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A fusion estimation approach for robust output feedback MPC of multi-sensor systems

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Abstract— **This paper presents a fusion-based output feedback MPC (FOFMPC) approach for multi-sensor systems with state and control constraints. Our approach consists of a fusion estimation procedure and a robust output feedback MPC scheme. For fusion estimation, we adopt a two-layer structure where local observers are designed for all the sensors and operate in parallel and a fusion center collects the states of the individual observers and produces a fusion estimate based on certain weighted fusion criterion. The weights are computed by minimizing the weighted Minkowski sum of the local robust positively invariant (RPI) sets. This fusion estimation procedure is then integrated into the framework of robust output feedback MPC (ROFMPC). We verify the effectiveness of the proposed approach using a three-zone building model.**

I. INTRODUCTION

The goal of fusion estimation is to observe the states of the same uncertain system using measurements from multiple sensors, see [1] and the references therein. To achieve this, local estimators are often designed for individual sensors and the global state estimate is obtained with certain fusion criterion, see, e.g., [2]–[4]. A well-known and efficient fusion estimation technique [5]–[7] uses a weighted sum of the local estimates to provide an overall estimate, where the weights are computed in a way that system uncertainties are effectively tackled. In general, there are two paradigms to model uncertainties: the stochastic paradigm and the set-membership paradigm. Early works typically rely on the stochastic one and fusion criteria are designed according to statistical properties of the system [8], [9]. The set-membership paradigm deals with unknown but bounded uncertainties and it has been widely used in state estimation in recent years [10], [11]. There also exist set-membership approaches for fusion estimation, see, e.g., [12]. However, to the best of our knowledge, existing set-membership fusion criteria only consider one-step estimation error. In this work, we use robust positively invariant (RPI) sets to confine state estimation errors in order to ensure feasibility of control design.

We are focused on robust output feedback model predictive control (ROFMPC) for multi-sensor systems. The ROFMPC framework is first introduced in [13], [14] using the tube-based MPC technique [15] which computes a RPI set as a tube to bound the prediction errors of the disturbed system. In this framework, a Luenberger-structure observer is often designed to obtain estimations of the system states. For this reason, the construction of the tube in ROFMPC involves both the prediction errors and the estimation errors. Note that the performance of the ROFMPC framework is significantly influenced by the tube confining uncertainties, which motivates us to design to obtain an RPI set as small as posible. For multi-sensor systems, we take one more step into complexity as fusion estimation is needed.

When multiple sensors measure different outputs of the same system, there are basically two methods to design control with the measured output data. A simple idea is to collect all the measurements and feed them into a stand-alone centralized observer [16], which then allows to utilize the existing ROFMPC approach in [13], [14]. To improve closed-loop performance, one can leverage techniques that optimize the observer gain by minimizing the minimal RPI set, see, e.g., [17]. However, such a procedure is usually computationally demanding. Moreover, in the ROFMPC framework, we need to consider RPI sets for both the estimation errors and the prediction errors, which makes the computation even more complicated. The other method is to use a weighted fusion criterion, where the weights are optimized by minimizing a composite tube which is the weighted Minkowski sum of the local RPI sets.

In fact, in the context of tube-based MPC, it is also possible to reduce conservatism by formulating an augmented system combining the estimation errors and the prediction errors as shown in [18]. However, since we need to compute a RPI set for the augmented system, this significantly increases the computational burden and is thus not realistic for multi-sensor systems.

This paper considers the problem of output feedback MPC for multi-sensor systems under polytopic uncertainties. The contribution of the paper is twofold. First, we leverage a two-layer fusion estimation procedure into the ROFMPC approach using RPI sets as tubes. Second, we propose an optimization algorithm with guaranteed convergence to design the fusion criterion which minimizes the size of the overall tube.

The rest of the paper is organized as follows. This section ends with the notation, followed by Section II on the review of preliminary results on the ROFMPC framework. Section III presents the main results which include the description of a two-layer fusion estimation procedure, robustness analysis, the optimization-based fusion criterion design, and the proposed fusion-based MPC approach. In Section IV, we apply the proposed approach to a double-zone building thermal model.

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II. Preliminaries and Problem Statement

Consider the following discrete-time disturbed systems

$$x^+ = Ax + Bu + w, \tag{1}$$

$$y_i = C_i x + v_i,\ i = 1, 2, ..., M \tag{2}$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^m$ is the control input, x^+ is the successor state, $w \in \mathbb{W} \subset \mathbb{R}^n$ is the disturbance, $y_i \in \mathbb{R}^{p_i}$ is the measurement of ith sensor with the noise $v_i \in \mathbb{V}_i \subset \mathbb{R}^{p_i}$, and the sets $\mathbb{W}$ and $\{\mathbb{V}_i\}_{i=1}^M$ are all convex polytopic sets that contain the origin in their interiors. (A, B) is assumed to be controllable, and for all $i = 1, 2, \cdots, M$ the couple (A, C_i) is observable (i.e. the system is locally observable at each sensor). The assumption of local observability is quite standard for fusion estimation, see, e.g., [19], [20].

