output regulation for a SISO LTI plant is carried out, particularly highlighting the key property of invariance of the PE subspace under various filtering operations needed in the regulator design.

The key limitation of the μ-modification is that it assumes perturbation terms arising from estimation errors vanish independently. A more sophisticated stability analysis is needed to handle the general problem in which coupling terms between the adaptation dynamics and regressor estimates appear. Another avenue is to elaborate how robustness is provided by the μ-modification beyond simply exponential stability.

APPENDIX

Proof: [Lemma 2] Let A_d be given. We will construct all relevant quantities in reverse order, starting with the matrix H_f relating the state w to a filtered regressor w_f. Then we show this filtered regressor is in fact the state of an exosystem in the canonical form [19]. At last, we relate this system to the filtered disturbance d_f produced by filtering d.

Consider $C(sI - A_d)^{-1}B =: N_d(s) \,/\, D_d(s)$, where we select $N_d(\cdot)$ and $D_d(\cdot)$ as coprime polynomials. Define

$$H_f := D_d(F + G\psi^\intercal)^{-1} N_d(F + G\psi^\intercal)\,. \tag{19}$$

Note $D_d(F + G\psi^\intercal)$ is invertible because A_d is Hurwitz, $\sigma(A_d) \cap \sigma(F + G\psi^\intercal) = \emptyset$ by (A2), and by the Spectral Mapping Theorem. Also, note that $N_d(F + G\psi^\intercal)$ is invertible by (A3). To see this, (A3) states that no zero of $C(sI - A)^{-1}B$ is an eigenvalue of $F + G\psi^\intercal$ because $\sigma(F + G\psi^\intercal) = \sigma(S)$. Since (C, A, B) and $(B^\intercal, A^\intercal, C^\intercal)$ represent the same SISO transfer function and zeros are unaffected by state feedback, we conclude the same of $C(sI - A_d)^{-1}B$. As a result, $N_d(F + G\psi^\intercal)$ is invertible because the Spectral Mapping Theorem implies that it has no zero eigenvalue. Overall, (19) is invertible.

Define the state $w_f := H_f w$, the output $d_f := \psi^\intercal w_f$, and note that H_f and $F + G\psi^\intercal$ commute. Therefore

$$\dot{w}_f = H_f \dot{w} = H_f (F + G\psi^\intercal) w = (F + G\psi^\intercal) w_f\,,$$

meaning that w_f is generated by (14) with initial condition $w_f(t_0) = H_f w(t_0)$.

Next we construct a filter of the form (13), which we will show generates d_f. By definition of w_f, we have $D_d(F + G\psi^\intercal) w_f = N_d(F + G\psi^\intercal) w$. Since $\dot{w}_f = (F + G\psi^\intercal) w_f$ and $\dot{w} = (F + G\psi^\intercal) w$, equivalently

$$D_d\left(\frac{d}{dt}\right)[w_f^\intercal] = N_d\left(\frac{d}{dt}\right)[w^\intercal]\,.$$

Now consider the filter

$$\dot{Z}_d = A_d Z_d + B w^\intercal\,, \quad w_\star^\intercal = C Z_d$$

with $Z_d(t_0) \in \mathbb{R}^{n \times q}$ selected such that $C Z_d(t_0) = w_f^\intercal(t_0)$. By linear systems theory, it is known that $w_\star$ must satisfy

$$D_d\left(\frac{d}{dt}\right)[w_\star^\intercal] = N_d\left(\frac{d}{dt}\right)[w^\intercal]\,.$$

Given that $w_\star$ and w_f have identical initial conditions and satisfy the same ODE, it must be that $w_\star = w_f$. Thus $d_f = \psi^\intercal w_f = C Z_d \psi$, which suggests to define $z_d := Z_d \psi$. Differentiation shows z_d satisfies (13) and generates d_f with initial condition $z_d(t_0) = Z_d(t_0)\psi$. □

The following justifies why terms that vanish independently may be safely ignored. To the best of our knowledge, the presented result is new. However, related stability results are obtained in Center Manifold Theory [20, Appendix B].

