IEEE Control Systems Letters paper presented at
2023 62nd IEEE Conference on Decision and Control (CDC)
December 13-15, 2023. Marina Bay Sands, Singapore

Removing Two Fundamental Assumptions in Verifying Strong Periodic (D-)Detectability of Discrete-Event Systems

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***Abstract*—In this letter, in discrete-event systems modeled by labeled finite-state automata (LFSAs), we show new thinking on the tools of detector and concurrent composition and derive two new algorithms for verifying strong periodic detectability (SPD) without any assumption that run in NL; we also reconsider the tool of observer and derive a new algorithm for verifying strong periodic D-detectability (SPDD) without any assumption that runs in PSPACE. These results strengthen the NL upper bound on verifying SPD and the PSPACE upper bound on verifying SPDD for deadlock-free and divergence-free LFSAs in the literature. In our algorithms, the two assumptions are removed by verifying the *negations* of these properties.**

***Index Terms*—Discrete-event system, detectability, concurrent composition, detector.**

I. Introduction

A. Background

DETECTABILITY is a basic property of partially-observed dynamical systems: when it holds one can use an observed output/label sequence produced by a system to reconstruct its current states [1], [2], [3], [4]. This property plays a fundamental role in many related control problems such as observer design and controller synthesis.

For DESs modeled by LFSAs, the verification problems for different definitions of detectability have been widely studied [1], [2], [3], [4], [5], [6], [7], [8], in which several complexity lower bounds and upper bounds for these problems were obtained, but most of the upper bounds depend on two fundamental assumptions that a system is deadlock-free and divergence-free. These requirements are collected in Assumption 1: when it holds, a system will always run and generate an infinitely long label/output sequence. The two assumptions have been used in detectability studies for over 15 years (see [1], [2], [5], [6], [7]). Notice that there are only a small number of LFSAs that satisfy the two assumptions. If these two assumptions can be removed, the theory and applications of LFSAs will be largely extended. Without the two assumptions, detectability still makes sense. For example, when a DES enters an unobservable cycle, although one can observe nothing, the DES is still running, and it makes sense to study its detectability based on the observed label sequences. Consider the DES $\mathcal{S}_2$ in Fig. 1 for example, obviously $\mathcal{S}_2$ does not satisfy Assumption 1, because it does not generate any infinitely long label sequence. However, one can observe one label a, and hence can study whether one can use a to determine the current state. The answer is obviously no. Unfortunately, the verification methods used in [1], [2] (these methods were also used in [6], [7]) cannot verify detectability of $\mathcal{S}_2$ correctly. See Example 1 for details.

The first verification algorithm for detectability of DESs that does not depend on Assumption 1 was given in [3], [4] by developing a technique called *concurrent composition*, arising from verifying the *negation* of strong detectability. Prior to that, verification of variant notions of detectability depends on the two assumptions because these notions *themselves* were verified [1], [2], [6], [7]. In the current paper, we further use the concurrent composition and two other tools of observer and detector [2] to obtain verification algorithms for strong periodic detectability and strong periodic D-detectability that do not depend on any assumption.

B. Literature Review on Verification of Detectability in LFSAs

Results based on Assumption 1: In [1], by using an *observer* method, exponential-time algorithms were given to verify four notions of detectability: strong (periodic) detectability and weak (periodic) detectability. Strong detectability means that there is a delay k, for *each* infinite-length event sequence s generated by an LFSA, every prefix of the label/output sequence of s of length greater than k allows reconstructing the current state. Weak detectability relaxes strong detectability by changing the verbatim from *each* to *some*. Weak detectability is strictly weaker than strong detectability. Strong periodic detectability implies that at any time, after some observation delay no greater than a given value, the system states can be determined along *each* infinite-length transition sequence also by observing the corresponding output sequence. Weak

Manuscript received 13 March 2023; revised 13 April 2023; accepted 2 May 2023. Date of publication 5 May 2023; date of current version 12 May 2023. Recommended by Senior Editor A. P. Aguiar.

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Digital Object Identifier 10.1109/LCSYS.2023.3273472

periodic detectability relaxes strong periodic detectability by changing the verbatim from *each* to *some*. Later in [2], by using a *detector* method, where the detector is obtained from an observer by splitting all its states into subsets of cardinality 2, polynomial-time algorithms were designed for verifying strong (periodic) detectability. The problem of verifying weak (periodic) detectability of LFSAs was proven to be PSPACE-complete [5] and verifying strong (periodic) detectability was proven to be NL-complete [6].