System (1) is subject to the following state and control constraints:

$$x \in \mathbb{X},\ u \in \mathbb{U}, \tag{3}$$

where $\mathbb{X} \subset \mathbb{R}^n$ and $\mathbb{U} \subset \mathbb{R}^m$ are also convex polytopic sets that contain the origin in their interiors.

We review some known results on robust output feedback model predictive control (ROFMPC) from [13], [14]. Let $y \in \mathbb{R}^p$ denote the collection of all the outputs:

$$y = \begin{bmatrix} y_1^\top & y_2^\top & \cdots & y_M^\top \end{bmatrix}^\top, \tag{4}$$

where $p = \sum_{i=1}^M p_i$. With the overall output, a Luenberger observer is used to estimate the state:

$$\hat{x}^+ = A\hat{x} + Bu + L(y - \hat{y}),\ \hat{y} = C\hat{x}, \tag{5}$$

where $\hat{x} \in \mathbb{R}^n$ is the observer state, $\hat{x}^+$ is the successor state of the observer, $\hat{y}$ is the output of the observer and $L \in \mathbb{R}^{n \times p}$ is chosen such that $\rho(A - LC) < 1$. To ensure constraint satisfaction, a tube-based output feedback MPC approach is proposed in [13] which bounds the state estimation error by an invariant set. The general structure of this centralized estimation framework is depicted in Figure 1.

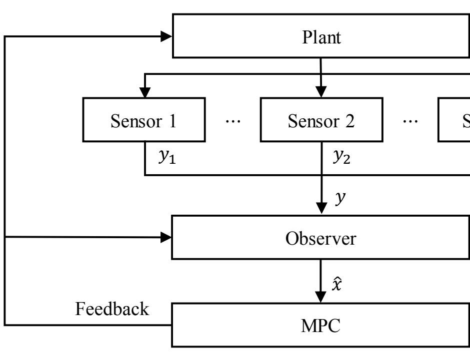


Fig. 1: Output feedback MPC with centralized estimation

For centralized estimation in ROFMPC, to improve estimation performance, one can optimize the observer gain by minimizing the minimal RPI set using the technique in [17]. However, this is computationally expensive. Instead, this paper proposes a fusion-based output feedback MPC (FOFMPC) approach where local observers are designed for individual sensors and a weighted fusion criterion is used to synthesize a fusion estimator.

III. Main Results

In this section, we present the proposed fusion-based output feedback MPC approach for systems with multiple sensors.

A. A two-layer fusion estimation procedure

Before we show the proposed MPC approach, we first introduce a two-layer fusion estimation procedure. The first fusion layer has a netted parallel structure where an individual observer is designed for each sensor as described below:

$$\hat{x}_i^+ = A\hat{x}_i + Bu + L_i(y_i - \hat{y}_i),\ \hat{y}_i = C_i\hat{x}_i, \tag{6}$$

where $\hat{x}_i$ and L_i are the estimated state and the gain of the ith observer for $i = 1, ..., M$. In the second fusion layer, a fusion center is used to fuse the estimates with weighting matrices as shown below:

$$\hat{x} = \sum_{i=1}^M \alpha_i \hat{x}_i, \tag{7}$$

where $\hat{x}$ is the overall state estimate after fusion and $\{\alpha_i \in \mathbb{R}^{n \times n}\}_{i=1}^M$ are the weighting matrices satisfying $\sum_{i=1}^M \alpha_i = I$. A fusion criterion will be designed later to optimize these weighting matrices. With this fusion estimation procedure, we propose a fusion-based output feedback MPC approach as shown in Figure 2.

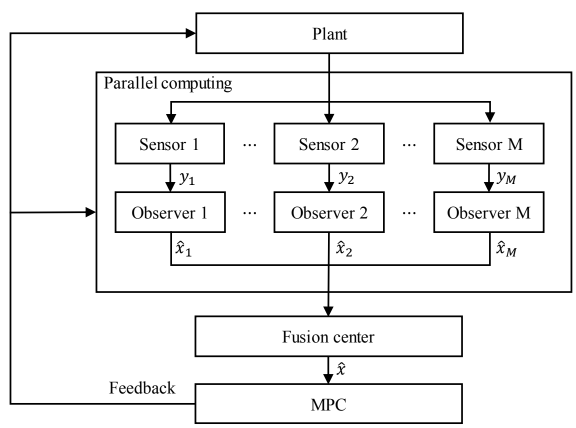


Fig. 2: Fusion-based output feedback MPC

B. Error analysis and set-membership bounding

We now discuss robustness properties of the proposed fusion estimation procedure. The nominal system that disregards all the disturbances is given by

$$\bar{x}^+ = A\bar{x} + B\bar{u}, \tag{8}$$

where $\bar{x}$ is the nominal system state and $\bar{u}$ is the input to the nominal system. Let $\phi(k; \bar{x}_0, \bar{\boldsymbol{u}})$ represent the solution of the nominal system (8) at time k given an initial state $\bar{x}_0$ and a control sequence $\bar{\boldsymbol{u}}$. The state sequence can be given as $\bar{\boldsymbol{x}} := \{\phi(k; \bar{x}_0, \bar{\boldsymbol{u}}), k = 0, 1, ...\}$, which is considered as the center of a tube in the tube-based MPC approach [15]. For the disturbed system (1) and the observers in (6), the control law is chosen in the form of

