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Global asymptotic stabilization of time-invariant bilinear non-homogeneous complex systems*

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Abstract— **The problem of global asymptotic stabilization by state feedback is considered for time-invariant bilinear non-homogeneous control systems in the complex space. For such systems, the possibility of applying the second Lyapunov method is proved, which is not valid for general nonlinear complex systems. The approach uses the Barbashin–Krasovsky theorem on global asymptotic stability. Sufficient conditions for global asymptotic stabilization of a bilinear non-homogeneous complex system by real state feedback are obtained. Finally, an example of using the obtained results is presented.**

I. INTRODUCTION

Bilinear systems are found in many areas of engineering and science: chemical, mechanical, electrical engineering, physics, biology etc. [1]. Bilinear systems are a intermediate subclass between linear and non-linear systems. On the one hand, the free dynamics of a bilinear system is described by a linear system, so methods of controlling linear systems are applicable to them. On the other hand, a closed loop system is non-linear, so non-linear analysis methods are required. Stabilization of bilinear systems is a fundamental issue in control theory and has been well investigated over the last few decades by many authors (see refs, e.g., in [2]). Sufficient conditions for global asymptotic stabilization of bilinear homogeneous real systems have been obtained in [3]. Subsequently these results have been generalized and extended to affine [4], [5] and general nonlinear systems [6]. Similar results have been obtained for bilinear [7], affine [8], and general nonlinear autonomous systems with the discrete time [9], [10]. Later these results have been extended to nonlinear systems with periodic coefficients with continuouos time [12], [13] and discrete time [14], [15]. The proofs of these results are based on the Barbashin–Krasovsky theorem on global asymptotical stability, which lies in the frameworks of the Lyapunov Second Method (Method of Lyapunov Functions).

Complex-valued differential systems have applications in many science problems and are attracting more and more attention (see [16], [17], [18] and refs therein). They are widely used in complex-variable neural networks [19] and quantum mechanics [20], [21], in particular, for bilinear systems [22]. Besides this, equations of many classical systems such as the Ginzburg–Landau equation, the Orr–Sommerfeld equation, the complex Riccati equation and the complex Lorenz equation are considered in the complex field (see refs in [16]). Therefore, the study of complex systems of differential equations is important and significant.

Note that the First Method of Lyapunov stability theory (the method of characteristic numbers) and its theorems (on stability in linear approximation) work both in the real field and in the complex field. In contrast, the Second Lyapunov Method (the Lyapunov Function Method) does not work in the complex field in the general case. In fact, let a nonlinear system $\dot{z} = f(z)$, $z \in \mathbb{C}^n$, $f(0) = 0$, be given. If some function $V(z)$, $z \in \mathbb{C}^n$, takes real values and is positive definite (i.e., $V(z) > 0$, $z \neq 0$; $V(0) = 0$), it may turn out that its derivative $L_f V(z) = (\partial V(z)/\partial z) f(z)$ does not take real values. Therefore, we are not able to use this method directly to complex systems of a general form. Nevertheless, for bilinear complex systems, we are able to do it. Here we solve the problem of global asymptotic stabilization of time-invariant complex-valued non-homogeneous bilinear systems.

II. PROBLEM STATEMENT AND PREVIOUS RESULTS

Suppose $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$, where $\mathbb{C}$ is the set of complex numbers, $\mathbb{R}$ is the set of real numbers; $\mathbb{K}^n = \{x = \mathrm{col}\,(x_1, \ldots, x_n) \colon x_i \in \mathbb{K}\}$ is the linear space of column vectors over $\mathbb{K}$; $M_{m,n}(\mathbb{K})$ is the space of $m \times n$-matrices over $\mathbb{K}$; $M_n(\mathbb{K}) := M_{n,n}(\mathbb{K})$; $I \in M_n(\mathbb{K})$ is the identity matrix; T is the transposition of a vector or a matrix; $*$ is the Hermitian conjugation, i.e., $A^* = \overline{A}^T$; the inequalities $P > (\geq) Q$ for Hermitian matrices P, Q are understood in the sense of quadratic forms; $|x| = \sqrt{x^* x}$ is the norm in $\mathbb{K}^n$.

Consider a time-invariant bilinear non-homogeneous control system in the complex space

$$\dot{z} = Az + \big(B(z) + L\big)u. \tag{1}$$

Here $z = x + iy \in \mathbb{C}^n$ is the state, $x, y \in \mathbb{R}^n$; $B(z) = [B_1 z, \ldots, B_r z] \in M_{n,r}(\mathbb{C})$, $L = [L_1, \ldots, L_r] \in M_{n,r}(\mathbb{C})$; $A = C + iD \in M_n(\mathbb{C})$, $B_j = F_j + iG_j \in M_n(\mathbb{C})$, $L_j = H_j + iJ_j \in M_{n,1}(\mathbb{C})$, $C, D, F_j, G_j \in M_n(\mathbb{R})$, $H_j, J_j \in M_{n,1}(\mathbb{R})$, $j = \overline{1,r}$. We will suppose that control $u = \mathrm{col}\,(u_1, \ldots, u_r)$ is real, i.e., $u \in \mathbb{R}^r$. The corresponding free dynamic system has the form

$$\dot{z} = Az. \tag{2}$$

We study the problem of global asymptotic stabilization of the origin of system (1): one needs to construct a feedback control

$$u = \widehat{u}(z) = \mathrm{col}\,\big(\widehat{u}_1(z), \ldots, \widehat{u}_r(z)\big), \tag{3}$$

$\widehat{u}(0) = 0$, in system (1) such that the equilibrium $z = 0$ of the closed-loop system is globally asymptotically stable.

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First, we present the known results on global asymptotic stabilization of real-valued systems. Let $X \in M_n(\mathbb{R})$. For an arbitrary matrix $Z \in M_n(\mathbb{R})$, define the operators $W_X Z$ and $T_X Z$ by the equalities: $W_X Z := X^T Z + ZX$, $T_X Z := ZX - XZ$. By definition, put $W_X^0 Z := Z$, $W_X^s Z := W_X(W_X^{s-1} Z)$, $T_X^0 Z := Z$, $T_X^s Z := T_X(T_X^{s-1} Z)$, $s \in \mathbb{N}$. If Z is symmetric, then $W_X^s Z$ is also symmetric for any $s \in \mathbb{N}$.

