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Switched Lyapunov Function-Based Controller Synthesis for Networked Control Systems: A Computationally Inexpensive Approach

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Abstract— **This paper presents a Lyapunov function-based control strategy for networked control systems (NCS) affected by variable time delays and data loss. A special focus is put on the reduction of the computational complexity. The crucial step to achieving computational efficiency involves defining a specific buffering mechanism that introduces an additional delay not larger than one sampling period. This allows representing the buffered NCS as a switched system, thus leading to a significant simplification of the NCS model and subsequent controller synthesis. The novel approach does not only circumvent the need for any over-approximation technique, but also leads to a strongly decreased number of optimization variables and linear matrix inequalities, allowing hereby greater flexibility with respect to additional degrees of freedom affecting the transient behavior. The performance and computational efficiency of the control strategy are demonstrated in a simulation example.**

I. Introduction

The great potential of wireless networked control systems (NCS) characterized by controlling the plant over a (shared) communication channel together with significant developments of communication technologies have moved these systems to the center of attention within many engineering industries. This has been motivated by increasing complexity of modern control systems leading to the rapidly growing interest in characteristics the NCSs possess, such as great flexibility and adaptability regarding the system topology. Nevertheless, the main challenges which arise in NCS are communication constraints such as limited transmission speed and unreliability of the communication channel. The resulting random time delays and data loss have a great impact on the control design and lead to the need for new control strategies which take these into account. An overview of the extensive literature considering the modeling of the network imperfections, the stability properties and control of NCS can be found, e.g., in [1], [2] and references therein.

The Lyapunov stability theory and analysis of dynamical systems in terms of Linear Matrix Inequalities (LMIs) has been essential in the control systems community, since it was proposed in literature [3]. The possibility to express a control problem as a standard convex optimization problem involving LMIs which can efficiently be solved numerically has been recognized in the field of NCS, as well. A very powerful stability characterization based on parameter-dependent Lyapunov functions is proposed in [4], [5], which leads to a control law ensuring the stability of NCS including time-varying delays, packet dropouts and variations in the sampling interval. However, the controller synthesis for time-variant NCS models combining all relevant network uncertainties results in a very high complexity of this method. First, a suitable polytopic model must be derived using over-approximation techniques based on the Jordan Form, Taylor series, etc., see [6]. Second, the number of LMIs which need to be solved to compute a control law is very high and increases exponentially when, e.g., the maximum allowable delay is increased.

To circumvent the high computational complexity of the aforementioned approach, a buffering mechanism has been proposed in literature, see, e.g., [7], [8] which adds an additional delay so, that a constant (worst-case) delay is ensured. The great advantage of this approach is that the resulting NCS model becomes time-invariant which significantly simplifies the control design. Nevertheless, even though the delay introduced by the network might be short for some data packets, they are not forwarded to the plant until the maximal delay is reached, therefore leading often to unnecessarily long delays, which are in general undesirable.

In this paper, we propose an LMI-based control strategy suitable for NCS affected by variable time delays and packet dropouts motivated by [5] but with a special focus on the reduction of the complexity of the method. The crucial step toward the computational simplicity represents a specific buffering mechanism [9]. In addition to considerably reducing additional buffer delays in comparison to the worst-case scenario from [7], it allows the formulation of the resulting buffered NCS as a switched system as well, allowing the control design to benefit from an extensive literature on LMI-based control design, see e.g. [10], [11] and [12]. As a result, the main contributions of the proposed control strategy are:

1) The stability of the feedback loop with time-varying delays and data loss is ensured.

2) The need for over-approximation techniques, which can be particularly challenging for NCS with delays larger than a sampling time, is circumvented.

3) The number of optimization variables and LMIs is reduced, resulting in significantly shorter computation times.

4) The substantial reduction of the algorithms' complexity allows the introduction of the additional degrees of freedom, which can be used to improve the transient behavior.

Moreover, the proposed control laws are compared to the

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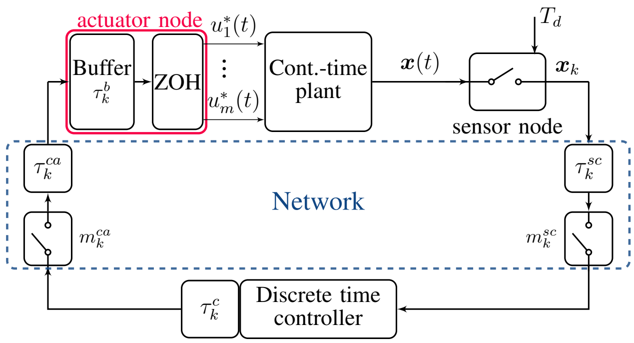

Fig. 1. Considered buffered networked control system

control law from [5] in simulations with respect to their performance and computational complexity.