$$u = \bar{u} + Ke, \tag{9}$$

where $e := \hat{x} - \bar{x}$ denotes the prediction error and K satisfies $\rho(A + BK) < 1$, Under this control law, for any $i = 1, 2, \cdots, M$, the closed-loop observer becomes:

$$\hat{x}_i^+ = A\hat{x}_i + B\bar{u} + BKe + L_iC_i\widetilde{x}_i + L_iv_i, \tag{10}$$

where $\widetilde{x}_i := x - \hat{x}_i$ is the state estimation error of the ith observer. This leads to the following prediction error dynamics: $i = 1, 2, \cdots, M$,

$$e_i^+ = A_Ke_i + (L_iC_i\widetilde{x}_i + L_iv_i) + BK(e - e_i), \tag{11}$$

where $A_K := A + BK$ and $e_i := \hat{x}_i - \bar{x}$. The dynamics of the estimation error of the ith observer can be described as:

$$\widetilde{x}_i^+ = A_{L_i}\widetilde{x}_i + (w - L_iv_i), \quad i = 1, 2, \cdots, M, \tag{12}$$

where $A_{L_i} := A - L_iC_i$ with $\rho(A_{L_i}) < 1$.

We then derive set-membership bounds for the prediction and estimation errors using robust positively invariant (RPI) sets, see the definition in [21], [22]. The dynamics of the estimation error given in (12) can be rewritten as follows:

$$\widetilde{x}_i^+ = A_{L_i}\widetilde{x}_i + \widetilde{\delta}_i, \quad i = 1, 2, \cdots, M, \tag{13}$$

where $\widetilde{\delta}_i := w - L_iv_i$ lies in the set $\widetilde{\Delta}_i$ defined by:

$$\widetilde{\Delta}_i := \mathbb{W} \oplus (-L_i\mathbb{V}_i). \tag{14}$$

Since $\rho(A_{L_i}) < 1$, it is shown in [22] that there exist a polytopic set $\widetilde{S}_i$ such that

$$A_{L_i}\widetilde{S}_i \oplus \widetilde{\Delta}_i \subseteq \widetilde{S}_i, \quad \forall i = 1, 2, \cdots, M \tag{15}$$

which means that $\widetilde{x}_i(0) \in \widetilde{S}_i$ implies $\widetilde{x}_i(k) \in \widetilde{S}_i$ for all $k \in \mathbb{N}$ and $i \in 1, 2, ..., M$. Combing the individual prediction error dynamics in (11), we obtain

$$\begin{aligned} e^+ &= \sum_{i=1}^{M} \alpha_i e_i^+ = \sum_{i=1}^{M} \alpha_i \left(A_K e_i + \bar{\delta}_i + BK(e - e_i) \right) \\ &= \sum_{i=1}^{M} \alpha_i (A_K e_i + \bar{\delta}_i) = A_K e + \sum_{i=1}^{M} \alpha_i \bar{\delta}_i \end{aligned} \tag{16}$$

where $\bar{\delta}_i := L_iC_i\widetilde{x}_i + L_iv_i$. Since $\widetilde{x}_i$ is bounded by $\widetilde{S}_i$, $\bar{\delta}_i$ lies in the set $\bar{\Delta}_i$ defined by:

$$\bar{\Delta}_i := L_iC_i\widetilde{S}_i \oplus L_i\mathbb{V}_i, \quad i = 1, 2, \cdots, M. \tag{17}$$

Since $\rho(A_K) < 1$, again from [22], there exists polytopic sets $\{\bar{S}_i\}$ such that

$$A_K\bar{S}_i \oplus \bar{\Delta}_i \subseteq \bar{S}_i, \quad i = 1, 2, \cdots, M. \tag{18}$$

For convenience, we call sets satisfying (15) estimation RPI sets and sets satisfying (18) prediction RPI sets respectively. Given $\widetilde{S}_i$ and $\bar{S}_i$ satisfying (15) and (18) respectively, we define

$$S_i := \bar{S}_i \oplus \widetilde{S}_i, \quad \forall i = 1, 2, ..., M. \tag{19}$$

Given the fusion estimate $\hat{x}$ in (7), the overall state estimation error $\widetilde{x} := x - \hat{x}$ is described as

$$\widetilde{x}^+ = \sum_{i=1}^{M} \alpha_i \left(A_{L_i}\widetilde{x}_i + (w - L_iv_i) \right). \tag{20}$$

With the results above, we are ready to discuss set-membership bounds for actual trajectories. In order to ensure that the actual control input is contained in the set $\mathbb{U}$, a standard technique is to tighten the constraint on the control of the nominal system. More precisely, consider the control law in (9), $u = \bar{u} + Ke \in \mathbb{U}$ given $\bar{u} \in \mathbb{U} \ominus K\bar{S}$, where

$$\bar{S} = \alpha_1\bar{S}_1 \oplus \alpha_2\bar{S}_2 ... \oplus \alpha_M\bar{S}_M. \tag{21}$$

Similarly, we impose a tightened constraint on the nominal state $\bar{x} \in \mathbb{X} \ominus S$, where

$$S = \alpha_1 S_1 \oplus \alpha_2 S_2 \oplus \cdots \oplus \alpha_M S_M. \tag{22}$$

The discussions above are now formalized as follows.