Consider the ad-operator (the commutator) to vector fields:

$$\mathrm{ad}_{f(x)} g(x) := [f(x), g(x)] := \frac{\partial g(x)}{\partial x} f(x) - \frac{\partial f(x)}{\partial x} g(x).$$

By definition, put

$$\mathrm{ad}^0_{f(x)} g(x) := g(x), \quad \mathrm{ad}^{\ell}_{f(x)} g(x) := [f(x), \mathrm{ad}^{\ell-1}_{f(x)} g(x)], \quad \ell \in \mathbb{N}.$$

Consider system (1), where $z \in \mathbb{R}^n$, $A, B_j \in M_n(\mathbb{R})$, $L_j \in M_{n,1}(\mathbb{R})$, $j = \overline{1,r}$. Suppose that system (2) is Lyapunov stable (non-asymptotically). This condition is necessary and sufficient (see, e.g., [11, Lemma 2]) for the existence of a matrix $Q = Q^T \in M_n(\mathbb{R})$ such that

$$Q > 0 \ \text{ and } \ W_A Q \leq 0. \tag{4}$$

Let us construct the control function

$$\widehat{u}(z) = -2(B(z) + L)^T Q z, \tag{5}$$

where Q satisfies conditions (4). Then

$$\begin{aligned} \widehat{u}_j(z) &= -2(B_j z + L_j)^T Q z \\ &= -\left[z^T (B_j^T Q + Q B_j) z + (L_j^T Q z + z^T Q L_j)\right], \quad j = \overline{1,r}. \end{aligned}$$

Denote $\Psi_j^{\ell}(z) := \mathrm{ad}^{\ell}_{Az}(B_j z + L_j)$, $j = \overline{1,r}$, $\ell = 0, 1, \ldots$. Then, obviously, we have

$$\begin{aligned} &\Psi_j^0(z) = B_j z + L_j, \\ &\Psi_j^1(z) = B_j A z - A B_j z - A L_j = (T_A B_j) z - A L_j, \quad \ldots, \\ &\Psi_j^{\ell}(z) = (T_A^{\ell} B_j) z + (-A)^{\ell} L_j, \quad \ldots. \end{aligned} \tag{6}$$

Let us construct the sets

$$\begin{aligned} &\Omega_0 = \{z \in \mathbb{R}^n : z^T (W_A Q) z = 0\}, \\ &\Omega_1 = \{z \in \mathbb{R}^n : z^T (W_A^s Q) z = 0,\ s \in \mathbb{N}\}, \\ &S_0 = \{z \in \mathbb{R}^n : z^T Q \big(B(z) + L\big) = 0\}, \\ &\Sigma_1 = \{z \in \mathbb{R}^n : z^T \left(W_A^s (Q B_j)\right) z + z^T (A^T)^s Q L_j = 0, \\ &\qquad j = \overline{1,r},\ s = 0, 1, \ldots\}, \\ &S_1 = \{z \in \mathbb{R}^n : z^T Q \big((T_A^{\ell} B_j) z + (-A)^{\ell} L_j\big) = 0, \\ &\qquad j = \overline{1,r},\ \ell = 0, 1, \ldots\}, \\ &S_2 = \{z \in \mathbb{R}^n : z^T W_A^s \big(Q (T_A^{\ell} B_j)\big) z + z^T (A^T)^s Q (-A)^{\ell} L_j = 0, \\ &\qquad j = \overline{1,r},\ \ell = 0, 1, \ldots,\ s = 0, 1, \ldots\}, \\ &E_0 = \Omega_0 \cap S_0, \quad E_1 = \Omega_1 \cap \Sigma_1, \quad E_2 = \Omega_1 \cap S_1, \\ &E_3 = \Omega_0 \cap S_1, \quad E_4 = \Omega_1 \cap S_2. \end{aligned}$$

Let us construct the linear subspace of $\mathbb{R}^n$:

$$\Delta(z) = \mathrm{span}\,\{Az, \Psi_j^{\ell}(z),\ j = \overline{1,r},\ \ell = 0, 1, \ldots\}.$$

The following theorems are true.

Theorem 1: Suppose that $z \in \mathbb{R}^n$, $A, B_j \in M_n(\mathbb{R})$, $L_j \in M_{n,1}(\mathbb{R})$, $j = \overline{1,r}$. Suppose that there exists a matrix $Q = Q^T \in M_n(\mathbb{R})$ satisfying conditions (4), and at least one of the following conditions holds: (a) $E_0 = \{0\}$; (b) $E_1 = \{0\}$; (c) $E_2 = \{0\}$; (d) $E_3 = \{0\}$; (e) $E_4 = \{0\}$. Then the state feedback control (3), (5) globally asymptotically stabilizes the zero solution of system (1).

Theorem 2: Suppose that $z \in \mathbb{R}^n$, $A, B_j \in M_n(\mathbb{R})$, $L_j \in M_{n,1}(\mathbb{R})$, $j = \overline{1,r}$. Suppose that there exists a matrix $Q = Q^T \in M_n(\mathbb{R})$ satisfying conditions (4), and the following condition holds: (f) $\Delta(z) = \mathbb{R}^n \quad \forall z \in E_0 \setminus \{0\}$. Then the state feedback control (3), (5) globally asymptotically stabilizes the zero solution of system (1).

Theorems 1 and 2 can be obtained as corollaries of the corresponding results for affine systems (see, e.g., [6], [12]). Under conditions (4), all of conditions (a), (b), (c), (d), and (e) in Theorem 1 are equivalent (see, e.g., [12, Theorem 8]).

The main purpose of the work is extending results on global asymptotical stabilization for system (1) (like Theorems 1, 2) from real-valued systems to complex-valued systems.

III. Auxiliary assertions

Let $X = U + iV \in M_n(\mathbb{C})$ be an arbitrary matrix, where $U, V \in M_n(\mathbb{R})$. We will write, by definition, $\mathbf{X} \rightsquigarrow X$ if $\mathbf{X} = \begin{bmatrix} U & -V \\ V & U \end{bmatrix} \in M_{2n}(\mathbb{R})$.

Lemma 1: Let $\mathbf{X} \rightsquigarrow X$ and $\mathbf{Y} \rightsquigarrow Y$. Then: (a) $\mathbf{X}^T \rightsquigarrow X^*$; (b) $\mathbf{X}^s \rightsquigarrow X^s$, $s \in \mathbb{N}$; (c) $\mathbf{X} + \mathbf{Y} \rightsquigarrow X + Y$; (d) $\mathbf{X} \cdot \mathbf{Y} \rightsquigarrow X \cdot Y$.

The proof is obtained by the direct computation.

Lemma 2: Let $P = Q + iR$, $Q, R \in M_n(\mathbb{R})$, and $P^* = P$ (i.e., $Q^T = Q$, $R^T = -R$). Let $\mathbf{P} \rightsquigarrow P$. Then, for the corresponding $z = x + iy \in \mathbb{C}^n$ and $\mathbf{z} = \mathrm{col}\,(x, y) \in \mathbb{R}^{2n}$, we have $z^* P z = \mathbf{z}^T \mathbf{P} \mathbf{z}$.