In order to make detectability adapt to more scenarios, one can weaken detectability to D-detectability in the sense of not exactly determining the states but making sure that the states cannot contain both states of any pair of states that are previously specified [2]. All above notions of detectability, including strong/weak detectability and strong/weak periodic detectability, can be extended to their D-versions. For example, strong D-detectability can be verified in polynomial time [2], while verifying strong periodic D-detectability is PSPACE-complete [7].

Note that all the above complexity upper bounds were obtained by the verification algorithms designed in [1], [2] based on Assumption 1. For an LFSA that does not satisfy Assumption 1, the algorithms may not return a correct answer. In [4, Remark 2], we gave a counterexample to show that neither the observer method [1] nor the detector method [2] correctly verifies its strong detectability. This counterexample also shows that neither Theorem 1 nor [9, Th. 2] is correct without Assumption 1. Therefore, the main results obtained in [9] are correct only under Assumption 1 but are not so general as claimed therein. Later in Remark 1 and Remark 2, we will give counterexamples to show that neither of the two methods correctly verifies their strong periodic detectability and strong periodic D-detectability.

Results which do not depend on assumptions: The two fundamental assumptions shown in Assumption 1 were for the first time removed in [3], [4] by developing a *concurrent-composition* method and verifying the *negation* of strong detectability. In [4], weak detectability was also verified without any assumption. In [10], the concurrent-composition method was used to verify the negation of strong detectability of a networked DES with a first-in, first-out single channel with a bounded delay and in which communication losses may occur, in polynomial time. Later in [8], an NL upper bound was given for the verification problem of strong detectability based on the concurrent-composition method. In addition, decentralized settings of strong detectability, diagnosability, and predictability were unified into one mathematical framework [8]. In [11], strong D-detectability was verified in polynomial time also by the concurrent-composition method.

C. Contribution of This Letter

1) We use the detector and concurrent composition to derive two new algorithms for verifying strong periodic detectability of LFSAs without any assumption, where both algorithms imply an NL upper bound for strong periodic detectability, which strengthens the NL upper bound given in [6] under Assumption 1.

TABLE I
COMPLEXITY RESULTS FOR VERIFYING DIFFERENT DEFINITIONS OF DETECTABILITY IN LFSAS, WHERE * MEANS THAT THE BOUNDS ONLY APPLY TO LFSAS SATISFYING ASSUMPTION 1

strong detectability	strong D-detectability
NL ([8]) NL-complete* ([6])	P ([11]) NL-complete* ([7])
strong periodic detectability	**strong periodic D-detectability**
NL (Thm. 3.7) NL-complete* ([6])	PSPACE (Thm. 3.12) PSPACE-complete* ([7])
weak detectability	**weak periodic detectability**
PSPACE-complete* ([5])	PSPACE-complete* ([5])

2) We use the observer to derive a new algorithm for verifying strong periodic D-detectability of LFSAs without any assumption, where the algorithm implies a PSPACE upper bound for strong periodic D-detectability, which strengthens the PSPACE upper bound given in [7] under Assumption 1. See Tab. I for a collection of related results.

Compared with giving an NL upper bound for strong detectability (see [8]), the process of obtaining the NL upper bound for strong periodic detectability is more difficult. Differently from verifying strong periodic detectability itself in [6], we verify its negation. Following such an opposite way, for an LFSA, we obtain two conditions on its detector such that at least one of them holds exactly violates its strong periodic detectability. Hence a polynomial-time verification algorithm based on the detector is obtained (Theorem 1), and then an NL algorithm naturally follows (Theorem 2). On the other hand, by developing more relationships between the notions of observer, detector, and concurrent composition (Proposition 4), we construct a variant of the concurrent composition by using which strong periodic detectability can also be verified in NL (Theorem 3). Similarly, we also obtain a polynomial-space verification algorithm for strong periodic D-detectability by verifying its negation.

The remainder is structured as follows. In Section II, basic notation and definitions in LFSAs are introduced. In Section III, the main results are shown. Section IV ends up with a short conclusion.

