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Tube-based Control Barrier Function with Integral Quadratic Constraints for Unknown Input Delay

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Abstract— **This paper proposes a Control Barrier Function (CBF)-based controller design to achieve safety for systems subjecting to unknown input delay and additive disturbance. Integral quadratic constraints characterizing the input-output behavior of the unmodeled dynamics caused by the unknown input delay are used to generate a bound of the error between the nominal model and the true uncertain system. The bound is incorporated into a tube-based CBF formulation to ensure robust system safety. The proposed method guarantees that the constraints are input affine, so the safe controller can be implemented by solving a quadratic programming problem in real-time. A simple example demonstrates the effectiveness of the tube-based CBF approach.**

I. INTRODUCTION

Control Barrier Functions (CBFs) combined controller is a popular approach that can guarantee stability and safety at the same time [1], [2]. CBF can be used as a constraint in a Quadratic Programming (QP) problem to filter the control actions for safety specifications. However, unmodeled input dynamics can cause safety violations and raise robustness concerns, leading to the necessity for a CBF design robust to unmodeled dynamics at the plant input [3]. For safety guarantee of uncertain systems, input-to-state CBF [4] and robust CBFs [5], [6], [7] have been proposed, which guarantee safety in the presence of $\mathcal{L}_\infty$ bounded disturbances or stochastic disturbances. In [8], a CBF design method is proposed to deal with known time delays. Furthermore, [3] presents an intuitive method to design CBFs for systems with unknown but bounded time delays, using Integral Quadratic Constraints (IQCs) to capture the effects of the unmodeled input dynamics. However, the IQC-CBF method in [3] introduces nonlinearity when formulating CBF-based constraints, leading to higher complexity in real-world applications. Also, it is reported that a Lagrangian multiplier λ as a tuning parameter in the IQC-CBF design affects the performance [3]. We found that with an improperly chosen λ, a discretized IQC-CBF controller shows aggressive behavior, which may violate the safety constraint even more than a nominal CBF.

In this paper, inspired by [3], we use IQC to describe the unmodeled dynamics caused by unknown input delay. Instead of directly combining the IQC with the CBF formulation and considering the worst case of the uncertainty [3], we first use the IQC to build a scalar system to bound the model error between the nominal model and the input-delayed system. A tube described by such a scalar system is then combined with CBF to ensure safety in the presence of unknown input delays. A similar formulation for the IQC-based tube has been used in [9], [10] to design robust MPC. In this paper, to guarantee that the constraints stay affine on the control input, a point-wise optimization combined with CBF is considered instead of an optimization with a receding horizon. Although distinct formulated discrete-time CBFs are proposed [11], [12], it is known that the discrete-time CBF is no longer affine in control inputs and only enforces constraints at discrete-time steps [13]. Thus, extended continuous-time CBF is needed to ensure a sufficient affine condition to guarantee safety with discrete-time nominal controllers. In this context, in this paper, we introduce tube-based CBF to guarantee a linear constraint where discretization error and uncertainty caused by unknown input delay are considered. The approach is demonstrated with a simple example and shows its efficiency with reduced complexity. However, the current paper proposes a method that only works for linear systems, while the IQC-CBF in [3] works equally for nonlinear affine systems.

Notation: A continuous function $\alpha : [0, a) \to [0, \infty)$ for some $a > 0$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. We denote the Euclidean norm as $\|x\| = \sqrt{x^T x}$ whereas the Euclidean norm w.r.t. $P = P^T \succ 0$ is denoted by $\|x\|_P^2 = x^T P x$. We also denote the set of exponentially stable systems with input dimension m and output dimension n as $\mathbb{RH}_\infty^{n\times m}$. The set of sequences x in $\mathbb{R}^n$ is denoted by $l_{2e}^n = \{(x_k)_{k\in\mathbb{N}} | x_k \in \mathbb{R}^n\}$. The Pontryagin set difference is defined by $X \ominus Y := \{z \in \mathbb{R}^n : z + y \in X, \forall y \in Y\}$.

II. PRELIMINARIES OF CONTROL BARRIER FUNCTION

Firstly, we consider a general case of a dynamic system with affine control inputs in a continuous-time domain:

$$\dot{x} = f(x) + g(x)u \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ is the system state, $u \in \mathcal{U} \subset \mathbb{R}^{n_u}$ is the control input, with $\mathcal{U}$ as a set of feasible control inputs, and $f : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ and $g : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x \times n_u}$ are locally Lipschitz functions. We consider a set $\mathcal{C}$ defined as the superlevel set of a continuously differentiable function

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$h : \mathbb{R}^{n_x} \rightarrow \mathbb{R}$, yielding

$$\begin{aligned} \mathcal{C} &= \{x \in \mathbb{R}^{n_x} : h(x) \geq 0\}, \\ \partial\mathcal{C} &= \{x \in \mathbb{R}^{n_x} : h(x) = 0\}, \\ \text{Int}(\mathcal{C}) &= \{x \in \mathbb{R}^{n_x} : h(x) > 0\}. \end{aligned} \tag{2}$$