II. Problem Formulation

In this work we consider systems described by a continuous-time multi-input linear time-invariant model

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_c\boldsymbol{x}(t) + \boldsymbol{B}_c\boldsymbol{u}^*(t) \tag{1}$$

with state vector $\boldsymbol{x}(t) \in \mathbb{R}^n$ and input $\boldsymbol{u}^*(t) \in \mathbb{R}^m$. The system to be controlled including sensor and actuator nodes at the plant side is connected over a wireless communication network to the controller side, where a discrete-time controller providing actuating signals for all m input channels is implemented, see Fig. 1. In the sensor node, the discrete time signal $\boldsymbol{x}_k$ is generated with a constant sampling period T_d. The k^{th}-data packet containing the sampled states $\boldsymbol{x}_k$ and a timestamp $t = kT_d$ ($k \in \mathbb{N}_0$) is sent over a network to the controller. Based on the received data the discrete-time control signal $\boldsymbol{u}_k$ is computed and forwarded over a network back to the plant side, where a zero-order hold block (ZOH) is used to generate the actuation signal $\boldsymbol{u}^*(t)$. The variable time needed for the execution of the control law is included by the variable τ_k^c. Furthermore, the network imperfections are modeled by variable network-induced sensor-to-controller delays τ_k^{sc} and controller-to-actuator delays τ_k^{ca}. In addition, the possibility of lossy networks is incorporated by variables m_k^{sc} and m_k^{ca} which denote whether the k^{th} data packet is lost ($m_k = 1$) or received ($m_k = 0$). For the NCS under consideration, the following assumptions hold:

Assumption 1: System (1) is controllable. The controllability is not lost due to sampling, i.e. the sampling time T_d is non-pathological in the sense of [13].

Assumption 2: The individual delays τ_k^{sc}, τ_k^{ca} and τ_k^c, $\forall k$ are assumed to be upper bounded (if no data is lost) by $\bar{\tau}^{sc}, \bar{\tau}^{ca}$ and $\bar{\tau}^c$, respectively. The upper bounds can be larger than T_d (large delay case). A specific distribution of the delays is not considered.

Assumption 3: The data loss in the network is modeled by defining the upper bound $\bar{p}$ for the number of subsequent packet dropouts. As the effect of packet dropouts on the implemented control updates remains the same regardless of where the packet was lost, the relation $\sum_{i=k-\bar{p}}^{k} \max\{m_i^{sc}, m_i^{ca}\} \leq \bar{p}$, $\forall k$ must hold. A specific distribution of the packet dropouts is not considered.

The delay τ_k, which denotes the complete time from the moment of sampling of $\boldsymbol{x}_k$ at $t = kT_d$ at the sensor node up to the moment when the control signal computed based on $\boldsymbol{x}_k$ is available at the actuator depends on the individual network delays, the NSC structure and the proposed strategy. In this work we first consider static controllers. Therefore, the delay introduced by the network for the k^{th} data packet is given as $\tau_k = \tau_k^{sc} + \tau_k^c + \tau_k^{ca}$, see [5], which is bounded due to Assumption 2, i.e. $\exists \bar{d} \in \mathbb{N}^+$ so that $\tau_k \leq \bar{d}T_d$ $\forall k$ holds. A parameter combining the considered imperfections of the network is defined as $\bar{\delta} = \bar{d} + \bar{p}$, which allows expressing the continuous-time actuation signal as $\boldsymbol{u}^*(t) = \boldsymbol{u}_{k+j-\bar{\delta}}$ for $kT_d + t_k^j \leq t < kT_d + t_k^{j+1}$ $(j = 0, 1, \ldots, \bar{\delta})$, see [5]. The parameters t_k^j denote time instants at which the corresponding $\boldsymbol{u}_{k+j-\bar{\delta}}$ become available at the actuator, as defined in [4], [5]. The discretization of (1) with a constant sampling time T_d leads to the discrete-time model

$$\boldsymbol{x}_{k+1} = \boldsymbol{A}_d\boldsymbol{x}_k + \sum_{j=0}^{\bar{\delta}} \boldsymbol{B}_j(t_k^j, t_k^{j+1})\boldsymbol{u}_{k+j-\bar{\delta}} \tag{2}$$

with $\boldsymbol{B}_j(t_k^j, t_k^{j+1}) = \int_{t_k^j}^{t_k^{j+1}} e^{\boldsymbol{A}_c(T_d - s)}\boldsymbol{B}_c ds$ and $\boldsymbol{A}_d = e^{\boldsymbol{A}_c T_d}$. From the complexity of the time-variant discrete-time model (2), which arises as a consequence of purely random network effects, it is clear that the conventional control approaches cannot be directly applied. Different over-approximation techniques have been proposed in the literature to deal with uncertainties appearing in an exponential form by embedding (2) in a larger but structurally simpler linear parameter varying polytopic model [6]. However, these methods increase the complexity of the approach. The aim of this paper is to define a control strategy suitable for the NCS affected by random network imperfections with a special focus on the practicability of the method and reduction of the computational complexity.

III. Proposed Approach

In the following, a switched Lyapunov function based control strategy is presented. Firstly, in order to simplify the discrete time-model (2) and avoid the necessity for the convex over-approximation and a large number of LMIs in the subsequent controller design, a buffering mechanism is defined. Based on this simplification, an LMI-based control law is proposed, with the focus on the practical usability in terms of computational simplicity and flexibility.