II. PRELIMINARIES

Notation: For a finite alphabet Σ, Σ^* and Σ^ω are used to denote the set of finite sequences (called *words*) of elements of Σ including the empty word ϵ and the set of infinite sequences (called *configurations*) of elements of Σ, respectively. $\Sigma^+ := \Sigma^* \setminus \{\epsilon\}$. For a word $s \in \Sigma^*$, $|s|$ stands for its length. For $s \in \Sigma^+$ and natural number k, s^k and s^ω denote the *concatenations* of k-copies and infinitely many copies of s, respectively. For a word (configuration) $s \in \Sigma^*(\Sigma^\omega)$, a word $s' \in \Sigma^*$ is called a *prefix* of s, denoted as $s' \sqsubset s$, if there exists another word (configuration) $s'' \in \Sigma^*(\Sigma^\omega)$ such that $s = s's''$. For two natural numbers $i \le j$, $[\![i, j]\!] := \{k$ are nonnegative integers$|i \le k \le j\}$; and for a set S, $|S|$ denotes its cardinality and 2^S its power set.

A DES modeled by an LFSA is a sextuple

$$\mathcal{S} = (X, T, X_0, \delta, \Sigma, \ell), \qquad (1)$$

where X is a finite set of *states*, T a finite set of *events*, $X_0 \subset X$ a set of *initial states*, $\delta \subset X \times T \times X$ the *transition relation*, Σ a finite set of *outputs (labels)*, and $\ell : T \to \Sigma \cup \{\epsilon\}$ the *labeling function*. ℓ can be recursively extended to $\ell : T^* \cup T^\omega \to \Sigma^* \cup \Sigma^\omega$ and particularly $\ell(\epsilon) = \epsilon$. The event set T can be rewritten as disjoint union of *observable* event set $T_o = \{t \in T | \ell(t) \in \Sigma\}$ and *unobservable* event set $T_{uo} = \{t \in T | \ell(t) = \epsilon\}$. Transition relation δ is recursively extended to $\delta \subset X \times T^* \times X$. We call a transition with an observable (unobservable) event an *observable (unobservable) transition*. We also denote a transition sequence $(x, s, x') \in \delta$ by $x \xrightarrow{s} x'$, where $x, x' \in X$, $s \in T^*$. For $x \in X$ and $s \in T^+$, (x, s, x) is called a *transition cycle* if $(x, s, x) \in \delta$. An *observable* (resp., *unobservable*) transition cycle is defined by a transition cycle with at least one (resp., with no) observable transition. Automaton $\mathcal{S}$ is called *deterministic* if $|X_0| = 1$ and for all $x, x', x'' \in X$ and $t \in T$, $(x, t, x'), (x, t, x'') \in \delta$ imply $x' = x''$. For deterministic $\mathcal{S}$, for all $x \in X$ and all $s \in T^*$, we also denote the unique state $x' \in X$ (if any) satisfying $x \xrightarrow{s} x'$ by $\delta(x, s)$.

For each $\sigma \in \Sigma^*$, we denote by $\mathcal{M}(\mathcal{S}, \sigma)$ the *current-state estimate*, i.e., the set of states that the system can be in after σ has been observed, i.e., $\mathcal{M}(\mathcal{S}, \sigma) := \{x \in X | (\exists x_0 \in X_0)(\exists s \in T^*)[(\ell(s) = \sigma) \wedge (x_0 \xrightarrow{s} x)]\}$. We use $L^\omega(\mathcal{S}) = \{t_1 t_2 \ldots \in T^\omega | (\exists x_0 \in X_0)(\exists x_1, x_2, \ldots, \in X)[x_0 \xrightarrow{t_1} x_1 \xrightarrow{t_2}, \ldots]\}$ to denote the set of infinite-length event sequences generated by $\mathcal{S}$.

For a state $x \in X$, its *unobservable reach* is defined by $\mathrm{UR}(x) := \{x' \in X | (\exists s \in (T_{uo})^*)[(x, s, x') \in \delta]\}$. For a subset $X' \subset X$, $\mathrm{UR}(X') = \bigcup_{x \in X'} \mathrm{UR}(x)$. Hence $\mathrm{UR}(X_0) = \mathcal{M}(\mathcal{S}, \epsilon)$. For a state $x \in X$, its *observable reach under* $\sigma \in \Sigma$ is defined by $\mathrm{Reach}_\sigma(x) := \{x' \in X | (\exists t \in T)[((x, t, x') \in \delta) \wedge (\sigma = \ell(t))]\}$. Analogously, for a subset $X' \subset X$, $\mathrm{Reach}_\sigma(X') = \bigcup_{x \in X'} \mathrm{Reach}_\sigma(x)$.