We denote $\mathcal{T}_{x_0} := [0, T_{\max})$ the maximum time interval for which (1) has a unique solution starting from $x(0) = x_0$. Referring $\mathcal{C}$ as the safe set, our control objective is to maintain the safety of the system (1), i.e., with $x(0) \in \mathcal{C}$, $x(t) \in \mathcal{C}$ for all $t \in \mathcal{T}_{x_0}$. Control Barrier Function (CBF) is one method to design a controller for such control objective, which ensures the system state remains in the safe set $\mathcal{C}$. This paper considers a general condition where the function h has a relative degree $r \geq 1$ w.r.t. the system (1). High-order CBFs were introduced in [14], [15] to derive necessary conditions for guaranteeing set invariance. We define the series of functions $\psi_i : \mathbb{R}^{n_x} \rightarrow \mathbb{R}, i \in \{1, \cdots, r\}$ and the corresponding sets $\mathcal{C}_i, i \in \{1, \cdots, r\}$ as follows:

$$\begin{aligned} \psi_i(x) &= \dot{\psi}_{i-1}(x) + \alpha_r(\psi_{i-1}(x)), \\ \mathcal{C}_i &= \{x \in \mathbb{R}^{n_x} : \psi_{i-1}(x) \geq 0\}, \end{aligned} \tag{3}$$

where $\alpha_i(\cdot), i \in \{1, \cdots, r\}$ are class $\mathcal{K}$ functions of their arguments, and $\psi_0(x) = h(x)$.

Definition 1: [15] A function $h : \mathbb{R}^{n_x} \rightarrow \mathbb{R}$ with a relative degree r is a high-order CBF for system (1) if there exist differentiable class $\mathcal{K}$ functions $\alpha_i, i \in \{1, \cdots, r\}$ such that for all $x \in \mathcal{C}_1 \cap \mathcal{C}_2 \cap \cdots \cap \mathcal{C}_r$:

$$\begin{aligned} \psi_r(x) = L_f^r h(x) + L_g L_f^{r-1} h(x) u + \mathcal{O}(h(x)) \\ + \alpha_r(\psi_{r-1}(x)) \geq 0 \end{aligned} \tag{4}$$

where $\mathcal{O}(.)$ denotes the remaining Lie derivatives along f with degree less than or equal to $r - 1$.

Given h and $\psi_i, i \in \{1, \cdots, r\}$, we define the set of control that satisfies the high-order CBF condition as

$$\begin{aligned} \mathcal{U}_{\text{cbf}}(x) := \{u \in \mathcal{U} : & L_f^r h(x) + L_g L_f^{r-1} h(x) u \\ & + \mathcal{O}(h(x)) + \alpha_r(\psi_{r-1}(x)) \geq 0\}. \end{aligned} \tag{5}$$

Theorem 1: [14], [15] Assume $h : \mathbb{R}^{n_x} \rightarrow \mathbb{R}$ has relative degree r w.r.t. the dynamics (1), and satisfies (4) for some $\alpha_i, i \in \{1, \cdots, r\}$. Then any continuous controller $k_{\text{safe}} : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_u}$ with $k_{\text{safe}} \in \mathcal{U}_{\text{cbf}}(x), \forall x \in \mathbb{R}^{n_x}$ renders the system safe, i.e., $x(0) \in \mathcal{C}_1 \cap \mathcal{C}_2 \cap ... \cap \mathcal{C}_r$ implies $x(t) \in \mathcal{C}$ for all $t \geq 0$.

Proof: Please refer to [14], [15] for details. ■

Then to generate a controller $k_{\text{safe}}(x)$ which ensures the closed-loop state remains in the safe set $\mathcal{C}$, the following optimization can be formulated as a Quadratic Programming (QP) problem and solved in real-time:

$$k_{\text{safe}}(x) := \arg \min_{u \in \mathcal{U}_{\text{cbf}}} \frac{1}{2} \left\| u - u_{\text{ref}} \right\|^2, \tag{6}$$

with some reference controller u_{ref}. The constraint (4) is affine in the control input u, which can be embedded into a QP problem. However, the CBF condition (4) assumes the control signal u is applied in continuous time, while controllers operate at discrete time instants in practice. Although distinct formulated discrete-time CBFs are proposed [11], [12], the discrete-time CBF is no longer affine in control inputs and only enforces constraints at discrete-time steps [13]. Thus, we use extended continuous-time CBFs accounting for discretization error to provide a sufficient affine condition to guarantee safety with discrete-time nominal controllers. Furthermore, the uncertainty introduced by the unknown input delay is considered such that the final safe controller is robust to unmodeled input dynamics.