Proposition 1: Consider the disturbed system (1)–(2), the observer (6) and the fusion strategy (7) with the weighting matrices $\{\alpha_i\}_{i=1}^M$. Let $x(k), \hat{x}_i(k), e_i(k), \widetilde{x}_i(k)$ denote the solution of (1), (10), (11) and (12) respectively and the set $\{\bar{S}_i, \widetilde{S}_i, S_i\}_{i=1}^M$ satisfy (15), (18) and (19) respectively. Suppose that $\widetilde{x}_i(0) \in \widetilde{S}_i$ for any $i = 1, 2, ..., M$ and $\sum_{i=1}^M \alpha_i e_i(0) \in \bar{S}$ and that $\bar{x}(k) \in \mathbb{X} \ominus S$ and $\bar{u}(k) \in \mathbb{U} \ominus K\bar{S}$ for all $k \in \mathbb{N}$, where $\bar{S}$ and S are defined as in (21) and (22). It holds that $x(k) \in \bar{x}(k) \oplus S \subseteq \mathbb{X}$ and $u(k) \in \bar{u}(k) \oplus \bar{S} \subseteq \mathbb{U}$ for all admissible disturbance sequences for all $k \in \mathbb{N}$.

Proof: We first show that $e(k) = \sum_{i=1}^M \alpha_i e_i(k) \in \bar{S}$ for any $k \in \mathbb{N}$. The proof goes by induction. Suppose $e(k) \in \bar{S}$ for some $k \in \mathbb{N}$. By the definition of $\bar{S}$ in (21), there exist $\{e_i' \in \bar{S}_i\}$ such that $e(k) = \sum_{i=1}^M \alpha_i e_i'$. From (16),

$$\begin{aligned} e(k+1) &= A_K e(k) + \sum_{i=1}^{M} \alpha_i \bar{\delta}_i = A_K \sum_{i=1}^{M} \alpha_i e_i' + \sum_{i=1}^{M} \alpha_i \bar{\delta}_i \\ &= \sum_{i=1}^{M} \alpha_i (A_K e_i' + \bar{\delta}_i) \in \sum_{i=1}^{M} \alpha_i \bar{S}_i = \bar{S} \end{aligned}$$

Since $e(0) = \sum_{i=1}^M \alpha_i e_i(0) \in \bar{S}$, $e(k) = \sum_{i=1}^M \alpha_i e_i(k) \in \bar{S}$ for any $k \in \mathbb{N}$. Similarly, we can show that $\widetilde{x}(k) \in \sum_{i=1}^M \alpha_i \widetilde{S}_i$ for any $k \in \mathbb{N}$. Indeed, from the invariance property in (18), $\widetilde{x}_i(0) \in \widetilde{S}_i$ implies that $\widetilde{x}_i(k) \in \widetilde{S}_i$ for any $k \in \mathbb{N}$ for any $i = 1, 2, \cdots, M$. Then, by definition, $x(k) = \bar{x}(k) + e(k) + \widetilde{x}(k) \in \bar{x}(k) \oplus S$ and $u(k) = \bar{u}(k) + Ke(k) \in \bar{u}(k) \oplus K\bar{S}$. This completes the proof. ■

C. Fusion criterion design

We now discuss the design of the weighting matrices in the fusion strategy in (7). The polyhedral volume can serve as a metric for measuring the size of RPI set. To improve closed-loop performance, we minimize the size of S as defined in (22) by formulating the following problem

$$\min_{\{\alpha_i\in\mathbb{R}^{n\times n}\}}\{\text{Vol}(\oplus_{i=1}^M \alpha_i S_i):\quad \text{s.t.}\quad \sum_{i=1}^M \alpha_i = I\} \tag{23}$$

where S_i is given in (19). However, computing the volume of a polytope is known to be expensive, see, e.g., [23]. Instead, we minimize the size of the smallest enclosing ellipsoid of S, expressed as $\mathcal{E}(\Omega) := \{x\in\mathbb{R}^n : x^\top \Omega x \le 1\}$ for some $\Omega\in\mathbb{S}_+^n$. Thus, an approximation of (23) is given as

$$\min_{\{\alpha_i\in\mathbb{R}^{n\times n}\},\Omega\in\mathbb{S}_+^n} -\log\det(\Omega) \tag{24a}$$

$$\text{s.t.}\quad \sum_{i=1}^M \alpha_i = I,\quad \oplus_{i=1}^M \alpha_i S_i \subset \mathcal{E}(\Omega), \tag{24b}$$

By utilizing the vertices of a polyhedron, (24) can be equivalently written as

$$\min_{\{\alpha_i\in\mathbb{R}^{n\times n}\},\Omega\in\mathbb{S}_+^n} -\log\det(\Omega) \tag{25a}$$

$$\text{s.t.}\quad \sum_{i=1}^M \alpha_i = I, \tag{25b}$$

$$(\sum_{i=1}^M \alpha_i v_i)^\top \Omega (\sum_{i=1}^M \alpha_i v_i) \le 1, \forall v_i\in\mathcal{V}(S_i). \tag{25c}$$

where $\mathcal{V}(\cdot)$ represents the vertices. We then solve the problem above via alternating minimization. The details are described in Algorithm 1. The convergence of Algorithm 1 is discussed in the following proposition.