Proof:

$$z^* P z = (x^T - i y^T)(Q + iR)(x + iy) = x^T Q x + y^T Q y - 2 x^T R y,$$

$$\mathbf{z}^T \mathbf{P} \mathbf{z} = \begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} Q & -R \\ R & Q \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x^T Q x + y^T Q y - 2 x^T R y.$$

■

Lemma 3: Let $P = Q + iR$, $Q, R \in M_n(\mathbb{R})$, and $P^* = P$ (i.e., $Q^T = Q$, $R^T = -R$). Let $U = U_1 + iU_2$, $Z = Z_1 + iZ_2$, $U_1, U_2, Z_1, Z_2 \in M_{n,1}(\mathbb{R})$, and $\mathbf{U} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$, $\mathbf{Z} = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$. Let $\mathbf{P} \rightsquigarrow P$. Then

$$U^* P Z + Z^* P U = \mathbf{U}^T \mathbf{P} \mathbf{Z} + \mathbf{Z}^T \mathbf{P} \mathbf{U}.$$

Proof:

$$\begin{aligned} &U^* P Z + Z^* P U = 2\,\mathrm{Re}\,\big((U_1^T - i U_2^T)(Q + iR)(Z_1 + i Z_2)\big) \\ &\quad = 2\big(U_1^T (Q Z_1 - R Z_2) + U_2^T (R Z_1 + Q Z_2)\big), \\ &\mathbf{U}^T \mathbf{P} \mathbf{Z} + \mathbf{Z}^T \mathbf{P} \mathbf{U} \\ &\quad = [U_1^T \quad U_2^T] \begin{bmatrix} Q & -R \\ R & Q \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} + [Z_1^T \quad Z_2^T] \begin{bmatrix} Q & -R \\ R & Q \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \\ &\quad = 2\big(U_1^T (Q Z_1 - R Z_2) + U_2^T (R Z_1 + Q Z_2)\big). \end{aligned}$$

■

Lemma 4: Let $X = U + iV$, $Y = W + iZ$, $U, V, W, Z \in M_n(\mathbb{R})$. Let $N = N_1 + iN_2$, $K = K_1 + iK_2$, $N_1, N_2, K_1, K_2 \in$

$M_{n,1}(\mathbb{R})$. Let $\mathbf{X} \rightsquigarrow X$, $\mathbf{Y} \rightsquigarrow Y$. Set $\mathbf{N} := \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$, $\mathbf{K} := \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}$, $\mathbf{E} := \mathbf{XN} + \mathbf{YK}$, $E := XN + YK$. Then $\mathbf{E} = \begin{bmatrix} \operatorname{Re} E \\ \operatorname{Im} E \end{bmatrix}$.

Proof: $E = (U+iV)(N_1+iN_2) + (W+iZ)(K_1+iK_2) = UN_1 - VN_2 + WK_1 - ZK_2 + i(VN_1 + UN_2 + ZK_1 + WK_2)$,

$$\mathbf{E} = \begin{bmatrix} U & -V \\ V & U \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} + \begin{bmatrix} W & -Z \\ Z & W \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \begin{bmatrix} UN_1 - VN_2 + WK_1 - ZK_2 \\ VN_1 + UN_2 + ZK_1 + WK_2 \end{bmatrix}.$$

■

Lemma 5: Let $X \in M_n(\mathbb{C})$, and $\mathbf{X} \rightsquigarrow X$. Then, for the corresponding $z = x + iy \in \mathbb{C}^n$ and $\mathbf{z} = \operatorname{col}(x,y) \in \mathbb{R}^{2n}$, the vector $\mathbf{z}_1 := \mathbf{Xz}$ corresponds to $z_1 := Xz$, i.e., $\mathbf{z}_1 = \operatorname{col}(x_1, y_1) \in \mathbb{R}^{2n}$, where $x_1 + iy_1 = z_1$.

Lemma 5 follows from Lemma 4 by taking $Y := 0$, $K := 0$, $N := z$ (considering $z \in \mathbb{C}^n$ as $n \times 1$-matrix).

Let $X \in M_n(\mathbb{C})$ (resp. $\mathbf{X} \in M_{2n}(\mathbb{R})$). For any $Z \in M_n(\mathbb{C})$ (for any $\mathbf{Z} \in M_{2n}(\mathbb{R})$) define the following operators by the equalities

$$\mathscr{W}_X Z := X^* Z + ZX, \qquad \mathscr{T}_X Z := ZX - XZ,$$
$$\mathbf{W_X Z} := \mathbf{X}^T \mathbf{Z} + \mathbf{ZX}, \qquad \mathbf{T_X Z} := \mathbf{ZX} - \mathbf{XZ}.$$

Lemma 6: Let $X, Z \in M_n(\mathbb{C})$, $\mathbf{X} \rightsquigarrow X$, and $\mathbf{Z} \rightsquigarrow Z$. Then: (a) $\mathbf{W}_\mathbf{X}^s \mathbf{Z} \rightsquigarrow \mathscr{W}_X^s Z$, $s \in \mathbb{N}$; (b) $\mathbf{T}_\mathbf{X}^s \mathbf{Z} \rightsquigarrow \mathscr{T}_X^s Z$, $s \in \mathbb{N}$.

Lemma 6 follows from Lemma 1.

IV. Global asymptotical stabilization of complex-valued systems

Consider the bilinear complex control system (1). From system (1), let us construct the equivalent real-valued system of dimension $2n$ (see [23, Sect. 2]. We get the following system:

$$\begin{cases} \dot{x} = Cx - Dy + \sum_{j=1}^{r} u_j (F_j x - G_j y + H_j), \\ \dot{y} = Dx + Cy + \sum_{j=1}^{r} u_j (G_j x + F_j y + J_j). \end{cases} \tag{7}$$

Denote

$$\mathbf{z} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^{2n}, \quad \mathbf{A} = \begin{bmatrix} C & -D \\ D & C \end{bmatrix} \in M_{2n}(\mathbb{R}), \tag{8}$$
$$\mathbf{B}_j = \begin{bmatrix} F_j & -G_j \\ G_j & F_j \end{bmatrix} \in M_{2n}(\mathbb{R}),\ \mathbf{L}_j = \begin{bmatrix} H_j \\ J_j \end{bmatrix} \in M_{2n,1}(\mathbb{R}).$$

System (7) has the form

$$\dot{\mathbf{z}} = \mathbf{Az} + \big(\mathbf{B}(\mathbf{z}) + \mathbf{L}\big) u, \tag{9}$$

where $\mathbf{B}(\mathbf{z}) = [\mathbf{B}_1\mathbf{z}, \ldots, \mathbf{B}_r\mathbf{z}] \in M_{2n,r}(\mathbb{R})$, $\mathbf{L} = [\mathbf{L}_1, \ldots, \mathbf{L}_r] \in M_{2n,r}(\mathbb{R})$.

There is one-to-one correspondence between solutions of (9) and (1): $z(t) = x(t) + iy(t) \in \mathbb{C}^n$ is a solution of (1) if and only if $\mathbf{z}(t) = \operatorname{col}\big(x(t), y(t)\big) \in \mathbb{R}^{2n}$ is a solution of (9).