III. PROBLEM STATEMENT

In this paper, we consider an uncertain discrete-time linear time-invariant system:

$$\begin{aligned} x_{k+1} &= \Phi x_k + d_k + \Gamma(u_k + w_k), \\ w_k &: = \Delta(u_k) \end{aligned} \tag{7}$$

where $x \in \mathbb{R}^{n_x}$ is the system state, $u \in \mathbb{R}^{n_u}$ is the control input with $\Phi \in \mathbb{R}^{n_x \times n_x}$ and $\Gamma \in \mathbb{R}^{n_x \times n_u}$. $d \in \mathbb{R}^{n_d}$ is the model error caused by discretization with $\|d\| \leq d_{\max}$, Δ accounts for the effects of the uncertain input delay, and $w \in \mathbb{R}^{n_w}$ with $n_w = n_u$ denotes the uncertainty representing the deviation from the nominal behavior due to uncertain time delay, which depends on the control signal through the dynamics of Δ.

The control objective is to find a control law u_k that ensures the system (7) remains safe for the disturbance d and the uncertainty w described by the dynamics of Δ, i.e., to find a u_k so that $x_0 \in \mathcal{C}$ implies $x_k \in \mathcal{C}$ for all $kT_s \in (0, \mathcal{T}_{x_0}]$, with some non-negative real number T_s as the sampling time. For safety insurance, the high-order CBF is considered. For the controller design, we consider an uncertainty-free system model, i.e., the nominal model in the discrete-time domain:

$$\xi_{k+1} = \Phi \xi_k + \Gamma \hat{u}_k, \tag{8}$$

with the nominal state ξ_k, and the nominal input $\hat{u}_k$. Inspired by the idea in [13] where the safety for the nominal state ξ_k is considered instead of the safety of the actual state x_k, we propose a tube-based CBF with the affine condition on $\hat{u}_k$ to guarantee the safety of ξ_k and x_k at the same time. Similar to tube MPC, we use an auxiliary controller to regulate the error between the nominal and actual states. With the true system in (7), and the nominal system in (8), We denote the error between the nominal state and the true state as

$$e_k \triangleq x_k - \xi_k. \tag{9}$$

The control input $\hat{u}_k$ is combined with a feedback control law for the error e_k to make sure the error will not diverge,

$$u_k = \hat{u}_k + K e_k, \tag{10}$$

where K is a fixed feedback control gain. The error e_k satisfies the dynamics

$$e_{k+1} = \Phi_K e_k + \Gamma w_k + d_k \tag{11}$$

with $\Phi_K \triangleq \Phi + \Gamma K$. The following sections show how to find a bound for the error e_k. With such a bound, a tube-based CBF is designed to guarantee the safety of the true state x_k.

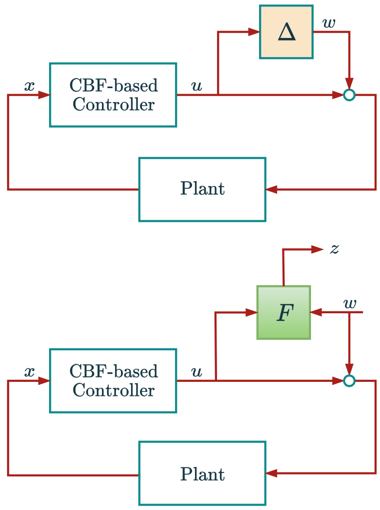


Fig. 1. IQC characterization allows us to replace Δ with the dynamics F and to consider the uncertainty output w as external input that satisfies the constraint (12).

IV. Tube-based Control Barrier Function

A. Integral Quadratic Constraints for Uncertain Input Delay

For the unmodeled input dynamics, here we aim to apply Integral Quadratic Constraints (IQCs) to bound the effect of the uncertain input delay. The idea is to replace the uncertainty with quadratic constraints on its inputs and outputs [16]. Firstly, a special case of ρ-hard IQC is considered for discrete-time systems.

Definition 2: [16] Let $\rho \in (0, 1]$, and $F(s) \in \mathbb{RH}_{\infty}^{n_z \times n_u}$. A bounded operator $\Delta : l_{2e}^{n_u} \rightarrow l_{2e}^{n_w}$ is said to satisfy the ρ-hard IQC defined by F if for all $K \geq 1$ and $w = \Delta(u)$, the following inequality holds:

$$\sum_{k=0}^{K-1} \rho^{-2k}(z_k^T z_k - \omega_k^T \omega_k) \geq 0, \tag{12}$$

where z is the output of F started from zero initial condition and driven by u.