The following two assumptions are commonly used in detectability studies (cf. [1], [2], [6], [7]), but are not needed in the current paper because we verify the negations of the properties.

Assumption 1: An LFSA $\mathcal{S}$ as in (1) satisfies

1) $\mathcal{S}$ is *deadlock-free*, i.e., for each reachable state $x \in X$, there exist $t \in T$ and $x' \in X$ such that $(x, t, x') \in \delta$;
2) $\mathcal{S}$ is *prompt* or *divergence-free*, i.e., for every reachable state $x \in X$ and every nonempty unobservable event sequence $s \in (T_{uo})^+$, there exists no transition sequence $x \xrightarrow{s} x$ in $\mathcal{S}$.

One sees (1) implies $L^\omega(\mathcal{S}) \neq \emptyset$ if $X_0 \neq \emptyset$; while (2) implies for all $s \in L^\omega(\mathcal{S})$, $\ell(s) \in \Sigma^\omega$. Condition (2) can be verified in polynomial time using Tarjan algorithm to compute all strongly connected components of $\mathcal{S}$. In addition, it is easy to see that there exist strongly periodically detectable LFSAs that do not satisfy (2).

III. Main Results

A. Preliminary Results

The definitions of strong periodic detectability and strong periodic D-detectability for LFSAs are as follows [2].

Definition 1 (SPD): An LFSA $\mathcal{S}$ is called *strongly periodically detectable* if there exists a positive integer k such that for each $s \in L^\omega(\mathcal{S})$ and each $s' \sqsubset s$, there is $s'' \in T^*$ such that $|\ell(s'')| < k$, $s's'' \sqsubset s$, and $|\mathcal{M}(\mathcal{S}, \ell(s's''))| = 1$.

In order to formulate strong periodic D-detectability, we specify a set $T_{\text{spec}} \subset X \times X$ of crucial state pairs that should be separated.

Definition 2 (T_{spec}-SPDD): An LFSA $\mathcal{S}$ is called *strongly periodically D-detectable with respect to* T_{spec} if there exists a positive integer k such that for each $s \in L^\omega(\mathcal{S})$ and each $s' \sqsubset s$, there is $s'' \in T^*$ such that $|\ell(s'')| < k$, $s's'' \sqsubset s$, and $(\mathcal{M}(\mathcal{S}, \ell(s's'')) \times \mathcal{M}(\mathcal{S}, \ell(s's''))) \cap T_{\text{spec}} = \emptyset$.

In order to verify detectability of an LFSA $\mathcal{S}$, an *observer*

$$\mathcal{S}_{obs} := (2^X \setminus \{\emptyset\}, \Sigma, \mathcal{M}(\mathcal{S}, \epsilon), \delta_{obs}) \tag{2}$$

as a deterministic LFSA was constructed in [1], where $\mathcal{M}(\mathcal{S}, \epsilon)$ is the unique initial state; for all $X' \in 2^X \setminus \{\emptyset\}$ and $\sigma \in \Sigma^*$, $\delta_{obs}(\mathcal{M}(\mathcal{S}, \epsilon), \sigma) = X'$ if and only if $X' = \mathcal{M}(\mathcal{S}, \sigma)$. The size of $\mathcal{S}_{obs}$ is exponential of that of $\mathcal{S}$.

Later in [2], a *detector*

$$\mathcal{S}_{det} := (Q, \Sigma, \mathcal{M}(\mathcal{S}, \epsilon), \delta_{det}) \tag{3}$$

that is a nondeterministic LFSA was used to provide polynomial-time algorithms for verifying strong detectability and strong periodic detectability under Assumption 1, where $Q \subset 2^X \setminus \{\emptyset\}$ consists of $\mathcal{M}(\mathcal{S}, \epsilon)$ and subsets of X with cardinality ≤ 2; for all $q, q' \in Q$, and $\sigma \in \Sigma$, $(q, \sigma, q') \in \delta_{det}$ if and only if either (1) $|(\mathrm{UR} \circ \mathrm{Reach}_\sigma)(q)| > 1$, $q' \subset (\mathrm{UR} \circ \mathrm{Reach}_\sigma)(q)$, and $|q'| = 2$, or (2) $|(\mathrm{UR} \circ \mathrm{Reach}_\sigma)(q)| = 1$ and $q' = (\mathrm{UR} \circ \mathrm{Reach}_\sigma)(q)$, where $\circ$ means the composition of two functions. The size of $\mathcal{S}_{det}$ is polynomial of that of $\mathcal{S}$.