For system (9), consider the corresponding free dynamic system

$$\dot{\mathbf{z}} = \mathbf{Az}. \tag{10}$$

Lemma 7: The following assertions are equivalent.

1. System (2) is Lyapunov stable.
2. There exists a matrix $P = Q + iR \in M_n(\mathbb{C})$ such that

$$P^* = P, \tag{11}$$
$$P > 0, \tag{12}$$
$$A^*P + PA \le 0. \tag{13}$$

3. System (10) with $\mathbf{A}$ of the form (8) is Lyapunov stable.
4. There exists a matrix $\mathbf{P} \in M_{2n}(\mathbb{R})$ of the form

$$\mathbf{P} = \begin{bmatrix} Q & -R \\ R & Q \end{bmatrix} \tag{14}$$

such that

$$\mathbf{P}^T = \mathbf{P}, \tag{15}$$
$$\mathbf{P} > 0, \tag{16}$$
$$\mathbf{A}^T\mathbf{P} + \mathbf{PA} \le 0. \tag{17}$$

Lemma 7 follows from Lemmas 2, 3, and 4 of [23], and besides, the matrices P and $\mathbf{P}$ can be found constructively. From the proof of [23, Lemma 4], it follows that the matrices $Q \in M_n(\mathbb{R})$ and $R \in M_n(\mathbb{R})$ can be chosen the same in items 2 and 4; the equalities $Q^T = Q$, $R^T = -R$ hold; and, for the corresponding $z = x + iy \in \mathbb{C}^n$ and $\mathbf{z} = \operatorname{col}(x,y) \in \mathbb{R}^{2n}$,

$$z^*(A^*P + PA)z = \mathbf{z}^T(\mathbf{A}^T\mathbf{P} + \mathbf{PA})\mathbf{z}. \tag{18}$$

Suppose (here and throughout) that the free dynamic system (2) is Lyapunov stable (non-asymptotically). Construct, according to Lemma 7, the corresponding matrices $Q, P \in M_n(\mathbb{R})$, the matrix $P = Q + iR$ satisfying (11), (12), (13), and the matrix $\mathbf{P}$ of (14) satisfying (15), (16), (17). We have $\mathbf{A} \rightsquigarrow A$, $\mathbf{P} \rightsquigarrow P$; hence, by Lemma 6 (a), $\mathbf{W}_\mathbf{A}^s \mathbf{P} \rightsquigarrow \mathscr{W}_A^s P$, $s \in \mathbb{N}$. Since $\big(\mathscr{W}_A^s P\big)^* = \mathscr{W}_A^s P$ for any $s \in \mathbb{N}$, it follows, by Lemma 2, that, for the corresponding $z = x + iy$ and $\mathbf{z} = \operatorname{col}(x,y)$,

$$z^*(\mathscr{W}_A^s P)z = \mathbf{z}^T(\mathbf{W}_\mathbf{A}^s \mathbf{P})\mathbf{z} \quad \forall s \in \mathbb{N}. \tag{19}$$

Let us construct the control function

$$\widehat{\mathbf{u}}(\mathbf{z}) = -2\big(\mathbf{B}(\mathbf{z}) + \mathbf{L}\big)^T \mathbf{Pz} \tag{20}$$

for system (9). Then

$$\widehat{\mathbf{u}}_j(\mathbf{z}) = -2\big(\mathbf{B}_j\mathbf{z} + \mathbf{L}_j\big)^T \mathbf{Pz} = -\big[\mathbf{z}^T(\mathbf{B}_j^T\mathbf{P} + \mathbf{PB}_j)\mathbf{z} + (\mathbf{L}_j^T\mathbf{Pz} + \mathbf{z}^T\mathbf{PL}_j)\big],\ j = \overline{1,r}. \tag{21}$$

Let us construct the control function

$$\widehat{u}_j(z) = -\big[z^*(B_j^*P + PB_j)z + (L_j^*Pz + z^*PL_j)\big],\ j = \overline{1,r}, \tag{22}$$

for system (1). Set

$$\mathbf{O}_0 = \{\mathbf{z} \in \mathbb{R}^{2n} : \mathbf{z}^T(\mathbf{W_A P})\mathbf{z} = 0\},$$
$$\mathbf{S}_0 = \{\mathbf{z} \in \mathbb{R}^{2n} : \mathbf{z}^T\mathbf{P}\big(\mathbf{B}(\mathbf{z}) + \mathbf{L}\big) = 0\},$$
$$\mathbf{E}_0 = \mathbf{O}_0 \cap \mathbf{S}_0,$$
$$\mathscr{O}_0 = \{z \in \mathbb{C}^n : z^*(\mathscr{W}_A P)z = 0\},$$
$$\mathscr{S}_0 = \{z \in \mathbb{C}^n : z^*(B_j^*P + PB_j)z + (L_j^*Pz + z^*PL_j) = 0, \quad j = \overline{1,r}\},$$
$$\mathscr{E}_0 = \mathscr{O}_0 \cap \mathscr{S}_0.$$

Theorem 3: Suppose that $z \in \mathbb{C}^n$, $A, B_j \in M_n(\mathbb{C})$, $L_j \in M_{n,1}(\mathbb{C})$, $j = \overline{1,r}$. Suppose that there exists a matrix $P \in M_n(\mathbb{C})$ satisfying conditions (11), (12), (13), and the following condition holds: (a) $\mathscr{E}_0 = \{0\}$. Then the state feedback control (3), (22) globally asymptotically stabilizes the zero solution of system (1).

Proof: For the corresponding $z = x+iy$ and $\mathbf{z} = \mathrm{col}\,(x,y)$, we have, for every $j = \overline{1,r}$: $\mathbf{B}_j^T\mathbf{P} + \mathbf{P}\mathbf{B}_j \rightsquigarrow B_j^*P + PB_j$, by Lemma 6 (a); hence, by Lemma 2,

$$z^*(B_j^*P + PB_j)z = \mathbf{z}^T(\mathbf{B}_j^T\mathbf{P} + \mathbf{P}\mathbf{B}_j)\mathbf{z}; \tag{23}$$

by Lemma 3 (considering $z \in \mathbb{C}^n$ as a complex $n \times 1$-matrix),

$$L_j^*Pz + z^*PL_j = \mathbf{L}_j^T\mathbf{P}\mathbf{z} + \mathbf{z}^T\mathbf{P}\mathbf{L}_j. \tag{24}$$

From (23), (24), (21), and (22), it follows that

$$\widehat{u}_j(z) = \widehat{\mathbf{u}}_j(\mathbf{z}), \quad j = \overline{1,r}, \tag{25}$$

in particular, $\widehat{u}_j(z) \in \mathbb{R}$, $j = \overline{1,r}$. Let us substitute

$$u = \widehat{\mathbf{u}}(\mathbf{z}) = \mathrm{col}\,\big(\widehat{\mathbf{u}}_1(\mathbf{z}), \ldots, \widehat{\mathbf{u}}_r(\mathbf{z})\big)$$

into system (9) and $u = \widehat{u}(z) = \mathrm{col}\,\big(\widehat{u}_1(z), \ldots, \widehat{u}_r(z)\big)$ into system (1). We obtain the closed-loop systems

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \big(\mathbf{B}(\mathbf{z}) + \mathbf{L}\big)\widehat{\mathbf{u}}(\mathbf{z}) \tag{26}$$

and

$$\dot{z} = Az + \big(B(z) + L\big)\widehat{u}(z). \tag{27}$$

Systems (26) and (27) are equivalent because the feedback control functions (21) and (22) coincide and are real.