With Definition 2, the notation $\Delta \in IQC(F, \rho)$ indicates that Δ satisfies the ρ-hard IQC defined by F, which is a constraint on the input/output pairs of Δ to make sure that the output of Δ has less energy than the input. The dynamics in F can be used to bound the effect of the uncertainty as a function of frequency. For an unknown input delay, we let D_τ denote a delay of τT_s seconds with $\tau \geq 0$, the actual plant input would be

$$D_\tau(u(kT_s)) := u(kT_s - \tau T_s). \tag{13}$$

The system uncertainty w is then given as

$$w(kT_s) := u(kT_s - \tau T_s) - u(kT_s) = \Delta(u(kT_s)), \tag{14}$$

with $\Delta := D_\tau - 1$.

For a delay τT_s, a ρ-hard IQC can be derived using frequency-domain relations. To find such ρ-hard IQC, firstly, let us define

$$\begin{aligned} \tilde{w}(kT_s) &:= e^{\alpha k T_s} w(kT_s), \\ \tilde{u}(kT_s) &:= e^{\alpha k T_s} u(kT_s), \\ \tilde{z}(kT_s) &:= e^{\alpha k T_s} z(kT_s). \end{aligned} \tag{15}$$

Multiplication by $e^{\alpha k T_s}$ in the time domain causes signals shifted in the frequency domain:

$$\tilde{W}(s) = W(s - \alpha). \tag{16}$$

Then by defining $\tilde{\Delta}(s) = \Delta(s - \alpha)$ and $\tilde{F}(s) = F(s - \alpha)$, the following relationship can be found satisfied

$$\tilde{W}(s) = \tilde{\Delta}(s)\tilde{U}(s), \tilde{Z}(s) = \tilde{F}(s)\tilde{U}(s), \tag{17}$$

The shifted filter $\tilde{F}(s)$ can be constructed to bound the frequency response of $\tilde{\Delta}(s)$, which can be formulated using Parseval's theorem [17] in the frequency domain:

$$\int_{\infty}^{\infty} (|\tilde{F}(j\tilde{\omega})|^2 - |\tilde{\Delta}(j\tilde{\omega})|^2) \cdot |\tilde{U}(j\tilde{\omega})|^2 d\tilde{\omega} \geq 0. \tag{18}$$

We aim to find a $\tilde{F}(s)$ such that $|\tilde{F}(j\tilde{\omega})| \geq |\tilde{\Delta}(j\tilde{\omega})|$ satisfies for all $\tilde{\omega}$, which can be realized by applying convex optimization. Given upper bound of τ such that $\tau \leq \tau_{\max}$, a stable, minimum-phase $\tilde{F}(s)$ can be constructed to bound the frequency responses of $\Delta(j\tilde{\omega})$ generated for many possible delay values with the range $\tau \in [0, \tau_{\max}]$. After constructing the shifted filter $\tilde{F}(s)$, the filter for the ρ-hard IQC is then obtained by shifting back:

$$F(s) = \tilde{F}(s + \alpha) \tag{19}$$

The state-space representation of $F(s)$ can be found as:

$$\begin{aligned} \dot{x}_F &= A_F x_F + B_F u, \\ z &= C_F x_F + D_F u, \end{aligned} \tag{20}$$

with $x_F(0) = 0$, such that the following relationship is satisfied:

$$\int_0^T e^{\alpha t}(z(t)^T z(t) - w(t)^T w(t))dt \geq 0. \tag{21}$$

With Euler discretization, the discrete-time system of the filter can be found as

$$\begin{aligned} x_{F\,k+1} &= \Phi_F x_{F\,k} + \Gamma_F u_k, \\ z_k &= C_F x_{F\,k} + D_F u_k, \end{aligned} \tag{22}$$

which satisfies the ρ-hard IQC in (12) with $\rho = e^{-\alpha T_s}$. This filter dynamics will be later utilized to bound the effect of the unknown input delay and incorporated into the tube formulation so that safety is guaranteed even with uncertainty via the tube-based CBF condition.

B. Error Bound System

In this sub-section, we show the method to find the bound for e_k when given the auxiliary controller in (10) with the following theorem, which will be used for the tube-based CBF design later. A Linear Matrix Inequality (LMI) is solved to help find the error bound with the IQC-described unknown input delay where the discretization error as an additive disturbance is considered. Notice a similar formulation for tube-based MPC is proposed in [9], [10]. In this paper, we modify the formulation, especially for point-wise optimization in the case of uncertain input delay and discretization error.