A. Buffering Mechanism

The buffer is implemented as a part of the actuator node at the receiving end of the feedback channel at the plant side, see Fig. 1, and is assumed to be synchronized with the sensor node. Upon arriving at the plant side, the control signal is first sent to the message rejection mechanism implemented in the buffer, which is necessary to deal with a possible data packet disorder. It ensures that older control data is discarded and only the most recent control data is forwarded to the buffering mechanism. The buffer then introduces an additional delay τ_k^b before forwarding $\boldsymbol{u}_k$ to the ZOH. The resulting round trip time (RTT) from the moment of sampling of $\boldsymbol{x}_k$ up to the moment of applying the corresponding control signal is therefore given by $\tau_k = \tau_k^{sc} + \tau_k^c + \tau_k^{ca} + \tau_k^b$. The delay τ_k^b is calculated based on the timestamp attached by the sensor, so that $\tau_k = \left\lceil \frac{\tau_k^{sc}+\tau_k^c+\tau_k^{ca}}{T_d} \right\rceil \cdot T_d = q_k T_d$ holds, where $\lceil h \rceil$ denotes the function calculating the smallest integer greater than or equal to h. The buffer delay is hereby given as $\tau_k^b = q_k T_d - (\tau_k^{sc} + \tau_k^c + \tau_k^{ca}) < T_d$. This ensures that the control signal is updated at $t = kT_d$ and that there is only one signal acting on the plant during one sampling period. In the case when more than one control signals arrive during one sampling period, the most recently received is applied, whereas in the case when no new signal is received, the previously applied $\boldsymbol{u}_k$ is kept active. Even though it is still unknown when a specific $\boldsymbol{u}_k$ will be applied to the system, the minimal time duration for which an applied control input is active is defined explicitly. The NCS can be therefore described by a simplified model

$$\boldsymbol{x}_{k+1} = \boldsymbol{A}_d \boldsymbol{x}_k + \boldsymbol{B}_d \boldsymbol{u}_{k-q_k} \tag{3}$$

with $\boldsymbol{B}_d = \int_0^{T_d} e^{\boldsymbol{A}_c(T_d - s)} \boldsymbol{B}_c ds$ and $q_k \in \{1, 2, \ldots, \bar{\delta}\}$ being the unknown input delay which depends on the network imperfections, see Assumptions 2 and 3.

In (2), infinitely many different input matrices $\boldsymbol{B}_j(t_k^j, t_k^{j+1})$ caused by unknown time-varying delays and packet dropouts can occur. In contrast, model (3) incorporates only a finite set of model variations as a consequence of a limited number of resulting RTTs achieved by the proposed buffering mechanism (i.e. $\tau_k \in \{T_d, 2T_d, \ldots, \bar{\delta} T_d\}$, although τ_k^{sc}, τ_k^c and τ_k^{ca} arbitrarily vary in given intervals). Due to this fact that there is only one control signal acting between two sampling instants, the buffered NCS can be written as a switched system. This is achieved by defining the lifted-state vector

$$\boldsymbol{\xi}_k = \begin{bmatrix} \boldsymbol{x}_k^T & \boldsymbol{u}_{k-1}^T & \boldsymbol{u}_{k-2}^T & \ldots & \boldsymbol{u}_{k-\bar{\delta}}^T \end{bmatrix}^T, \tag{4}$$

which results in the equivalent delay-free lifted model

$$\boldsymbol{\xi}_{k+1} = \hat{\boldsymbol{A}}(\boldsymbol{\alpha}_k) \boldsymbol{\xi}_k + \hat{\boldsymbol{B}} \boldsymbol{u}_k \quad \text{with} \tag{5}$$

$$\hat{\boldsymbol{A}}(\boldsymbol{\alpha}_k) = \begin{bmatrix} \boldsymbol{A_d} & \alpha_{1,k}\boldsymbol{B}_d \ \ldots \ \alpha_{\bar{\delta}-1,k}\boldsymbol{B}_d & \alpha_{\bar{\delta},k}\boldsymbol{B}_d \\ \boldsymbol{0}_{m\times n} & \boldsymbol{0}_m \ \ldots \ \boldsymbol{0}_m & \boldsymbol{0}_m \\ \boldsymbol{0}_{(\bar{\delta}-1)m\times n} & \boldsymbol{I}_{(\bar{\delta}-1)m} & \boldsymbol{0}_{(\bar{\delta}-1)m\times m} \end{bmatrix}, \tag{6}$$

$\hat{\boldsymbol{B}} = \begin{bmatrix} \boldsymbol{0}_{m\times n} & \boldsymbol{I}_{m\times m} & \boldsymbol{0}_{m\times(\bar{\delta}-1)m} \end{bmatrix}^T$ with $\boldsymbol{\alpha}_k = [\alpha_{1,k} \ \alpha_{2,k} \ \ldots \ \alpha_{\bar{\delta},k}]^T$ and $\alpha_{i,k} = 1$ if $\boldsymbol{u}_{k-i}$ is active for $kT_d \leq t < (k+1)T_d$ and $\alpha_{i,k} = 0$ otherwise, while $\sum_{i=1}^{\bar{\delta}} \alpha_{i,k} = 1$ holds. Therefore, when the proposed specific buffering mechanism is used, the network uncertainties are completely embedded in $\boldsymbol{\alpha}_k$, which is not known *a priori*, since it depends on the unknown network effects. As a result, even though the NCS does not directly involve switching, $\boldsymbol{\alpha}_k$ can be interpreted as a switching variable for the resulting switched system given as

$$\boldsymbol{\xi}_{k+1} = \sum_{i=1}^{\bar{\delta}} \alpha_{i,k} \hat{\boldsymbol{A}}_i \boldsymbol{\xi}_k + \hat{\boldsymbol{B}} \boldsymbol{u}_k. \tag{7}$$

The matrices $\hat{\boldsymbol{A}}_i$ in (7) can be determined by evaluating (6) for $\boldsymbol{\alpha}_k = \boldsymbol{e}_i$ with $\boldsymbol{e}_i$ being the vector whose components are all zero, except the i^{th} element which equals 1. Therefore, the usually complex over-approximation used to obtain the polytopic model for (2) suitable for stability analysis and control synthesis [4], [5] is no longer necessary. The infinite set of allowable values of the dynamic matrix is hereby reduced to a finite set containing only $\bar{\delta}$ elements.