Proposition 1: Suppose $\eta = 0$ is GES for the system

$$\begin{bmatrix} \dot{\eta} \\ \dot{\eta}_\perp \end{bmatrix} = \begin{bmatrix} f(t, \eta) \\ 0 \end{bmatrix}, \tag{20}$$

where $f(\cdot)$ is piecewise continuous in t and globally Lipschitz in η uniformly in t. Consider $p(t, \eta, \eta_\perp, \nu)$ piecewise continuous in t satisfying

$$\|p(t, \eta, \eta_\perp, \nu)\| \leq \bar{p}_1(\nu) \|(\eta, \eta_\perp)\| + \bar{p}_2(\nu)$$

for some $\bar{p}_i(\cdot) \geq 0$ locally Lipschitz with $\bar{p}_i(0) = 0$. Given $\Delta(t, \nu)$ piecewise continuous in t and globally Lipschitz in ν uniformly in t, consider the perturbed system

$$\begin{bmatrix} \dot{\eta} \\ \dot{\eta}_\perp \end{bmatrix} = \begin{bmatrix} f(t, \eta) \\ 0 \end{bmatrix} + p(t, \eta, \eta_\perp, \nu) \tag{21a}$$

$$\dot{\nu} = \Delta(t, \nu)\,, \tag{21b}$$

where $\nu = 0$ is GES. Then all states are uniformly bounded and $\eta \to 0$ exponentially. ◇

Proof: We first prove uniform boundedness of all states. Since $\nu = 0$ is GES, there exists c_ν, $\lambda > 0$ such that $\|\nu(t)\| \leq c_\nu \|\nu(t_0)\| \mathrm{e}^{-\lambda(t - t_0)}$ for all $t \geq t_0$ and clearly ν is uniformly bounded. Since $\eta = 0$ is GES for (20), there exists a converse Lyapunov function $V_f(t, \eta)$ satisfying the conclusions of [21, Theorem 4.14][1] for the η dynamics. Consider the function $V_0(t, \eta, \eta_\perp) := V_f(t, \eta) + \|\eta_\perp\|^2$ and take its Lie derivative with respect to (21a). Then

$$\dot{V}_0(t, \eta, \eta_\perp) \leq \gamma_1(\nu(t)) V_0(t, \eta, \eta_\perp) + \gamma_2(\nu(t)) \tag{22}$$

for some appropriate $\gamma_i(\cdot)$ after a couple applications of Young's inequality and by writing ν as an exogenous signal. Because the $\bar{p}_i(\cdot)$ are locally Lipschitz with $\bar{p}_i(0) = 0$, so are the $\gamma_i(\cdot)$. As a result, pick any $\delta > 0$ and suppose the initial conditions satisfy $\|(\eta, \eta_\perp, \nu)(t_0)\| \leq \delta$. Then there exists $c_{(\gamma,i)}(\delta) \geq 0$ such that $\gamma_i(\nu(t)) \leq c_{(\gamma,i)} \|\nu(t)\|$, implying

$$\gamma_i(\nu(t)) \leq c_\nu c_{(\gamma,i)} \|\nu(t_0)\| \mathrm{e}^{-\lambda(t - t_0)}$$

for all $t \geq t_0 \geq 0$. Substituting into (22), an application of the Comparison Lemma to upper bound $V_0(\cdot)$ makes it clear that η and $\eta_\perp$ are uniformly bounded.

We now outline the proof for exponential convergence, which follows a standard Lyapunov argument. Again because $\nu = 0$ is GES, there exists a converse Lyapunov function $V_\Delta(t, \nu)$ satisfying the conclusions of [21, Theorem 4.14] for the ν dynamics. Let $V(t, \eta, \nu) := V_f(t, \eta) + \epsilon V_\Delta(t, \nu)$

[1]The continuous differentiability assumption in [21, Theorem 4.14] may be relaxed by considering upper Dini derivatives, leaving the proof virtually unchanged.

for some $\epsilon > 0$ to be selected sufficiently large. Taking its time derivative with respect to trajectories of (21), after some algebra we get

$$\dot{V}(t,\eta,\nu) \leq -a_3\|\eta\|^2 - \epsilon b_3\|\nu\|^2 + a_4\bar{p}_1(\nu(t))\|\eta\|^2 + a_4\left(c_1\|\eta_\perp(t)\|_{\mathcal{L}_\infty} + c_2\right)\|\eta\|\|\nu\|,$$

where we have omitted the time dependence of some variables, a_3, $a_4 > 0$ are bounding constants from $V_f(\cdot)$, $b_3 > 0$ is a bounding constant from $V_\Delta(\cdot)$, and the $c_i(\delta)$ are selected to satisfy $\bar{p}_i(\nu(t)) \leq c_i\|\nu(t)\|$. The $\|\eta\|\|\nu\|$ cross-term can be dealt with using the Peter-Paul inequality, whereas the $a_4\bar{p}_1(\nu(t))$ term vanishes uniformly in the initial conditions and so the technique in [8, Proposition 2] can be applied to deal with it. This concludes the proof. □

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