By (18), $\mathbf{O}_0 = \{\mathbf{z} = \mathrm{col}\,(x,y) \in \mathbb{R}^{2n} : z = x+iy \in \mathscr{O}_0\}$. By (25), (20), (22), we have $\mathbf{S}_0 = \{\mathbf{z} = \mathrm{col}\,(x,y) \in \mathbb{R}^{2n} : z = x+iy \in \mathscr{S}_0\}$. Thus,

$$\mathbf{E}_0 = \{\mathbf{z} = \mathrm{col}\,(x,y) \in \mathbb{R}^{2n} : z = x+iy \in \mathscr{E}_0\}. \tag{28}$$

Hence, the condition $\mathscr{E}_0 = \{0\}$ is equivalent to the condition $\mathbf{E}_0 = \{0\}$. If $\mathbf{E}_0 = \{0\}$ then, by applying Theorem 1 to the real-valued system (9), we obtain that the zero solution of the closed-loop system (26) is globally asymptotically stable. Thus, the zero solution of the equivalent complex closed-loop system (27) is globally asymptotically stable. ∎

Let us construct the sets

$$\begin{aligned}
&\mathscr{O}_1 = \{z \in \mathbb{C}^n : z^*(\mathscr{W}_A^s P)z = 0,\ s \in \mathbb{N}\},\\
&\Theta_1 = \{z \in \mathbb{C}^n : z^*\mathscr{W}_A^s(B_j^*P + PB_j)z\\
&\quad + \big(L_j^*PA^sz + z^*(A^*)^sPL_j\big) = 0,\ j = \overline{1,r},\ s = 0,1,\ldots\},\\
&\mathscr{S}_1 = \{z \in \mathbb{C}^n : z^*\big((\mathscr{T}_A^\ell B_j)^*P + P(\mathscr{T}_A^\ell B_j)\big)z\\
&\quad + (L_j^*(-A^*)^\ell Pz + z^*P(-A)^\ell L_j = 0,\\
&\qquad j = \overline{1,r},\ \ell = 0,1,\ldots\},\\
&\mathscr{S}_2 = \{z \in \mathbb{C}^n : z^*\mathscr{W}_A^s\big((\mathscr{T}_A^\ell B_j)^*P + P(\mathscr{T}_A^\ell B_j)\big)z\\
&\quad + (L_j^*(-A^*)^\ell PA^sz + z^*(A^*)^sP(-A)^\ell L_j = 0,\\
&\qquad j = \overline{1,r},\ \ell = 0,1,\ldots,\ s = 0,1,\ldots\},\\
&\mathscr{E}_1 = \mathscr{O}_1 \cap \Theta_1, \quad \mathscr{E}_2 = \mathscr{O}_1 \cap \mathscr{S}_1,\\
&\mathscr{E}_3 = \mathscr{O}_0 \cap \mathscr{S}_1, \quad \mathscr{E}_4 = \mathscr{O}_1 \cap \mathscr{S}_2.
\end{aligned}$$

Let us construct the sets

$$\begin{aligned}
&\mathbf{O}_1 = \{\mathbf{z} \in \mathbb{R}^{2n} : \mathbf{z}^T(\mathbf{W}_\mathbf{A}^s\mathbf{P})\mathbf{z} = 0,\ s \in \mathbb{N}\},\\
&\mathfrak{S}_1 = \{\mathbf{z} \in \mathbb{R}^{2n} : \mathbf{z}^T\mathbf{W}_\mathbf{A}^s(\mathbf{B}_j^T\mathbf{P} + \mathbf{P}\mathbf{B}_j)\mathbf{z}\\
&\quad + \big(\mathbf{L}_j^T\mathbf{P}\mathbf{A}^s\mathbf{z} + \mathbf{z}^T(\mathbf{A}^T)^s\mathbf{P}\mathbf{L}_j\big) = 0,\ j = \overline{1,r},\ s = 0,1,\ldots\},\\
&\mathbf{S}_1 = \{\mathbf{z} \in \mathbb{R}^{2n} : \mathbf{z}^T\big((\mathbf{T}_\mathbf{A}^\ell\mathbf{B}_j)^T\mathbf{P} + \mathbf{P}(\mathbf{T}_\mathbf{A}^\ell\mathbf{B}_j)\big)\mathbf{z}\\
&\quad + \big(\mathbf{L}_j^T(-\mathbf{A}^T)^\ell\mathbf{P}\mathbf{z} + \mathbf{z}^T\mathbf{P}(-\mathbf{A})^\ell\mathbf{L}_j\big) = 0,\\
&\qquad j = \overline{1,r},\ \ell = 0,1,\ldots\},\\
&\mathbf{S}_2 = \{\mathbf{z} \in \mathbb{R}^{2n} : \mathbf{z}^T\mathbf{W}_\mathbf{A}^s\big((\mathbf{T}_\mathbf{A}^\ell\mathbf{B}_j)^T\mathbf{P} + \mathbf{P}(\mathbf{T}_\mathbf{A}^\ell\mathbf{B}_j)\big)\mathbf{z}\\
&\quad + \big(\mathbf{L}_j^T(-\mathbf{A}^T)^\ell\mathbf{P}\mathbf{A}^s\mathbf{z} + \mathbf{z}^T(\mathbf{A}^T)^s\mathbf{P}(-\mathbf{A})^\ell\mathbf{L}_j\big) = 0,\\
&\qquad j = \overline{1,r},\ \ell = 0,1,\ldots,\ s = 0,1,\ldots\},\\
&\mathbf{E}_1 = \mathbf{O}_1 \cap \mathfrak{S}_1, \quad \mathbf{E}_2 = \mathbf{O}_1 \cap \mathbf{S}_1,\\
&\mathbf{E}_3 = \mathbf{O}_0 \cap \mathbf{S}_1, \quad \mathbf{E}_4 = \mathbf{O}_1 \cap \mathbf{S}_2.
\end{aligned}$$