Proposition 2: Give the sets $\{S_i\}_{i=1}^M$ as defined in (22), let the sequence $\{\Omega^k\}$ be generated from Algorithm 1. It hold that Vol $\left(\mathcal{E}(\Omega^{k+1})\right) \le$ Vol $\left(\mathcal{E}(\Omega^k)\right)$ for any $k\in\mathbb{N}$.

Proof: For any $k\in\mathbb{N}$, by definition, Ω^{k+1} is the optimal solution of (26) for the given $\{\alpha_i^k\}$, meaning that $\mathcal{E}(\Omega^{k+1})$ is the smallest ellipsoid that encloses $\oplus_{i=1}^M \alpha_i^{k+1} S_i$. From (27), $\oplus_{i=1}^M \alpha_i^{k+1} S_i \subseteq \mathcal{E}(\Omega^k/\gamma^{k+1})$, where γ^{k+1} denotes the optimal value function of (27), which means that Ω^k/γ^{k+1} is a feasible solution to (26) for the given $\{\alpha_i^{k+1}\}$. Hence, Vol $\left(\mathcal{E}(\Omega^k/\gamma^{k+1})\right) \ge$ Vol $\left(\mathcal{E}(\Omega^{k+1})\right)$. We then show that $\gamma^{k+1}\le 1$. Since $(\{\alpha_i^k\},1)$ is always a feasible solution to (27), by optimality, $\gamma^{k+1}\le 1$. Finally, Vol $\left(\mathcal{E}(\Omega^{k+1})\right) \le$ Vol $\left(\mathcal{E}(\Omega^k/\gamma^{k+1})\right) \le$ Vol $\left(\mathcal{E}(\Omega^k)\right)$. This completes the proof. ■

D. The proposed MPC approach: Recursive feasibility and stability

With the fusion estimation procedure above, we are ready to present the proposed fusion-based MPC approach. Essentially, our approach is an extension of [13] to the case of multi-sensor fusion.

Algorithm 1 Fusion criterion design

Input: $\{S_i\}_{i=1}^M$, ϵ
Output: $\{\alpha_i\}_{i=1}^M$

Initialization: Let $k\leftarrow 0$ and the initial weighting matrices be $\alpha_i^0 \leftarrow \frac{1}{M}I$ for all $i=1,2,\cdots,M$;

1: For the given $\{S_i\}_{i=1}^M$ and α_i^0, find the smallest enclosing ellipsoid of S. Compute Ω^k by solving

$$\min_{\Omega\in\mathbb{S}_+^n} -\log\det(\Omega) \tag{26a}$$

$$\text{s.t.}\quad (\sum_{i=1}^M \alpha_i^k v_i)^\top \Omega (\sum_{i=1}^M \alpha_i^k v_i) \le 1, \forall v_i\in\mathcal{V}(S_i). \tag{26b}$$

2: Fix the variable Ω and update the weights $\{\alpha_i^{k+1}\}$ which can be solved by

$$\min_{\{\alpha_i\in\mathbb{R}^{n\times n}\},\gamma\le 1} \gamma \tag{27a}$$

$$\text{s.t.}\quad \sum_{i=1}^M \alpha_i = I, \tag{27b}$$

$$(\sum_{i=1}^M \alpha_i v_i)^\top \Omega^k (\sum_{i=1}^M \alpha_i v_i) \le \gamma, \forall v_i\in\mathcal{V}(S_i). \tag{27c}$$

3: If $k\ge 1$ and $|\text{Vol}\left(\mathcal{E}(\Omega^k)\right) - \text{Vol}\left(\mathcal{E}(\Omega^{k-1})\right)| < \epsilon$, stop; Otherwise, let $k\leftarrow k+1$ and return to Step 1.