B. LMI based Controller for Switched NCS

In this section we propose a static state feedback control law of the form

$$\boldsymbol{u}_k = -\boldsymbol{K}_x \boldsymbol{x}_k = -\hat{\boldsymbol{K}} \boldsymbol{\xi}_k = -\begin{bmatrix} \boldsymbol{K}_x & \boldsymbol{0} \end{bmatrix} \boldsymbol{\xi}_k, \tag{8}$$

which ensures asymptotic stability of the switched hybrid system (7) leading to the closed loop system

$$\boldsymbol{\xi}_{k+1} = \sum_{i=1}^{\bar{\delta}} \left(\alpha_{i,k} \hat{\boldsymbol{A}}_i - \hat{\boldsymbol{B}} \hat{\boldsymbol{K}} \right) \boldsymbol{\xi}_k. \tag{9}$$

Defining (8) solely as a linear combination of $\boldsymbol{x}_k$ avoids more restrictive assumptions and potential deadlocks which can occur when using a dynamic controller, see [5].

Theorem 1: Consider the NCS consisting of a continuous-time plant (1) and a static state feedback discrete-time controller (8) interconnected over a network. Let the network be affected by uncertain, time-varying delays τ_k^{sc}, τ_k^c and τ_k^{ca}, as well as packet dropouts and let Assumptions 1-3 hold. Furthermore, consider the proposed buffering mechanism, which introduces an additional buffer delay $\tau_k^b < T_d, \forall k$, resulting in the equivalent representation (7) of the NCS. If there exist symmetric positive definite matrices $\boldsymbol{Y}_i \in \mathbb{R}^{(n+\bar{\delta}m)\times(n+\bar{\delta}m)}$, a matrix $\boldsymbol{Z} \in \mathbb{R}^{m\times n}$, matrices $\boldsymbol{X}_i = \begin{bmatrix} \boldsymbol{X}_1 & \boldsymbol{0}_{n\times\bar{\delta}m} \\ \boldsymbol{X}_{2,i} & \boldsymbol{X}_{3,i} \end{bmatrix}$ with $\boldsymbol{X}_1 \in \mathbb{R}^{n\times n}$, $\boldsymbol{X}_{2,i} \in \mathbb{R}^{\bar{\delta}m\times n}$, $\boldsymbol{X}_{3,i} \in \mathbb{R}^{\bar{\delta}m\times\delta m}$ for $i \in \{0, 1 \ldots, \bar{\delta}\}$ and a scalar $0 \leq \gamma < 1$ such that

$$\begin{bmatrix} \boldsymbol{X}_i + \boldsymbol{X}_i^T - \boldsymbol{Y}_i & \boldsymbol{X}_i^T \hat{\boldsymbol{A}}_i^T - \hat{\boldsymbol{Z}}^T \hat{\boldsymbol{B}}^{\boldsymbol{T}} \\ \hat{\boldsymbol{A}}_i \boldsymbol{X}_i - \hat{\boldsymbol{B}} \hat{\boldsymbol{Z}} & (1-\gamma)\boldsymbol{Y}_j \end{bmatrix} > \boldsymbol{0} \tag{10}$$

with $\hat{\boldsymbol{Z}} = \begin{bmatrix} \boldsymbol{Z} & \boldsymbol{0}_{m\times\bar{\delta}} \end{bmatrix}$ is satisfied for $\forall i, j \in \{1, 2, \ldots, \bar{\delta}\}$, then the buffered NCS affected by unknown time-varying network delays and dropouts is asymptotically stable. Moreover, the state feedback gain matrix is determined by $\boldsymbol{K}_x = \boldsymbol{Z}\boldsymbol{X}_1^{-1}$ and the corresponding Lyapunov function is given as $V(\boldsymbol{\xi}_k, \boldsymbol{\alpha}_k) = \boldsymbol{\xi}_k^T \hat{\boldsymbol{P}}(\boldsymbol{\alpha}_k) \boldsymbol{\xi}_k$ with the parameter dependent Lyapunov matrix $\hat{\boldsymbol{P}}(\boldsymbol{\alpha}_k) = \sum_{i=1}^{\bar{\delta}} \alpha_{i,k} \boldsymbol{P}_i$ for $\boldsymbol{P}_i = \boldsymbol{Y}_i^{-1}$, which is positive definite for all values of $\boldsymbol{\alpha}_k$.

Proof: The relation between the matrix $\hat{\boldsymbol{Z}}$ from (10) and the control gain is given by $\hat{\boldsymbol{Z}} = \hat{\boldsymbol{K}} \boldsymbol{X}_i$ leading to the LMIs

$$\begin{bmatrix} \boldsymbol{X}_i + \boldsymbol{X}_i^T - \boldsymbol{Y}_i & \left(\left(\hat{\boldsymbol{A}}_i - \hat{\boldsymbol{B}} \hat{\boldsymbol{K}} \right) \boldsymbol{X}_i \right)^T \\ \left(\hat{\boldsymbol{A}}_i - \hat{\boldsymbol{B}} \hat{\boldsymbol{K}} \right) \boldsymbol{X}_i & (1-\gamma)\boldsymbol{Y}_j \end{bmatrix} > \boldsymbol{0}, \tag{11}$$

incorporating the closed loop dynamic matrices for individual modes of (7). For $\boldsymbol{Y}_i^{-1} = \boldsymbol{P}_i$ this is equivalent to