Remark 1: For reasons of symmetry, the sets $\mathfrak{S}_1$, $\mathbf{S}_1$, $\mathbf{S}_2$ can be rewritten in the form

$$\begin{aligned}
&\mathfrak{S}_1 = \{\mathbf{z} \in \mathbb{R}^{2n} : \mathbf{z}^T\mathbf{W}_\mathbf{A}^s(\mathbf{P}\mathbf{B}_j)\mathbf{z} + \mathbf{z}^T(\mathbf{A}^T)^s\mathbf{P}\mathbf{L}_j = 0,\\
&\qquad j = \overline{1,r},\ s = 0,1,\ldots\},\\
&\mathbf{S}_1 = \{\mathbf{z} \in \mathbb{R}^{2n} : \mathbf{z}^T\mathbf{P}\big((\mathbf{T}_\mathbf{A}^\ell\mathbf{B}_j)\mathbf{z} + (-\mathbf{A})^\ell\mathbf{L}_j\big) = 0,\\
&\qquad j = \overline{1,r},\ \ell = 0,1,\ldots\},\\
&\mathbf{S}_2 = \{\mathbf{z} \in \mathbb{R}^{2n} : \mathbf{z}^T\mathbf{W}_\mathbf{A}^s\big(\mathbf{P}(\mathbf{T}_\mathbf{A}^\ell\mathbf{B}_j)\big)\mathbf{z} + \mathbf{z}^T(\mathbf{A}^T)^s\mathbf{P}(-\mathbf{A})^\ell\mathbf{L}_j = 0,\\
&\qquad j = \overline{1,r},\ \ell = 0,1,\ldots,\ s = 0,1,\ldots\},
\end{aligned}$$

which is similar to Σ_1, S_1, S_2.

Theorem 4: Suppose that $z \in \mathbb{C}^n$, $A, B_j \in M_n(\mathbb{C})$, $L_j \in M_{n,1}(\mathbb{C})$, $j = \overline{1,r}$. Suppose that there exists a matrix $P = P^* \in M_n(\mathbb{C})$ satisfying conditions (11), (12), (13), and at least one of the following conditions holds: (b) $\mathscr{E}_1 = \{0\}$; (c) $\mathscr{E}_2 = \{0\}$; (d) $\mathscr{E}_3 = \{0\}$; (e) $\mathscr{E}_4 = \{0\}$. Then the state feedback control (3), (22) globally asymptotically stabilizes the zero solution of system (1).

Proof: By (19),

$$\mathbf{O}_1 = \{\mathbf{z} = \mathrm{col}\,(x,y) \in \mathbb{R}^{2n} : z = x+iy \in \mathscr{O}_1\}.$$

Consider the sets Θ_1 and $\mathfrak{S}_1$. Let $Z_j = B_j^*P + PB_j$ and $\mathbf{Z}_j = \mathbf{B}_j^T\mathbf{P} + \mathbf{P}\mathbf{B}_j$. By Lemma 6 (a), we have $\mathbf{Z}_j \rightsquigarrow Z_j$; hence, by Lemma 6 (a) again, $\mathbf{W}_\mathbf{A}^s\mathbf{Z}_j \rightsquigarrow \mathscr{W}_A^sZ_j$. Moreover, since $Z_j^* = Z_j$, it follows that $(\mathscr{W}_A^sZ_j)^* = \mathscr{W}_A^sZ_j$, $j = \overline{1,r}$, $s = 0,1,\ldots$. Hence, by Lemma 2, for the corresponding $z = x+iy \in \mathbb{C}^n$ and $\mathbf{z} = \mathrm{col}\,(x,y) \in \mathbb{R}^{2n}$, for any $j = \overline{1,r}$, $s = 0,1,\ldots$, we have

$$z^*\mathscr{W}_A^s(B_j^*P + PB_j)z = \mathbf{z}^T\mathbf{W}_\mathbf{A}^s(\mathbf{B}_j^T\mathbf{P} + \mathbf{P}\mathbf{B}_j)\mathbf{z}. \tag{29}$$

For every $s = 0,1,\ldots$, for the corresponding $z = x+iy \in \mathbb{C}^n$ and $\mathbf{z} = \mathrm{col}\,(x,y) \in \mathbb{R}^{2n}$, by Lemma 5, the vectors $\mathbf{z}_s := \mathbf{A}^s\mathbf{z}$ correspond to $z_s := A^sz$ (in the sense of Lemma 5). Now, by applying Lemma 3 to $U := L_j$ and $Z := A^sz$, we obtain that, for all $s = 0,1,\ldots$, $j = \overline{1,r}$,

$$L_j^*PA^sz + z^*(A^*)^sPL_j = \mathbf{L}_j^T\mathbf{P}\mathbf{A}^s\mathbf{z} + \mathbf{z}^T(\mathbf{A}^T)^s\mathbf{P}\mathbf{L}_j. \tag{30}$$

From (29) and (30), it follows that

$$\mathfrak{S}_1 = \{\mathbf{z} = \mathrm{col}\,(x,y) \in \mathbb{R}^{2n} : z = x+iy \in \Theta_1\}.$$

Consider the sets $\mathscr{S}_1$ and $\mathbf{S}_1$. Let

$$X_{j,\ell} = (\mathscr{T}_A^\ell B_j)^* P + P(\mathscr{T}_A^\ell B_j),$$
$$\mathbf{X}_{j,\ell} = (\mathbf{T}_\mathbf{A}^\ell \mathbf{B}_j)^T \mathbf{P} + \mathbf{P}(\mathbf{T}_\mathbf{A}^\ell \mathbf{B}_j). \qquad (31)$$

By Lemma 6 (b), $\mathbf{T}_\mathbf{A}^\ell \mathbf{B}_j \rightsquigarrow \mathscr{T}_A^\ell B_j$; hence, by Lemma 6 ($a$), $\mathbf{X}_{j,\ell} \rightsquigarrow X_{j,\ell}$. Since $X_{j,\ell}^* = X_{j,\ell}$, we have, by Lemma 2, for the corresponding $z = x+iy \in \mathbb{C}^n$ and $\mathbf{z} = \mathrm{col}\,(x,y) \in \mathbb{R}^{2n}$,

$$z^*\big((\mathscr{T}_A^\ell B_j)^* P + P(\mathscr{T}_A^\ell B_j)\big)z = \mathbf{z}^T\big((\mathbf{T}_\mathbf{A}^\ell \mathbf{B}_j)^T \mathbf{P} + \mathbf{P}(\mathbf{T}_\mathbf{A}^\ell \mathbf{B}_j)\big)\mathbf{z}, \quad (32)$$