Theorem 2: If there exists $\rho \in (0,1]$, $P \succ 0$, $\tau > 0$, $\gamma > 0$, such that the following LMI

$$\begin{bmatrix} A^T PA - \rho^2 P & A^T PB_w & A^T PB_d & A^T PB_u \\ B_w^T PA & B_w^T PB_w & B_w^T PB_d & B_w^T PB_u \\ B_d^T PA & B_d^T PB_w & B_d^T PB_d & B_d^T PB_u \\ B_u^T PA & B_u^T PB_w & B_u^T PB_d & B_u^T PB_u \end{bmatrix} + C_s^T MC_s \prec \gamma \begin{bmatrix} \mathbf{0}_{(n_x+n_w)\times(n_x+n_w)} & \mathbf{0}_{(n_x+n_w)\times(n_d+n_u)} \\ \mathbf{0}_{(n_d+n_u)\times(n_x+n_w)} & I_{(n_d+n_u)\times(n_d+n_u)} \end{bmatrix}$$
$$\text{with } M \succ \alpha \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{23}$$

is feasible with

$$A = \begin{bmatrix} \Phi_K & 0 \\ B_F K & A_F \end{bmatrix}, B = \begin{bmatrix} \Gamma \\ 0 \end{bmatrix}, B_d = \begin{bmatrix} I \\ 0 \end{bmatrix}, B_u = \begin{bmatrix} 0 \\ B_F \end{bmatrix},$$
$$C_s = \begin{bmatrix} D_F K & C_F & 0 & 0 & D_F \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Then, for all $k \geq 0$, it holds that

$$\left\| \begin{matrix} e_k \\ x_{Fk} \end{matrix} \right\|_P^2 \leq \sigma_k \tag{24}$$

where

$$\begin{aligned} \sigma_0 &= \left\| \begin{matrix} e_0 \\ x_{F0} \end{matrix} \right\|_P^2 \\ \sigma_{k+1} &= \rho^2 \sigma_k + \|\hat{u}_k\|^2 + \|d_k\|^2 \end{aligned} \tag{25}$$

Proof: z_k defined in (22) can be rewritten as

$$z_k = C_F x_{Fk} + D_F K e_k + D_F \hat{u}_k. \tag{26}$$

Suppose that

$$\sum_{\kappa=0}^{k-1} \rho^{-2\kappa} \begin{bmatrix} z_\kappa^T & w_\kappa^T \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} z_\kappa \\ w_\kappa \end{bmatrix} \geq 0. \tag{27}$$

Then

$$\sum_{\kappa=0}^{k-1} \rho^{-2\kappa} \left\| \begin{bmatrix} z_\kappa \\ w_\kappa \end{bmatrix} \right\|_M^2 \geq 0 \tag{28}$$

is satisfied with some $\alpha > 0$,

$$M \succ \alpha \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{29}$$

We define $s_\kappa = \begin{bmatrix} e_\kappa^T & x_{F\kappa}^T & w_\kappa^T & d_\kappa^T & \hat{u}_\kappa^T \end{bmatrix}^T$ such that

$$\begin{bmatrix} z_\kappa \\ w_\kappa \end{bmatrix} = C_s s_\kappa, \tag{30}$$

with

$$C_s = \begin{bmatrix} D_F K & C_F & 0 & 0 & D_F \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Augmented state $\begin{bmatrix} e_\kappa^T & x_{F\kappa}^T \end{bmatrix}^T$ produces the dynamics

$$\begin{aligned} \begin{bmatrix} e_{\kappa+1} \\ x_{F\kappa+1} \end{bmatrix} &= \underbrace{\begin{bmatrix} \Phi_K & 0 \\ B_F K & A_F \end{bmatrix}}_{A} \begin{bmatrix} e_\kappa \\ x_{F\kappa} \end{bmatrix} \\ &+ \underbrace{\begin{bmatrix} \Gamma \\ 0 \end{bmatrix}}_{B_w} w_\kappa + \underbrace{\begin{bmatrix} I \\ 0 \end{bmatrix}}_{B_d} d_\kappa + \underbrace{\begin{bmatrix} 0 \\ B_F \end{bmatrix}}_{B_u} \hat{u}_\kappa \end{aligned} \tag{31}$$

Then with (30) and (31), multiplying s_k and its transpose on the right and left sides of (23) leads to the following inequality

$$\left\| \begin{bmatrix} e_{\kappa+1} \\ x_{F\kappa+1} \end{bmatrix} \right\|_P^2 - \rho^2 \left\| \begin{bmatrix} e_\kappa \\ x_{F\kappa} \end{bmatrix} \right\|_P^2 + \left\| \begin{bmatrix} z_\kappa \\ w_\kappa \end{bmatrix} \right\|_M^2 - \gamma \|d_\kappa\|^2 - \gamma \|\hat{u}_\kappa\|^2 \leq 0. \tag{32}$$