$$\begin{bmatrix} \boldsymbol{P}_i & \left(\hat{\boldsymbol{A}}_i - \hat{\boldsymbol{B}}\hat{\boldsymbol{K}}\right)^T \boldsymbol{P}_j \\ \boldsymbol{P}_j \left(\hat{\boldsymbol{A}}_i - \hat{\boldsymbol{B}}\hat{\boldsymbol{K}}\right) & (1-\gamma)\boldsymbol{P}_j \end{bmatrix} = \boldsymbol{Q}_{ij} > \boldsymbol{0}, \quad (12)$$

see [11], [12]. From the Schur complement of $\sum_{i=1}^{\bar{\delta}} \alpha_{i,k} \sum_{j=1}^{\bar{\delta}} \alpha_{j,k} \boldsymbol{Q}_{ij}$, the inequality

$$\hat{\boldsymbol{P}} - \left(\hat{\boldsymbol{A}}(\boldsymbol{\alpha}_k) - \hat{\boldsymbol{B}}\hat{\boldsymbol{K}}\right)^T \hat{\boldsymbol{P}}^+ \left(\hat{\boldsymbol{A}}(\boldsymbol{\alpha}_k) - \hat{\boldsymbol{B}}\hat{\boldsymbol{K}}\right) > \gamma \hat{\boldsymbol{P}} \quad (13)$$

can be derived with $\hat{\boldsymbol{P}}^+ = \hat{\boldsymbol{P}}(\boldsymbol{\alpha}_{k+1}) = \sum_{j=1}^{\bar{\delta}} \alpha_{j,k} \boldsymbol{P}_j$ and $0 \leq \gamma < 1$, see [11]. The equilibrium of (9) is globally uniformly asymptotically stable if

$$\begin{aligned} -\Delta V(\boldsymbol{\xi}_k, \boldsymbol{\alpha}_k) &= V(\boldsymbol{\xi}_k, \boldsymbol{\alpha}_k) - V(\boldsymbol{\xi}_{k+1}, \boldsymbol{\alpha}_{k+1}) = \quad (14) \\ &= \boldsymbol{\xi}_k^T \left(\hat{\boldsymbol{P}} - \left(\hat{\boldsymbol{A}}(\boldsymbol{\alpha}_k) - \hat{\boldsymbol{B}}\hat{\boldsymbol{K}}\right)^T \hat{\boldsymbol{P}}^+ \left(\hat{\boldsymbol{A}}(\boldsymbol{\alpha}_k) - \hat{\boldsymbol{B}}\hat{\boldsymbol{K}}\right)\right) \boldsymbol{\xi}_k > \boldsymbol{0} \end{aligned}$$

holds. Due to the fact that $\hat{\boldsymbol{P}}$ is positive definite, this condition is ensured when (13) is fulfilled. ■

The impact of γ becomes clear from (13) and (14), since increasing its value results in the larger value of $|\Delta V(\boldsymbol{\xi}_k, \boldsymbol{\alpha}_k)|$ and therefore the larger lower bound for the transient decay rate of the system states. Hence, γ represents an additional variable which can be used to influence the transient performance and resulting settling time, see [4].

The total number of LMI conditions in the form (11) which should be solved is $\bar{\delta}^2$. This represents a significant result in reduction of the computational complexity in comparison to the approach presented in [4] and [5], where the number of LMI to be solved is $2^{2\bar{\delta}\nu}$ with $\nu \leq n$ depending on the geometric multiplicity of the eigenvalues, see [4]. Unlike [5], where the computation complexity is greatly influenced not only by the network utilized but also by the order of the system, the number of LMIs in the proposed approach is completely independent on the considered system.

IV. Extensions of the Control Design

This section addresses extensions of the previously proposed static feedback control law which can be applied to enhance the performance of the complete NCS.

A. Switched State Feedback Control Law

Theorem 2: Consider the buffered NCS (3) as in Theorem 1 with a switched state feedback controller

$$\boldsymbol{u}_k = -\boldsymbol{K}_{x,i}\boldsymbol{x}_k = -\hat{\boldsymbol{K}}_i\boldsymbol{\xi}_k = -\begin{bmatrix} \boldsymbol{K}_{x,i} & \boldsymbol{0} \end{bmatrix} \boldsymbol{\xi}_k. \quad (15)$$

Furthermore, consider the scalar γ and the matrices as in the Theorem 1 with the difference that the constant matrices $\boldsymbol{X}_1$ and $\boldsymbol{Z}$ are replaced by matrices $\boldsymbol{X}_{1,i}$ and $\boldsymbol{Z}_i$ of the same size ($\forall i = 1, 2, \ldots, \bar{\delta}$), respectively. If the resulting LMIs of the form (11) are satisfied $\forall i, j \in \{1, 2, \ldots, \bar{\delta}\}$, then the buffered NCS affected by unknown time-varying network delays and dropouts is asymptotically stable. The switched control law (15) is obtained by $\boldsymbol{K}_{\boldsymbol{x},\boldsymbol{i}} = \boldsymbol{Z}_i\boldsymbol{X}_{1,i}^{-1}$ leading to $\bar{\delta}$ values for $\boldsymbol{u}_k$ computed based on single data packet containing $\boldsymbol{x}_k$.