Proposition 1 [2]: Consider an LFSA $\mathcal{S}$. Under Assumption 1, $\mathcal{S}$ is strongly periodically detectable if and only if in $\mathcal{S}_{det}$, every reachable transition cycle contains at least one singleton; $\mathcal{S}$ is strongly periodically D-detectable if and only if in $\mathcal{S}_{obs}$, every reachable transition cycle contains at least one state q such that $(q \times q) \cap T_{\text{spec}} = \emptyset$.

In [3], in order to verify (delayed) strong detectability of $\mathcal{S}$, the *self-composition*

$$\mathrm{CC_A}(\mathcal{S}) = (X', T', X'_0, \delta') \tag{4}$$

of $\mathcal{S}$ (i.e., the concurrent composition of $\mathcal{S}$ and itself) was constructed as follows:

- $X' = X \times X$;
- $T' = T'_o \cup T'_{uo}$, where $T'_o = \{(\breve{t}, \breve{t}') | \breve{t}, \breve{t}' \in T, \ell(\breve{t}) = \ell(\breve{t}') \in \Sigma\}$, $T'_{uo} = \{(\breve{t}, \epsilon) | \breve{t} \in T, \ell(\breve{t}) = \epsilon\} \cup \{(\epsilon, \breve{t}) | \breve{t} \in T, \ell(\breve{t}) = \epsilon\}$;
- $X'_0 = X_0 \times X_0$;
- for all $(\breve{x}_1, \breve{x}'_1), (\breve{x}_2, \breve{x}'_2) \in X'$, $(\breve{t}, \breve{t}') \in T'_o$, $(\breve{t}'', \epsilon) \in T'_{uo}$, and $(\epsilon, \breve{t}''') \in T'_{uo}$,
  - $((\breve{x}_1, \breve{x}'_1), (\breve{t}, \breve{t}'), (\breve{x}_2, \breve{x}'_2)) \in \delta'$ if and only if $(\breve{x}_1, \breve{t}, \breve{x}_2), (\breve{x}'_1, \breve{t}', \breve{x}'_2) \in \delta$,
  - $((\breve{x}_1, \breve{x}'_1), (\breve{t}'', \epsilon), (\breve{x}_2, \breve{x}'_2)) \in \delta'$ if and only if $(\breve{x}_1, \breve{t}'', \breve{x}_2) \in \delta$, $\breve{x}'_1 = \breve{x}'_2$,
  - $((\breve{x}_1, \breve{x}'_1), (\epsilon, \breve{t}'''), (\breve{x}_2, \breve{x}'_2)) \in \delta'$ if and only if $\breve{x}_1 = \breve{x}_2$, $(\breve{x}'_1, \breve{t}''', \breve{x}'_2) \in \delta$.

For an event sequence $s' \in (T')^*$, $s'(L)$ and $s'(R)$ denote its left and right components, respectively. Similarly for $x' \in X'$, denote $x' =: (x'(L), x'(R))$. In addition, for every $s' \in (T')^*$,

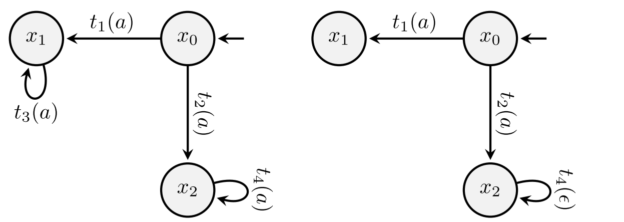

Fig. 1. FSA $\mathcal{S}_1$ (left) and LFSA $\mathcal{S}_2$ (right), where a state with an input arrow from nowhere is initial (e.g., x_0), the letters beside arrows outside () denote events, the letters in () denote the corresponding labels/outputs.