$j = \overline{1,r}$, $\ell = 0,1,\ldots$. By applying Lemma 4 to $X := (-A)^\ell$, $N := L_j$, $Y := 0$, $K := 0$, and then, by applying Lemma 3 to $U := (-A)^\ell L_j$, $Z := z$, we get

$$L_j^*(-A^*)^\ell P z + z^* P(-A)^\ell L_j = \mathbf{L}_j^T(-\mathbf{A}^T)^\ell \mathbf{P}\mathbf{z} + \mathbf{z}^T\mathbf{P}(-\mathbf{A})^\ell \mathbf{L}_j, \quad (33)$$

$j = \overline{1,r}$, $\ell = 0,1,\ldots$. From (32) and (33), it follows that

$$\mathbf{S}_1 = \{\mathbf{z} = \mathrm{col}\,(x,y) \in \mathbb{R}^{2n} : z = x+iy \in \mathscr{S}_1\}.$$

Consider the sets $\mathscr{S}_2$ and $\mathbf{S}_2$. Define $X_{j,\ell}$ and $\mathbf{X}_{j,\ell}$ by (31). Again we have $\mathbf{X}_{j,\ell} \rightsquigarrow X_{j,\ell}$. Hence, by Lemma 6 ($a$), $\mathbf{W}_\mathbf{A}^s \mathbf{X}_{j,\ell} \rightsquigarrow \mathscr{W}_A^s X_{j,\ell}$. Since $X_{j,\ell}^* = X_{j,\ell}$, it follows that $(\mathscr{W}_A^s X_{j,\ell})^* = \mathscr{W}_A^s X_{j,\ell}$. By Lemma 2, for the corresponding $z = x+iy \in \mathbb{C}^n$ and $\mathbf{z} = \mathrm{col}\,(x,y) \in \mathbb{R}^{2n}$,

$$z^*\mathscr{W}_A^s\big((\mathscr{T}_A^\ell B_j)^* P + P(\mathscr{T}_A^\ell B_j)\big)z = \mathbf{z}^T\mathbf{W}_\mathbf{A}^s\big((\mathbf{T}_\mathbf{A}^\ell \mathbf{B}_j)^T \mathbf{P} + \mathbf{P}(\mathbf{T}_\mathbf{A}^\ell \mathbf{B}_j)\big)\mathbf{z}, \quad (34)$$

$j = \overline{1,r}$, $\ell = 0,1,\ldots$, $s = 0,1,\ldots$. By applying Lemma 4 to $X := (-A)^\ell$, $N := L_j$, $Y := 0$, $K := 0$, then, by applying Lemma 5 to $X := A^s$, and then, by applying Lemma 3 to $U := (-A)^\ell L_j$, $Z := A^s z$, we get

$$L_j^*(-A^*)^\ell P A^s z + z^*(A^*)^s P(-A)^\ell L_j = \mathbf{L}_j^T(-\mathbf{A}^T)^\ell \mathbf{P}\mathbf{A}^s\mathbf{z} + \mathbf{z}^T(\mathbf{A}^T)^s\mathbf{P}(-\mathbf{A})^\ell \mathbf{L}_j, \quad (35)$$

$j = \overline{1,r}$, $\ell = 0,1,\ldots$, $s = 0,1,\ldots$. From (34) and (35), it follows that

$$\mathbf{S}_2 = \{\mathbf{z} = \mathrm{col}\,(x,y) \in \mathbb{R}^{2n} : z = x+iy \in \mathscr{S}_2\}.$$

Thus, we have $\mathbf{E}_j = \{\mathbf{z} = \mathrm{col}\,(x,y) \in \mathbb{R}^{2n} : z = x+iy \in \mathscr{E}_j\}$, $j = 1,2,3,4$. The equality $\mathscr{E}_j = \{0\}$ is equivalent to $\mathbf{E}_j = \{0\}$ for every $j = 1,2,3,4$. Taking into account Remark 1 and by applying Theorem 1 to the real-valued system (9), we obtain that the zero solution of the closed-loop system (26) is globally asymptotically stable. Thus, the zero solution of the equivalent complex closed-loop system (27) is globally asymptotically stable. ■

Denote by ad ($\mathbf{ad}$) the ad-operator for vector fields in $\mathbb{C}^n$ (in $\mathbb{R}^{2n}$ respectively). For systems (1) and (9) denote $\Xi_j^\ell(z) = ad_{Az}^\ell(B_j z + L_j)$ and $\mathfrak{P}_j^\ell(\mathbf{z}) = \mathbf{ad}_{\mathbf{Az}}^\ell(\mathbf{B}_j\mathbf{z} + \mathbf{L}_j)$, respectively, $j = \overline{1,r}$, $\ell = 0,1,\ldots$. Then, similarly to (6), we obtain that

$$\Xi_j^\ell(z) = (\mathscr{T}_A^\ell B_j)z + (-A)^\ell L_j,$$
$$\mathfrak{P}_j^\ell(\mathbf{z}) = (\mathbf{T}_\mathbf{A}^\ell \mathbf{B}_j)\mathbf{z} + (-\mathbf{A})^\ell \mathbf{L}_j.$$

Next, consider the following sets of vectors in $\mathbb{C}^n$ and in $\mathbb{R}^{2n}$, respectively:

$$\{\mathtt{h}_1, \mathtt{h}_2, \ldots\} = \{Az, \Xi_1^\ell(z), \ldots, \Xi_r^\ell(z),\ \ell = 0,1,\ldots\}, \quad (36)$$
$$\{\mathbf{h}_1, \mathbf{h}_2, \ldots\} = \{\mathbf{Az}, \mathfrak{P}_1^\ell(\mathbf{z}), \ldots, \mathfrak{P}_r^\ell(\mathbf{z}),\ \ell = 0,1,\ldots\}, \quad (37)$$

for the corresponding $z = x+iy \in \mathbb{C}^n$ and $\mathbf{z} = \mathrm{col}\,(x,y) \in \mathbb{R}^{2n}$. Denote by $\mathrm{span}_\mathbb{R}\{\mathtt{h}_1, \mathtt{h}_2, \ldots\}$ the linear span of vectors $\mathtt{h}_j \in \mathbb{C}^n$ over $\mathbb{R}$, i.e., $\mathtt{h} \in \mathrm{span}_\mathbb{R}\{\mathtt{h}_1, \mathtt{h}_2, \ldots\}$ if there exist $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ such that $\mathtt{h} = \sum_{j=1}^k \lambda_j \mathtt{h}_{i_j}$. Construct the linear subspaces of $\mathbb{C}^n$ and $\mathbb{R}^{2n}$ respectively:

$$\Gamma(z) := \mathrm{span}_\mathbb{R}\{Az, \Xi_1^\ell(z), \ldots, \Xi_r^\ell(z),\ \ell = 0,1,\ldots\},$$
$$\Lambda(\mathbf{z}) := \mathrm{span}\{\mathbf{Az}, \mathfrak{P}_1^\ell(\mathbf{z}), \ldots, \mathfrak{P}_r^\ell(\mathbf{z}),\ \ell = 0,1,\ldots\}.$$

Theorem 5: Suppose that $z \in \mathbb{C}^n$, $A, B_j \in M_n(\mathbb{C})$, $L_j \in M_{n,1}(\mathbb{C})$, $j = \overline{1,r}$. Suppose that there exists a matrix $P = P^* \in M_n(\mathbb{C})$ satisfying conditions (11), (12), (13), and the following condition holds:

$$(\mathtt{f}) \qquad \Gamma(z) = \mathbb{C}^n \quad \forall z \in \mathscr{E}_0 \setminus \{0\}. \quad (38)$$

Then the state feedback control (3), (22) globally asymptotically stabilizes the zero solution of system (1).