Proof: The proof follows using similar arguments as in the proof of Theorem 1. ■

Please note that in this case, an additional mechanism implemented in the buffer is necessary. Since it is not known at the moment of the computation of the controller which $\hat{\boldsymbol{A}}_i$ mode is active, the data packet containing all the $\bar{\delta}$ values for $\boldsymbol{u}_k$ is sent to the actuator node. Based on the timestamp attached by the sensor, the total delay can be calculated allowing the buffer to determine which control law (15) should be applied.

B. Extended State Feedback Control Law

Theorem 3: Consider the buffered NCS (3) as in Theorem 1 with an extended state feedback controller

$$\boldsymbol{u}_k = -\hat{\boldsymbol{K}}\boldsymbol{\xi}_k = -\begin{bmatrix} \boldsymbol{K}_x & \boldsymbol{K}_u \end{bmatrix} \boldsymbol{\xi}_k, \quad (16)$$

including not only the state variables but also part of the control input history. Furthermore, consider the scalar γ and the matrices as in the Theorem 1 with the difference that $\boldsymbol{X}_i$ and $\hat{\boldsymbol{Z}}$ are replaced by one constant matrix $\boldsymbol{X}$ and a full matrix $\hat{\boldsymbol{Z}} = \begin{bmatrix} \boldsymbol{Z}_x & \boldsymbol{Z}_u \end{bmatrix}$ ($\forall i = 1, \ldots, \bar{\delta}$), respectively. If the resulting LMIs of the form (11) are satisfied $\forall i, j \in \{1, 2, \ldots, \bar{\delta}\}$, then the buffered NCS affected by unknown time-varying network delays and dropouts is asymptotically stable for the extended control law (16) obtained by $\boldsymbol{K} = \hat{\boldsymbol{Z}}\boldsymbol{X}^{-1}$.

Proof: The proof follows using similar arguments as in the proof of Theorem 1. ■

It is important to note that (16) relies on the past control inputs being available, thus requiring that there may be no dropouts in the connection path from the plant to the controller (i.e. $m_k^{sc} = 0, \forall k$). Otherwise, a deadlock can occur leading to a situation where the control signal is not updated at all. Furthermore, the previously computed control inputs should be stored locally in the controller after the computation. The controller must wait until all the previous control signals are available before computing $\boldsymbol{u}_k$, which could add an additional delay in the system and hence leads to the reformulation of the sensor-to controller delay to $\tau_k^{sc} = \max\{kT_d + \tau_k^{sc}, jT_d + \tau_j^{sc}\}, \forall j < k$. However, the maximal round trip time, the value of $\bar{\delta}$ and therefore the model of the NCS remain unchanged, see Assumption 2.

C. Additional Influence on the Transient Performance

The advantage of the previously proposed control laws regarding the significantly reduced computational effort allows further extensions of the control law proposed in Theorems 1, 2 and 3 by introducing additional degrees of freedom. This is done by replacing one scalar γ by γ_i for $i = 1, 2, \ldots, \bar{\delta}$, which impact the transient performance of the NCS.

Theorem 4: Consider the buffered NCS (3) as in Theorem 1 with a static state feedback controller (8) and the matrices $\boldsymbol{Y}_i$, $\boldsymbol{X}_i$ and $\boldsymbol{Z}$ as in the Theorem 1. Furthermore, consider scalars $0 \leq \gamma_i < 1$ ($i = 1, 2, \ldots, \bar{\delta}$) such that

$$\begin{bmatrix} \boldsymbol{X}_i + \boldsymbol{X}_i^T - \boldsymbol{Y}_i & \boldsymbol{X}_i^T \hat{\boldsymbol{A}}_i^T - \hat{\boldsymbol{Z}}^T \hat{\boldsymbol{B}}^{\boldsymbol{T}} \\ \hat{\boldsymbol{A}}_i \boldsymbol{X}_i - \hat{\boldsymbol{B}}\hat{\boldsymbol{Z}} & (1-\gamma_i)\boldsymbol{Y}_j \end{bmatrix} > \boldsymbol{0} \quad (17)$$

is satisfied for $\forall i, j \in \{1, 2, \ldots, \bar{\delta}\}$, then the buffered NCS affected by unknown time-varying network delays and dropouts is asymptotically stable for the control law (8).

Proof: The LMIs (17) can analogously to Theorem 1 be brought to the form $\boldsymbol{Q}_{ij}$ as in (12) with γ_i. Evaluating $\sum_{i=1}^{\bar{\delta}} \alpha_{i,k} \sum_{j=1}^{\bar{\delta}} \alpha_{j,k} \boldsymbol{Q}_{ij}$ from (12) leads to

$$\begin{bmatrix} \hat{\boldsymbol{P}} & \left(\hat{\boldsymbol{A}}(\boldsymbol{\alpha}_k) - \hat{\boldsymbol{B}}\hat{\boldsymbol{K}}\right)^T \hat{\boldsymbol{P}}^{+} \\ \hat{\boldsymbol{P}}^{+}\left(\hat{\boldsymbol{A}}(\boldsymbol{\alpha}_k) - \hat{\boldsymbol{B}}\hat{\boldsymbol{K}}\right) & \left(1 - \sum_{i=1}^{\bar{\delta}} \alpha_{i,k}\gamma_i\right) \hat{\boldsymbol{P}}^{+} \end{bmatrix} > 0 \tag{18}$$

from which the Schur complement can be computed

$$\hat{\boldsymbol{P}} - \left(\hat{\boldsymbol{A}}(\boldsymbol{\alpha}_k) - \hat{\boldsymbol{B}}\hat{\boldsymbol{K}}\right)^T \hat{\boldsymbol{P}}^{+} \left(\hat{\boldsymbol{A}}(\boldsymbol{\alpha}_k) - \hat{\boldsymbol{B}}\hat{\boldsymbol{K}}\right) > \sum_{i=1}^{\bar{\delta}} \alpha_{i,k}\gamma_i \hat{\boldsymbol{P}}. \tag{19}$$