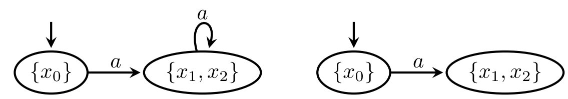

Fig. 2. Detectors $\mathcal{S}_{1det}$ (left, the same as observer $\mathcal{S}_{1obs}$ coincidentally) and $\mathcal{S}_{2det}$ (right, the same as observer $\mathcal{S}_{2obs}$ coincidentally) of LFSA $\mathcal{S}_1$ and LFSA $\mathcal{S}_2$ shown in Fig. 1.

$\ell(s')$ denotes $\ell(s'(L))$ or $\ell(s'(R))$, since $\ell(s'(L)) = \ell(s'(R))$. In the above construction, $\mathrm{CC_A}(\mathcal{S})$ aggregates every pair of transition sequences of $\mathcal{S}$ producing the same label sequence. The size of $\mathrm{CC_A}(\mathcal{S})$ is polynomial of that of $\mathcal{S}$.

B. Verifying Strong Periodic Detectability

In order to verify strong periodic detectability without any assumption, we first characterize its negation. By directly observing Definition 1, the following result follows.

Proposition 2: An LFSA $\mathcal{S}$ is not strongly periodically detectable if and only if for every positive integer k, there exists $s_k \in L^\omega(\mathcal{S})$ and prefix $s' \sqsubset s_k$ such that for all $s'' \in T^*$, $s's'' \sqsubset s_k$ and $|\ell(s'')| < k$ imply $|\mathcal{M}(\mathcal{S}, \ell(s's''))| > 1$.

Proposition 3 [12]: Consider an LFSA $\mathcal{S}$. For every transition $(q, \sigma, q') \in \delta_{obs}$, for every $\emptyset \neq \bar{q}' \subset q'$ satisfying $|\bar{q}'| = 2$ if $|q'| \geq 2$ and $|\bar{q}'| = 1$ otherwise, there is $\bar{q} \subset q$ such that $(\bar{q}, \sigma, \bar{q}') \in \delta_{det}$, where $|\bar{q}| = 2$ if $|q| \geq 2$.

Theorem 1 [12]: An LFSA $\mathcal{S}$ is not strongly periodically detectable if and only if in its detector $\mathcal{S}_{det}$ as in (3), at least one of the two following conditions holds.

(i) There is a reachable state $q' \in Q$ and $x \in q'$ such that $|q'| > 1$ and there is a transition sequence $x \xrightarrow{s_1} x' \xrightarrow{s_2} x'$ in $\mathcal{S}$ for some $s_1 \in (T_{uo})^*$, $s_2 \in (T_{uo})^+$, $x' \in X$.

(ii) There is a reachable transition cycle such that all states in the cycle have cardinality 2.

In order to check condition (ii), one could firstly use Tarjan algorithm to compute all reachable strongly connected components of $\mathcal{S}_{det}$, which takes time linear in the size of $\mathcal{S}_{det}$; secondly at each component, remove all singletons and then check whether there is a cycle. If and only if in some reachable component, such a cycle exists, (ii) holds. Hence Theorem 1 provides a polynomial-time algorithm for verifying strong periodic detectability. Moreover, Theorem 1 also implies an NL upper bound for strong periodic detectability.

Theorem 2: The problem of verifying strong periodic detectability of LFSA $\mathcal{S}$ belongs to NL.

Example 1: We give two examples to illustrate Theorem 1. Consider two LFSAs $\mathcal{S}_1$ and $\mathcal{S}_2$ shown in Fig. 1. One sees that $\mathcal{S}_1$ satisfies Assumption 1. However, $\mathcal{S}_2$ does not satisfy Assumption 1, as x_1 is a deadlock (violating (1) of Assumption 1), and there is a reachable unobservable transition cycle $x_2 \xrightarrow{t_4} x_2$ (violating (2) of Assumption 1).