Proof: Consider the vectors (36) and (37). By applying Lemma 4 to $X := \mathscr{T}_A^\ell B_j$, $Y := (-A)^\ell$, $N := z$, and $K := L_j$, we obtain that the corresponding vectors $\mathtt{h}_k$ and $\mathbf{h}_k$ in (36) and (37) satisfy the equality

$$\mathbf{h}_k = \mathrm{col}\,(\mathrm{Re}\,\mathtt{h}_k, \mathrm{Im}\,\mathtt{h}_k) \in \mathbb{R}^{2n}. \quad (39)$$

Due to (39) and (28), condition (38) is equivalent to (**f**) $\Lambda(\mathbf{z}) = \mathbb{R}^{2n} \quad \forall \mathbf{z} \in \mathbf{E}_0 \setminus \{0\}$. Thus, from Theorem 2, we get the required. ■

Remark 2: Suppose that $L = 0$ in (1). Then system (1) is bilinear homogeneous. In this case Theorems 3, 4, and 5 coincide with Theorems 3, 4, and 5 in [23]. Thus, the presented results extend the sufficient conditions of global asymptotic stabilization from complex time-invariant bilinear homogeneous systems to non-homogeneous ones.

Remark 3: Under conditions (11), (12), and (13), each of conditions (a) (of Theorem 3), or (b), or (c), or (d), or (e) (of Theorem 4) is equivalent to other. Thus, Theorem 3 is equivalent to Theorem 4. Under conditions (11), (12), and (13), condition (f) of Theorem 5 is stronger than any of conditions (a)–(e). Thus, Theorem 5 is weaker than Theorem 3 or Theorem 4.

Corollary 1: Suppose that $z \in \mathbb{C}^n$, $A, B_j \in M_n(\mathbb{C})$, $L_j \in M_{n,1}(\mathbb{C})$, $j = \overline{1,r}$. Suppose that the free system (2) is Lyapunov stable and at least one of the following conditions holds: (a) or (b), or (c), or (d), or (e), or (f), where $P = P^* \in M_n(\mathbb{C})$ is an arbitrary matrix satisfying conditions (11), (12), (13). Then the state feedback control (3), (22) globally asymptotically stabilizes the zero solution of system (1).

Corollary 1 follows from Theorems 3, 4, 5, and Lemma 7.

V. Example

Consider system (1) with $n = 2$, $r = 1$,

$$A = \begin{bmatrix} -1 & 1-i \\ 0 & i \end{bmatrix}, \quad B = \begin{bmatrix} i & 1 \\ -i & 2 \end{bmatrix}, \quad L = \begin{bmatrix} 1 \\ i \end{bmatrix}. \quad (40)$$

The eigenvalues of A are i, -1. The free dynamical system is Lyapunov stable. Let us construct a matrix $P \in M_2(\mathbb{C})$ satisfying (11), (12), (13) (e.g., by using [24, Lemma 3]). We obtain $P = \begin{bmatrix} 1 & i \\ -i & 2 \end{bmatrix}$, $A^*P + PA = \begin{bmatrix} -2 & -2i \\ 2i & -2 \end{bmatrix}$. Thus, (11), (12), and (13) hold.

Suppose that $z = \operatorname{col}(z_1, z_2) \in \mathscr{S}_2$. Writing the equalities from the definition of the set $\mathscr{S}_2$ at $\ell = 0$, $s = 0$ and at $\ell = 1$, $s = 1$, we get:

$$(\ell = 0, s = 0): \; z^* \begin{bmatrix} 2 & 2+4i \\ 2-4i & 8 \end{bmatrix} z + i(\bar{z}_2 - z_2) = 0, \quad (41)$$

$$(\ell = 1, s = 1): \; z^* \begin{bmatrix} 0 & -2-2i \\ -2+2i & -4 \end{bmatrix} z + i(z_2 - \bar{z}_2) = 0. \quad (42)$$

By adding (41) and (42), we get

$$z^* \begin{bmatrix} 2 & 2i \\ -2i & 4 \end{bmatrix} z = 0. \quad (43)$$

Since the matrix of quadratic form in (43) is positive definite, it follows from (43) that $z = 0$. Hence, $\mathscr{S}_2 = \{0\}$. Thus, $\mathscr{E}_4 = \{0\}$. So, the conditions of Theorem 4 are fulfilled. Let us construct the feedback control function (22). We obtain

$$\begin{aligned} \widehat{u}(z) &= -\left[z^*(B^*P + PB)z + (L^*Pz + z^*PL)\right] \\ &= -\left[2|z_1|^2 + (2+4i)\bar{z}_1 z_2 + (2-4i) z_1 \bar{z}_2 + 8|z_2|^2 \right. \\ &\quad \left. + i(\bar{z}_2 - z_2)\right] \\ &= -\left[2(x_1^2 + y_1^2) + 2(2x_1x_2 + 4y_1x_2 - 4x_1y_2 + 2y_1y_2) \right. \\ &\quad \left. + 8(x_2^2 + y_2^2) + 2y_2\right]. \end{aligned} \quad (44)$$

By Theorem 4, feedback control $u = \widehat{u}(z)$, (44) globally asymptotically stabilizes the origin of system (1), (40).

VI. Conclusion

We have obtained sufficient conditions for global asymptotic stabilization of bilinear non-homogeneous complex-valued systems generalizing similar conditions for real-valued systems. For real systems, these conditions were obtained earlier using the Barbashin–Krasovsky theorem. For complex-valued systems, this theorem cannot be applied in the general case, since the derivative of the Lyapunov function along a complex-valued vector field may not take real values. By passing to the equivalent real system of dimension $2n$, these difficulties were overcome. The key properties for being able to do this were: the linearity of the free dynamic system; necessity and sufficiency in Lemma 7 (if the free dynamic system is not linear, then condition like 2, sufficient for 1, in Lemma 7, is no longer necessary); the choice of control u in the system (1) is real. Actually, these properties ensure that the derivative of the Lyapunov function is real and allow the Barbashin–Krasovsky theorem to work for complex-valued systems. For nonlinear affine complex systems, these difficulties have not yet been overcome. We plan to overcome them in future works. These results can be used to study the stability and synchronization of complex-valued dynamical networks.

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