By letting

$$\sigma_{\kappa+1} = \rho^2 \sigma_\kappa + \|\hat{u}_\kappa\|^2 + \|d_\kappa\|^2, \tag{33}$$

considering $\kappa \in [0, k-1]$, the following inequalities can be obtained by multiplying (32) with $\rho^{2(k-\kappa-1)}$

$$\left(\left\| \begin{bmatrix} e_k \\ x_{Fk} \end{bmatrix} \right\|_P^2 - \sigma_k \right) + \left\| \begin{bmatrix} z_k \\ w_k \end{bmatrix} \right\|_M^2 - \rho^2 \left(\left\| \begin{bmatrix} e_{k-1} \\ x_{Fk-1} \end{bmatrix} \right\|_P^2 - \sigma_{k-1} \right) \leq 0,$$
$$\vdots$$
$$\rho^{2(k-1)} \left(\left\| \begin{bmatrix} e_1 \\ x_{F1} \end{bmatrix} \right\|_P^2 - \sigma_2 \right) + \rho^{2(k-1)} \left\| \begin{bmatrix} z_1 \\ w_1 \end{bmatrix} \right\|_M^2 - \rho^{2(k-1)} \rho^2 \left(\left\| \begin{bmatrix} e_0 \\ x_{F0} \end{bmatrix} \right\|_P^2 - \sigma_0 \right) \leq 0.$$

Summing up the inequalities above leads to

$$\left\| \begin{bmatrix} e_k \\ x_{Fk} \end{bmatrix} \right\|_P^2 - \sigma_k + \sum_{\kappa=1}^{k-1} \rho^{2(k-\kappa-1)} \left\| \begin{bmatrix} z_\kappa \\ w_\kappa \end{bmatrix} \right\|_M^2 - \rho^{2(k-1)} \left(\left\| \begin{bmatrix} e_0 \\ x_{F0} \end{bmatrix} \right\|_P^2 - \sigma_0 \right) \leq 0. \tag{34}$$

Then, we find that with (28), by choosing $\sigma_0 = \left\| \begin{matrix} e_0 \\ x_{F0} \end{matrix} \right\|_P$, there exists

$$\left\| \begin{matrix} e_k \\ x_{Fk} \end{matrix} \right\|_P^2 < \sigma_k. \tag{35}$$

■

Now with the theorem, we find the bound for e_k, i.e., the tube later used for the tube-based CBF design. Let P be decomposed into $P = \begin{bmatrix} P_{11} & P_{21}^T \\ P_{21} & P_{22} \end{bmatrix}$ with $P_{11} \in \mathbb{R}^{n_x \times n_x}$. Using the Schur complement for

$$P_{21}^T P_{22}^{-1} P_{21} - P_{21}^T P_{22}^{-1} P_{21} = 0 \succeq 0 \tag{36}$$

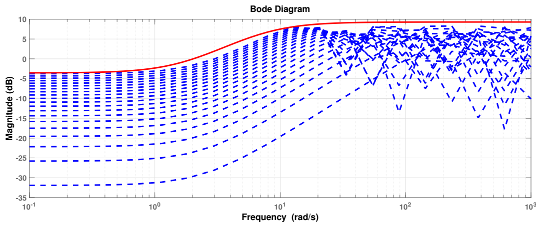


Fig. 2. Frequency responses of input delay $\tilde{\Delta}(s)$ with 20 samples of delay with evenly spaced τ restricted to $[0, \tau_{\max}]$, and a bound $\tilde{F}$.

and $P_{22} \succ 0$, we obtain $\begin{bmatrix} P_{21}^T P_{22}^{-1} P_{21} & P_{21}^T \\ P_{21} & P_{22} \end{bmatrix} \succeq 0$ and thus $\begin{bmatrix} P_e & 0 \\ 0 & 0 \end{bmatrix} = P - \begin{bmatrix} P_{21}^T P_{22}^{-1} P_{21} & P_{21}^T \\ P_{21} & P_{22} \end{bmatrix} \preceq P$, with $P_e = P_{11} - P_{21}^T P_{22}^{-1} P_{21}$. Finally, this yields

$$\|e_k\|_{P_e}^2 \leq \left\| \begin{matrix} e_k \\ x_{Fk} \end{matrix} \right\|_P^2 \leq \sigma_k \tag{37}$$

C. Tube-based CBF

With the bound σ_k, we define the set Ω_k at the sampling time k

$$\Omega_k = \{x \in \mathbb{R}^{n_x} : \|x - \xi_k\|_{P_e}^2 \leq \sigma_k\}. \tag{38}$$

Then with Ω_k, a reduced safe set $\mathcal{C}_k'$ at time k is defined as

$$\mathcal{C}_k' := \mathcal{C} \ominus \Omega_k. \tag{39}$$

With the reduced safe set $\mathcal{C}_k'$, we can find the corresponding function h_k' such that the following statement holds:

$$\mathcal{C}_k' = \{x \in \mathbb{R}^{n_x} : h_k'(x) \geq 0\}. \tag{40}$$

Then with $h_k'(\xi_k)$ and corresponding $\psi_i(\xi_k), i \in \{1, \cdots, r\}$, a set of control is defined as

$$\begin{aligned} \mathcal{U}_{\text{cbf}}'(x) := \{u \in \mathcal{U} : L_f^r h_k'(\xi_k) + L_g L_f^{r-1} h_k'(\xi_k) u_k \\ + \mathcal{O}(h_k'(\xi_k)) + \alpha_r(\psi_{r-1}(\xi_k)) \geq 0\}. \end{aligned} \tag{41}$$