The equilibrium of (9) is globally uniformly asymptotically stable if $|\Delta V(\boldsymbol{\xi}_k, \boldsymbol{\alpha}_k)| < 0$ which is, by the definition of the matrix $\hat{\boldsymbol{P}}$ and scalars γ_i, ensured if (19) holds. ■

By varying parameters γ_i, the lower bound for the transient decay rate of the system states can be increased possibly leading to even faster transient performance as in the case of one scalar γ in Theorem 1. Please note that control laws proposed in Theorems 2 and 3 can be extended analogously to Theorem 4. Furthermore, the additional degrees of freedom γ_i represent only a slight modification of the implemented optimization problem since they don't introduce any new optimization variables or increase the number of LMIs.

V. Illustrative Example

The performance and efficiency of the proposed algorithms with respect to the reduced complexity are highlighted in a simulation study on the basis of TrueTime [14], in which a rotary servo plant is considered. By choosing the angle and rotational speed as the system states, the mathematical model (1) with $\boldsymbol{A}_c = \begin{bmatrix} 0 & 1 \\ 0 & -72.5 \end{bmatrix}$ and $\boldsymbol{b}_c = \begin{bmatrix} 0 \\ 75.3 \end{bmatrix}$ is obtained, see [9]. The sampling time of $T_d = 20$ms is chosen. The network-induced delays τ_k^{sc}, τ_k^{ca} are randomly time-varying within the intervals $0 \leq \tau_k^{sc} < 2T_d$, $0 \leq \tau_k^{ca} < 2T_d, \forall k$. The computation time of the controller is neglected, i.e. $\tau_k^c = 0, \forall k$ and the network ensures the safe transmission of data, i.e. $m_k^{sc} = m_k^{ca} = 0, \forall k$. This network configuration results in $\bar{d} = 4T_d$, $\bar{p} = 0$ and consequently $\bar{\delta} = 4T_d$.

A. Simulation study with one constant γ

In the first simulation study, the existing approach from [5] is compared with the proposed control laws from Theorems 1 and 3 for one γ. To ensure the fair comparison, the scenario with the fastest achievable decay rate is selected by determining the largest admissible value of γ for which a feasible solution can be derived for each control law. Figure 2 illustrates the simulation results for an initial state $\boldsymbol{x}_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$, while Table I provides an overview of the obtained control laws. Additionally, in order to compare the computational effort, the number of optimization variables, the number of LMIs of the form (11) and the time needed to solve the optimization problem (Intel(R) Core(TM) i7-8565U CPU @ 1.80GHz) are listed as well.

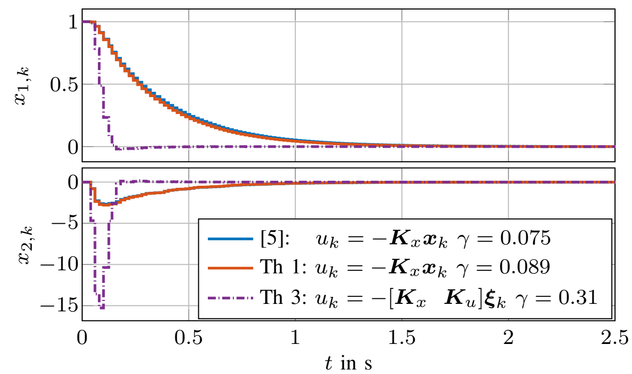

Fig. 2. Simulation results for $\boldsymbol{x}_k$ with the control laws from Table I

TABLE I
COMPUTATIONAL EVALUATION FOR ONE γ

Control Law		γ	opt.var. / LMIs	Time in s
[5]	$\boldsymbol{u}_k = -\begin{bmatrix}\boldsymbol{K}_x & \boldsymbol{0}\end{bmatrix}\xi_k$ $\boldsymbol{K}_x = \begin{bmatrix}2.62 & 0.04\end{bmatrix}$	0.075	11526 65536	4648.7
Th 1	$\boldsymbol{u}_k = -\begin{bmatrix}\boldsymbol{K}_x & \boldsymbol{0}\end{bmatrix}\xi_k$ $\boldsymbol{K}_x = \begin{bmatrix}2.73 & 0.034\end{bmatrix}$	0.089	186 16	0.39
Th 3	$\boldsymbol{u}_k = -\begin{bmatrix}\boldsymbol{K}_x & \boldsymbol{K}_u\end{bmatrix}\xi_k$ $\boldsymbol{K}_x = \begin{bmatrix}15.69 & 0.20\end{bmatrix}$ $\boldsymbol{K}_u = \begin{bmatrix}0.36 & 0.22 & 0.17 & 0.13\end{bmatrix}$	0.31	118 16	0.27

The control law proposed in Theorem 1 shows slightly faster transient decay rate of the system states in comparison to the existing approach from [5]. The extended formulation of the control algorithm (purple curve) can yield much faster response as with the static control methods, but, as previously mentioned, additional restrictions must hold. The most striking aspect of the comparison can be seen in Table I in the number of optimization variables and LMIs as well as the time needed to solve the corresponding optimization problem. Not only the complexity with respect to necessary over-approximation techniques from [5] is avoided by reformulation of the buffered NCS as a switched system, but the resulting optimization problem is greatly simplified. The computation effort is therefore significantly reduced and the solution can be obtained in the range of milliseconds.
Please note, that increasing $\bar{\delta}$ to 5 would result in 46086 optimization variables and 1048576 LMIs necessary to obtain the control law from [5], which greatly increases the complexity for the over-approximation of the NCS and computational effort for the control synthesis. For the proposed control law from, e.g., Theorem 1, $\bar{\delta} = 5$ would, however, lead to only 321 optimization variables and 25 LMIs. The complexity of defining the NCS as a switched system and the resulting computational effort would not be affected at all.