Given the fusion state estimate $\hat{x}$, we solve the following online optimization problem:

$$\min_{\bar{x}_0,\bar{\boldsymbol{u}}} V_N(\bar{\boldsymbol{x}},\bar{\boldsymbol{u}}) := \sum_{k=0}^{N-1} \ell(\bar{x}_k,\bar{u}_k) + V_f(\bar{x}_N) \tag{28a}$$

$$\text{s.t.}\ \hat{x}\in\bar{x}_0\oplus\bar{S}, \bar{x}_N\in X_f \tag{28b}$$

$$\bar{x}_{k+1} = A\bar{x}_k + B\bar{u}_k,\quad k=1,2,\cdots,N-1 \tag{28c}$$

$$\bar{x}_k\in\mathbb{X}\ominus S, \bar{u}_k\in\mathbb{U}\ominus K\bar{S} \tag{28d}$$

where N is the prediction horizon, $\bar{\boldsymbol{x}} := \{\bar{x}_0,\bar{x}_1,\cdots,\bar{x}_N\}$, $\bar{\boldsymbol{u}} := \{\bar{u}_0,\bar{u}_1,\cdots,\bar{u}_{N-1}\}$, $\ell(\bar{x}_k,\bar{u}_k) := \bar{x}_k^\top Q\bar{x} + \bar{u}_k^\top R\bar{u}$ is the stage cost for some $Q\in\mathbb{S}_+^n$ and $R\in\mathbb{S}_+^m$, $V_f(\bar{x}_N) := \bar{x}_N^\top P\bar{x}_N$ is the terminal cost for some $P\in\mathbb{S}_+^n$, and X_f is the terminal constraint set satisfying the standard axioms in [24]. Let $(\bar{x}_0^*(\hat{x}),\bar{\boldsymbol{u}}^*(\hat{x}))$ denote the minimizer of (28). The control law $\kappa_N(\cdot)$ becomes

$$\kappa_N(\hat{x}) := \bar{u}_0^*(\hat{x}) + K(\hat{x}-\bar{x}_0^*(\hat{x})) \tag{29}$$

where $\bar{u}_0^*(\hat{x})$ is the first element in $\bar{\boldsymbol{u}}^*(\hat{x})$. Under this control law, Combing the local observations in (10), we obtain

$$\hat{x}^+ = \sum_{i=1}^M \alpha_i\hat{x}_i^+ = \sum_{i=1}^M \alpha_i(A\hat{x}_i + B\kappa_N(\hat{x}) + \bar{\delta}_i)$$
$$= A\hat{x} + B\kappa_N(\hat{x}) + \sum\nolimits_{i=1}^M \alpha_i\bar{\delta}_i \tag{30}$$

where $\bar{\delta}_i$ is given in (16). Since $\bar{\delta}_i$ lies in the set $\bar{\Delta}_i$ which is defined in (17), So $\sum_{i=1}^M \alpha_i\bar{\delta}_i$ lies in the set $\sum_{i=1}^M \alpha_i\bar{\Delta}_i = \sum_{i=1}^M \alpha_i(L_iC_i\widetilde{S}_i \oplus L_i\mathbb{V}_i)$.

We then characterize the domain of attraction of the closed-loop system (30). For the nominal system (8), given any initial state $\bar{x} \in \mathbb{X} \ominus S$, the set of admissible control sequences is

$$\mathcal{U}_N^*(\bar{x}) = \{\bar{\boldsymbol{u}} \,|\, \bar{u}_k \in \mathbb{U} \ominus K\bar{S},\, \bar{x}_k \in \mathbb{X} \ominus S, \quad \forall k = 1, 2, \cdots, N-1, \bar{x}_0 = \bar{x}, \bar{x}_N \in X_f\}. \tag{31}$$

Let $\bar{X}_N := \{\bar{x} \in \mathbb{X} \ominus S \,|\, \mathcal{U}_N^*(\bar{x}) \neq \emptyset\}$. With these definitions, it can be shown that $\bar{X}_N \oplus \bar{S}$ is a domain of attraction for System (30). Putting all the pieces together, we arrive at the key result of this paper.

Theorem 1: Consider the disturbed system (1)–(2), the observer (6) and the fusion strategy (7) with the weighting matrices $\{\alpha_i\}_{i=1}^M$. Let the set $\bar{S}$, S, $\{\widetilde{S}_i\}_{i=1}^M$ satisfy (21), (22) and (15) respectively and $\widetilde{x}_i(k)$ denote the solution of (12). Given any $\hat{x} \in \bar{X}_N \oplus \bar{S}$, let $(\bar{x}_0^*(\hat{x}), \bar{\boldsymbol{u}}^*(\hat{x}))$ denote the minimizer of (28) and the control law $\kappa_N(\hat{x})$ is given as in (29). Then, the following results hold:
(i) $\bar{S}$ is robustly exponentially stable for $\hat{x}^+ = A\hat{x} + B\kappa_N(\hat{x}) + \sum_{i=1}^M \alpha_i \bar{\delta}_i$ with a region of attraction $\bar{X}_N \oplus \bar{S}$.
(ii) For any state $x(0) = \hat{x}(0) + \widetilde{x}(0)$ where $\widetilde{x}(0) = \sum_{i=1}^M \alpha_i \widetilde{x}_i(0)$ and $(\hat{x}(0), \widetilde{x}_i(0)) \in (\bar{X}_N \oplus \bar{S}) \times \widetilde{S}_i$ for any $i = 1, 2, ..., M$, System (1) is robustly steered to S exponentially fast while satisfying the state and control constraints.