Then a nominal controller $\hat{u}_k$ that renders the nominal state ξ_k stays in the reduced safe set $\mathcal{C}_k'$ such that $\xi_k \in \mathcal{C}_k'$ can be found by solving an optimization problem

$$\hat{u}_k^* = \arg \min_{\hat{u} \in \mathcal{U}_{\text{cbf}}'} \frac{1}{2} \|\hat{u}_k - u_{\text{ref}k}\|^2. \tag{42}$$

With (42), the safety of x_k such that $x_k \in \mathcal{C}$ can be provided by applying the feedback combined control law (10). Notice that with the exact value of ξ_k and σ_k known, (41) is a linear constraint affine on the nominal control input $\hat{u}$. With the affine tube-based CBF condition, the optimization (42) can be solved as a QP problem. The computation complexity is reduced compared to the result in [3], where the quadratic form of control input is introduced into the formulation of the CBF condition. Note that using σ_k defined in (25) requires the value of d_k at each sample time, which is impractical in real-world applications. Therefore, we introduce an upper bound $\bar{\sigma}_k$ of σ_k, in which $d_{\max}$ is utilized instead of d_k. The update law for $\bar{\sigma}_k$ is then

$$\bar{\sigma}_{k+1} = \rho^2 \bar{\sigma}_k + \gamma \|d_{\max}\|^2 + \gamma \|\hat{u}_k\|^2. \tag{43}$$

Each upper bound σ_k can be readily computed.

V. Example

To demonstrate the performance of the proposed algorithm, we performed simulations where the two-dimensional point mass dynamics is considered [3]

$$\dot{x}(t) = \begin{bmatrix} 0_2 & I_2 \\ 0_2 & 0_2 \end{bmatrix} x(t) + \begin{bmatrix} 0_2 \\ I_2 \end{bmatrix} u(t), x(t) = \begin{bmatrix} p(t) \\ \dot{p}(t) \end{bmatrix} \in \mathbb{R}^4, \tag{44}$$

with position $p \in \mathbb{R}^2$ and velocity $\dot{p} \in \mathbb{R}^2$. Baseline controller to track the reference position command $p_{\text{ref}k} \in \mathbb{R}^4$ is given:

$$u_{\text{ref}} = K_{\text{ref}} \cdot \left(\begin{bmatrix} p_{\text{ref}} \\ 0 \end{bmatrix} - \begin{bmatrix} p \\ \dot{p} \end{bmatrix} \right) \tag{45}$$

where K_{ref} is computed using linear quadratic regulator. Here we consider the obstacle of radius $r_o = 1.5$ at the position $p_o = \begin{bmatrix} 2 & 0.2 \end{bmatrix}^T$, the safe set $\mathcal{C}$ is then defined with (2) by the $h(x) := (p - p_o)^T (p - p_o) - r_o^2 \geq 0$. The sampling time is chosen as $T_s = 0.01$sec. We consider uncertain input delay with τ, where the true delay τ is unknown but restricted to $[0, \tau_{\max}]$ with $\tau_{\max} = 0.2$sec. With $\alpha = 5$, the shifted filter dynamics $\tilde{F}(s)$ is found via `fitmagfrd` in MATLAB as $\tilde{F}(s) := \frac{2.92s + 4.93}{s + 7.43}$, which satisfies $|\tilde{F}(j\tilde{\omega})| \geq |\tilde{\Delta}(j\tilde{\omega})|$ for every delay sample. Then by shifting back the frequency such that $F(s) = \tilde{F}(s + \alpha)$, the state-space representation of $F(s)$ in discrete-time domain is found as $(\Phi_F, \Gamma_F, C_F, D_F) = (0.9055, 0.0095, -16.72, 2.92)$ via the Euler method. With the dynamics of F, the ρ-hard IQC condition (12) is satisfied with $\rho = 0.95$. Fig. 2 shows the frequency responses for $\tilde{\Delta}(s) = \Delta(s - \alpha)$ with 20 values of delay evenly spaced between $(0, \tau_{\max}]$.