B. Simulation study with multiple γ_i

Furthermore, to underpin the impact of additional degrees of freedom in the control design, the control laws from

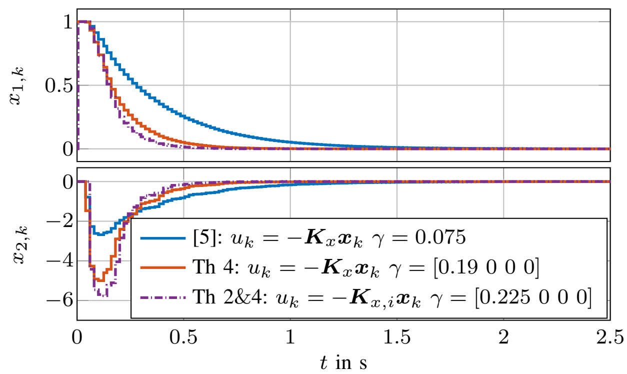

Fig. 3. Simulation results for $\boldsymbol{x}_k$ with the control laws from Table II

TABLE II

COMPUTATIONAL EVALUATION FOR MULTIPLE γ_i

Control Law		γ	opt.var. / LMIs	Time in s
[5]	$\boldsymbol{u}_k = -\begin{bmatrix}\boldsymbol{K}_x & \boldsymbol{0}\end{bmatrix} \xi_k$ $\boldsymbol{K}_x = \begin{bmatrix}2.62 & 0.04\end{bmatrix}$	0.075	11526 65536	4648.7
Th 4	$\boldsymbol{u}_k = -\begin{bmatrix}\boldsymbol{K}_x & \boldsymbol{0}\end{bmatrix} \xi_k$ $\boldsymbol{K}_x = \begin{bmatrix}4.93 & 0.09\end{bmatrix}$	$\begin{bmatrix}0.19\\0\\0\\0\end{bmatrix}$	186 16	0.32
Th 4 switched control law	$\boldsymbol{u}_k = -\begin{bmatrix}\boldsymbol{K}_{x,i} & \boldsymbol{0}\end{bmatrix} \xi_k$ $\boldsymbol{K}_{x,1} = \begin{bmatrix}5.56 & 0.086\end{bmatrix}$ $\boldsymbol{K}_{x,2} = \begin{bmatrix}5.6 & 0.08\end{bmatrix}$ $\boldsymbol{K}_{x,3} = \begin{bmatrix}5.87 & 0.08\end{bmatrix}$ $\boldsymbol{K}_{x,4} = \begin{bmatrix}6.09 & 0.096\end{bmatrix}$	$\begin{bmatrix}0.225\\0\\0\\0\end{bmatrix}$	204 16	0.24

Theorems 1 and 2 with $\boldsymbol{\gamma} = \begin{bmatrix}\gamma_1 & \gamma_2 & \gamma_3 & \gamma_4\end{bmatrix}^T$ are compared with the existing approach from [5]. Figure 3 and Table II summarize the results in the scenario with the fastest achievable transient decay rate obtained for $\boldsymbol{x}_0 = \begin{bmatrix}1 & 0\end{bmatrix}^T$, as in Section V-A.

Already in the case of a static controller, the specific choice of γ_i leads to the further increase of the system states decay rate. The even more significant improvement in the convergence speed is achieved using a switching control law, where the buffer applies the control signal depending on the active switching mode. Hence, the proposed control laws using multiple γ_i result in a considerably faster decay rate compared to [5], which may be desirable in many applications. The computation effort and time needed remain the same as in the case with a single γ.

VI. CONCLUSION AND OUTLOOK

This paper presents a control strategy for NCS affected by random variable delays and packet dropouts with a special focus on the practical usability in terms of computational simplicity and flexibility. This is achieved by first introducing a new buffering mechanism, which not only simplifies the discrete-time model of the NCS significantly but limits the additional buffer delay to one sampling time as well. The resulting buffered NCS can therefore be described as a switched system, which reduces the complexity of the control design greatly, since over-approximation techniques for obtaining a polytopic model suitable for stability analysis as in [4], [5] are no longer necessary. Hence, the stability conditions and controller synthesis can be defined in terms of strongly reduced number of LMIs and the computational time and effort required to solve the optimization problem is substantially reduced. As a result, it is possible to further exploit the potential of the control design with respect to its impact on the decay rate by introducing additional degrees of freedom γ_i, $i = 1, 2, \ldots, \bar{\delta}$. The performance and computational effort of the proposed control approaches are evaluated in a simulation which not only highlights the achieved computational simplicity but demonstrates the greater flexibility of the proposed laws with respect to the transient behavior as well.

Future developments will focus on the impact on the transient behavior and systematic choice for γ_i. Furthermore, a generalization of the proposed strategies to more complex multi-hop or spatially distributed network structures is of interest.

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