Proof: (i) Since $\widetilde{x}_i(0) \in \widetilde{S}_i$, it follows from Proposition 1 that $\widetilde{x}_i(k) \in \widetilde{S}_i$ for any $k \in \mathbb{N}$ and any $i = 1, 2, \cdots, M$. Let $\hat{x}(k)$ denote the solution of $\hat{x}^+ = A\hat{x} + B\kappa_N(\hat{x}) + \sum_{i=1}^M \alpha_i \bar{\delta}_i$ given its initial state is $\hat{x}(0)$. It follows from [15, Theorem 1] that there exists a c and a $\gamma \in (0, 1)$ such that $|\bar{x}_0^*(\hat{x}(k))| \leq c\gamma^i |\bar{x}_0^*(\hat{x}(0))|$ for all $\hat{x}(0) \in \bar{X}_N \oplus \bar{S}$ and every $\sum_{i=1}^M \alpha_i \bar{\delta}_i$. From (28), We can know that $\hat{x}(k) \in \bar{x}_0^*(\hat{x}(k)) \oplus \bar{S}$. Then, the distance of point $\hat{x}(k)$ and $\hat{x}(0)$ from the set $\bar{S}$ satisfies $d(\hat{x}(k), \bar{S}) \leq c\gamma^i d(\hat{x}(0), \bar{S})$ for all $\hat{x}(0) \in \bar{X}_N \oplus \bar{S}$ and every $\sum_{i=1}^M \alpha_i \bar{\delta}_i$. So the set $\bar{S}$ is robustly exponentially stable for the controlled uncertain system with a region of attraction of $\bar{X}_N \oplus \bar{S}$.
(ii) Since $\widetilde{x}_i(0) \in \widetilde{S}_i$ for any $i = 1, 2, ..., M$, it follows from Proposition 1 that $\widetilde{x}(k) \in \sum_{i=1}^M \alpha_i \widetilde{S}_i$ for any $k \in \mathbb{N}$. Since $x(k) \in \hat{x}(k) \oplus \sum_{i=1}^M \alpha_i \widetilde{S}_i$ for all $k \in N$. Then System (1) is robustly steered to S exponentially fast. ■

Based on the results above, the details of the proposed FOFMPC approach are described in Algorithm 2.

IV. SIMULATION

Consider the following building thermal model with three zones [25] [26]: $c_1\dot{T}_1 = \frac{T_2 - T_1}{R_{12}} + \frac{T_3 - T_1}{R_{13}} + \frac{T_o - T_1}{R_1^o} + q_1 + w_1$, $c_2\dot{T}_2 = \frac{T_1 - T_2}{R_{12}} + \frac{T_3 - T_2}{R_{23}} + \frac{T_o - T_2}{R_2^o} + q_2 + w_2$, $c_3\dot{T}_3 = \frac{T_1 - T_3}{R_{13}} + \frac{T_2 - T_3}{R_{23}} + \frac{T_o - T_3}{R_3^o} + w_3$, where T_i, c_i, R_i^o, and w_i are the temperature of zone i, the thermal capacitance of zone i, the thermal resistance between zone i and the outside environment, and the thermal disturbance of zone i respectively, T_0 is the temperature of outside air, R_{12}, R_{13}, R_{23} denote the thermal resistances between zones, and q_1, q_2 are the energy inputs into zone 1 and zone 2. We consider the case where only zone 1 and zone 2 are equipped with thermostats. The outside air temperature is $38°C$. The disturbances are uniformly sampled from the corresponding bounding sets: $\mathbb{W} = \{w \in \mathbb{R}^3 : \|w\|_\infty \leq 0.1\}$, $\mathbb{V}_1 = \mathbb{V}_2 = \{v \in \mathbb{R} : v \in [-0.1, 0.1]\}$. Other system parameters are given as follows: c_1= $1.375 \times 10^3 kJ/K$, c_2= $2.0625 \times 10^3 kJ/K$, c_3= $1.7187 \times 10^3 kJ/K$, $R_{12} = R_{21} = 1.5K/kW$, $R_{13} = R_{31} = R_{23} = R_{32} = 1.2K/kW$, $R_1^o = R_2^o = 3K/kW$, $R_3^o = 2.7K/kW$.

Algorithm 2 Fusion-based output feedback MPC

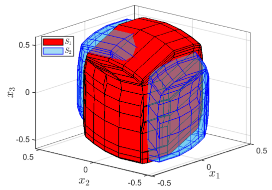
Input: $A, B, \{C_i, L_i, \alpha_i\}$, R, Q, K, P, S, $\bar{S}$, $\mathbb{X}$, $\mathbb{U}$, X_f, N
Output: $\{\hat{x}(k), u(k)\}$ *Initialization*: Let $k \leftarrow 0$, set the initial local estimates $\{\hat{x}_i(k)\}_{i=1}^M$;
1: Compute the overall estimate $\hat{x}(k)$ from (7);
2: Solve the optimization problem (28) and obtain the solution $(\bar{x}_0^*(\hat{x}(k)), \bar{\boldsymbol{u}}^*(\hat{x}(k)))$;
3: Obtain the control input $u(k) = \kappa_N(\hat{x}(k))$ from (9) and apply it to System (1);
4: Obtain the measurement $\{y_i(k)\}_{i=1}^M$ of the current time through sensors;
5: Compute the local estimates $\{\hat{x}_i(k+1)\}_{i=1}^M$ from (6);
6: Set $k \leftarrow k+1$ and go to Step 1 until some stopping criterion is met.