For the comparative study, we compare the proposed tube-based CBF method with the nominal CBF and the IQC-CBF in [3]. In the paper, with $I(z, w) := z^T z - w^T w$, the IQC-CBF condition is introduced by

$$\begin{aligned} \mathcal{U}_{\text{iqc-cbf}}(x) := \{u \in \mathcal{U} : L_f^r h(x) + L_g L_f^{r-1} h(x)(u + w) \\ + \mathcal{O}(h(x)) + \alpha_r(\psi_{r-1}(x)) - \lambda I(z, w) \geq 0\}. \end{aligned}$$

In the real-time implementation, the worst-case value of w is considered. Note that, thus, the Lagrangian multiplier λ for the combination of the IQC and CBF is a vital tuning parameter in the design of the IQC-CBF, which decides the performance of the IQC-CBF. For further details, readers are referred to [3]. It is reported that a heuristic choice of Lagrange multiplier and IQC filter F provides some robustness to unmodeled dynamics [3]. We choose λ for the IQC-CBF as $\lambda = 0.05, 0.1, 0.3, 0.5$. Firstly, with an unknown time delay, the trajectory generated by the nominal CBF-based controller has safety violations around the obstacle. The IQC-CBF-based controllers with $\lambda = 0.05$ and $\lambda = 0.1$ take more cautious paths around the obstacle than the nominal CBF-based controller. The performance, however, largely depends on the choice of the Lagrangian multiplier λ in the CBF design. It can be shown in Fig. 3 that the IQC-CBF controller with $\lambda = 0.05$ takes a much more conservative path around the vehicle than other values. In contrast, by choosing a larger multiplier $\lambda = 0.5$, which

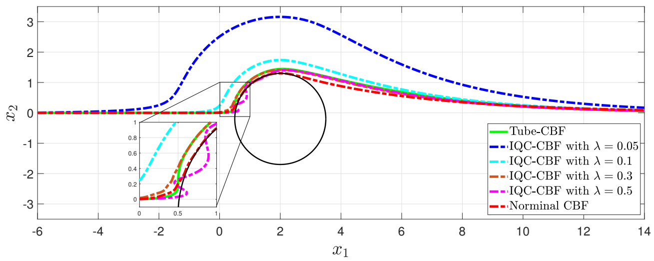
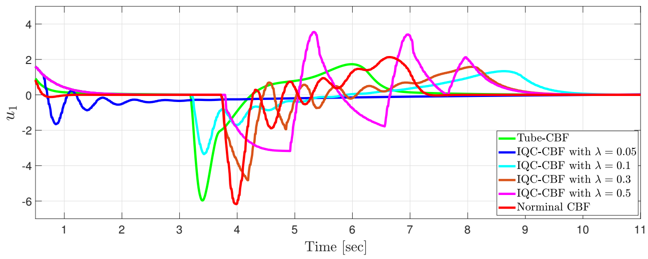
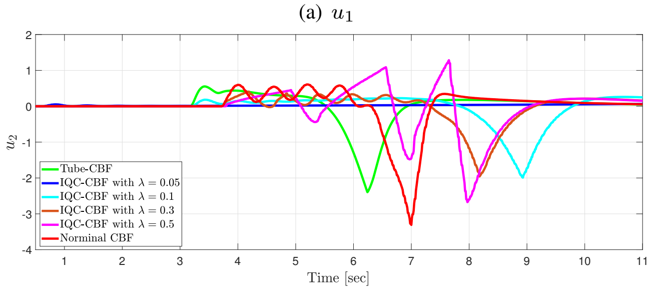


Fig. 3. Position for proposed tube-based CBF, IQC-CBF and nominal CBF on point mass with delay $\tau \leq 0.2$sec.



(a) u_1



(b) u_2

Fig. 4. Control inputs for proposed tube-based CBF, IQC-CBF and nominal CBF on point mass with delay $\tau \leq 0.2$sec.

is improper, i.e., not satisfying the robust CBF condition, it turns out that the discretized IQC-CBF controller shows aggressive behavior, which violates the safety constraint even more than the nominal CBF. The proposed tube-based CBF does not require the Lagrangian multipliers λ so that the true state is maintained in the safe set less conservatively without tuning the parameter. Also, Fig. 4 shows the control inputs generated with the proposed method, the IQC-CBF, and the nominal CBF. The unknown delays cause the inputs generated by the nominal CBF to oscillate when the nominal CBF is activated. It can be observed that the IQC-CBF still generates control inputs with unavoidable oscillations. At the same time, the proposed methods provide a smoother control performance because the disturbed state is no longer used to generate the barrier function. We consider the nominal state in the CBF design, while the safety of the true state can be guaranteed. We also found that by choosing a proper λ, the conservativeness of the IQC-CBF is reduced to the same level as the proposed controller. However, the control inputs generated by the IQC-CBF still suffer from more oscillations than the proposed method.

VI. Conclusions

This paper proposed a safety-ensuring algorithm based on CBF robust to unknown input delay and discretization error. We explored the IQCs to bound the input-output behavior of the unmodeled dynamics and generate an LMI-based scalar system that gives the error bound between the nominal and true system state, where discretization error is considered an external disturbance. With the bound, a tube-based CBF was proposed, which guaranteed control input affine constraints, and a QP problem with the CBF condition can be solved in real-time to ensure safety.

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