The temperature set points of zone 1 and zone 2 are $T_1^s = 23°C$ and $T_2^s = 24°C$ respectively. It can be obtained that the steady temperature of zone 3 is $T_3^s = 26.14°C$ and the steady inputs are $q^s := (q_1^s, q_2^s) = (-8.2803, -5.7803)$. We consider the following constraints: $16 \leq T_i \leq 38, \forall i = 1, 2, 3$ and $-16 \leq q_i \leq 0, \forall i = 1, 2$. We discretize the continuous-time system by the zero-order-hold method with the sampling time $\Delta t = 5\,\text{min}$, obtaining the state space model: $A = [0.6849,\ 0.1148,\ 0.1322; 0.0765,\ 0.7783,\ 0.0960; 0.1057,\ 0.1152, 0.7170]$, $B = [0.1810,\ 0.0090; 0.0090,\ 0.1283; 0.0128,\ 0.0091]$, $C_1 = [1,\ 0,\ 0]$, $C_2 = [0,\ 1,\ 0]$, $C = [C_1; C_2]$.

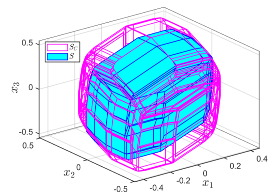
Let the state be $x = T - T^s$ and the control input be $u = q - q^s$ where $T = (T_1, T_2, T_3)$ and $q = (q_1, q_2)$. The state and input constraint sets are $\mathbb{X} = \{x \in \mathbb{R}^3 : x_1 \in [-7, 15],\ x_2 \in [-8, 14],\ x_3 \in [-10.1364, 11.8636]\}$ and $\mathbb{U} = \{u \in \mathbb{R}^2 : u_1 \in [-7.7197, 8.2803],\ u_2 \in [-10.2197, 5.7803]\}$. For fair comparison, the local observer gains are obtained from the optimal Luenberger gain of centralized estimation and compute K from LQR. The weights for fusion estimation are obtained through Algorithm 1.

Under different estimation strategies, we compute the tubes for ROFMPC as shown in Figure 3. It is expected that our fusion estimation strategy produces a smaller tube. We also show the evolution of the temperature of the three zones in Figure 3c from the initial state $x_0 = (3, 5, 3)^T$ under the fusion estimation strategy and observe that the input constraint are satisfied in Figure 3d.

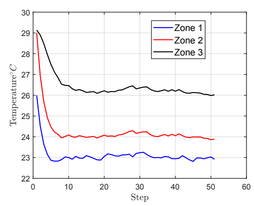
In the rest of the section, we also make comparison with the approach in [18] which reduce conservatism in [13] by using an augmented system. We scale the measurement noise sets as follows: $\mathbb{V}_i = \{v : |v| \leq V_{\max}\}, i = 1, 2$, where $V_{\max}$ is the scaling parameter. We consider the LQ



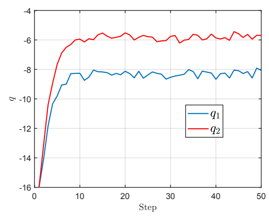
(a) RPI set with observer 1 and 2



(b) RPI set with fusion and centralized



(c) Temperature of zone



(d) The energy input of zone

Fig. 3: Invariant sets under different control strategies and the temperatures of with the controller: S_c denotes the tube for ROFMPC with centralized estimation.

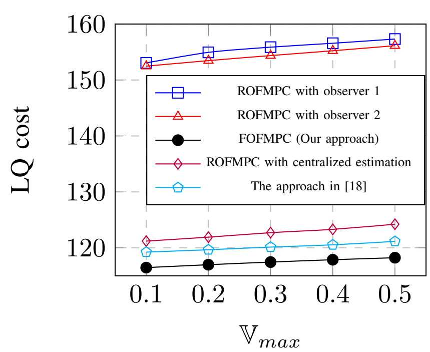


Fig. 4: LQ costs under different approaches

cost $\sum_{t=0}^{50} x^\top(t)Qx(t) + u^\top(t)Ru(t)$ for 50 steps from the same initial state under the same disturbances (generated randomly) when $N = 6$. As can be seen in Figure 4, our approach outperforms [18] as measurement noise increases, showing the effectiveness of fusion estimation.

V. Conclusion

This paper proposes a fusion estimation approach for robust output feedback MPC of multi-sensor systems. For each sensor, a local observer is designed with a Luenberger structure. We adopt a weighted fusion criterion where the weights are computed by minimizing the weighted Minkowski sum of the RPI sets. This fusion estimation procedure is then integrated into the design of tube-based MPC which guarantees recursive feasibility and stability. Finally, we show by numerical simulation the advantage of the proposed approach.

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