Their detectors $\mathcal{S}_{1det}$ and $\mathcal{S}_{2det}$ are shown in Fig. 2. One sees $\mathcal{S}_{1det}$ satisfies (ii) of Theorem 1 because there is a self-loop on reachable state $\{x_1, x_2\}$, but does not satisfy (i) because $\{x_1, x_2\}$ is the unique reachable state of cardinality 2 and in $\mathcal{S}_1$ there is no infinitely long unobservable transition sequence starting at x_1, the same for x_2. $\mathcal{S}_{2det}$ satisfies (i) because $\{x_1, x_2\}$ is reachable in $\mathcal{S}_{2det}$ and in $\mathcal{S}_2$, starting at x_2 there is an infinite-length unobservable transition sequence, but does not satisfy (ii) because there is no cycle all of whose states are of cardinality 2. Hence by Theorem 1, neither $\mathcal{S}_1$ nor $\mathcal{S}_2$ is strongly periodically detectable.

Remark 1: By Example 1, one sees that (i) and (ii) do not imply each other. So they cannot take the place of each other when verifying strong periodic detectability. Let us compare Theorem 1 with Proposition 1. One directly sees that the necessary and sufficient condition for strong periodic detectability under Assumption 1 shown in Proposition 1 is exactly the negation of (ii). By Proposition 1, $\mathcal{S}_2$ is strongly periodically detectable vacuously. Then Proposition 1 does not always work correctly if Assumption 1 is not satisfied.

Next we show that a slight variant of the concurrent-composition structure can also provide an NL upper bound for strong periodic detectability. The concurrent-composition structure has essentially different features compared with the detector structure. On the one hand, a detector tracks output sequences and collects all states between only unobservable transitions and divides them into subsets of cardinality 2. So a detector does not reflect information in unobservable transitions. However, the concurrent-composition structure can do that. On the other hand, a concurrent composition collects all pairs of transition sequences generating the same output sequence, but sometimes does not collect different transitions starting at the same state. However, a detector can do that. For example, consider states x_1, x_2, x_3, x_4 such that $x_1 \neq x_2$ and $x_3 \neq x_4$, there exist transitions $x_1 \xrightarrow{t_1} x_3$, $x_1 \xrightarrow{t_2} x_4$ satisfying $\ell(t_1) = \ell(t_2) \neq \epsilon$, but there is no transition $x_2 \xrightarrow{t_3} x_4$ satisfying $\ell(t_3) = \ell(t_1)$. Then in $\mathcal{S}_{det}$ there is a transition $\{x_1, x_2\} \xrightarrow{\ell(t_1)} \{x_3, x_4\}$, but in $\mathrm{CC_A}(\mathcal{S})$ there is no transition $(x_1, x_2) \xrightarrow{t'} (x_3, x_4)$ for any $t' \in T'$ satisfying $\ell(t') = \ell(t_1)$. Next we add additional transitions into $\mathrm{CC_A}(\mathcal{S})$ to make it depict both the above advantages of a concurrent composition and a detector.

Consider an LFSA $\mathcal{S}$ as in (1) and its self-composition $\mathrm{CC_A}(\mathcal{S})$ as in (4). We construct a variant

$$\mathrm{CC_A^{\leftarrow\epsilon}}(\mathcal{S}) = (X', T' \cup \{\varepsilon\}, X'_0, \delta'_{\leftarrow_\epsilon}) \tag{5}$$

from $\mathrm{CC_A}(\mathcal{S})$ as follows: For all $x_1, x_2, x_3, x_4 \in X$, and $t' \in T'$ such that $x_1 \neq x_2$, $((x_1, x_1), t', (x_3, x_4)) \in \delta'$ (resp., $((x_2, x_2), t', (x_3, x_4)) \in \delta'$), but $((x_1, x_2), \bar{t}', (x_3, x_4)) \notin \delta'$ for any $\bar{t}' \in T'$, add transition $((x_1, x_2), \varepsilon, (x_1, x_1))$ (resp., $((x_1, x_2), \varepsilon, (x_2, x_2))$), where we let $\ell(\varepsilon) = \epsilon$. We call $\mathrm{CC_A^{\leftarrow\epsilon}}(\mathcal{S})$ *ε-extended self-composition* of $\mathcal{S}$.

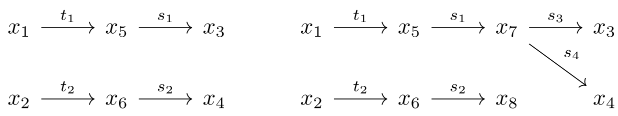

Fig. 3. Case 1 (left). Case 2 (right).

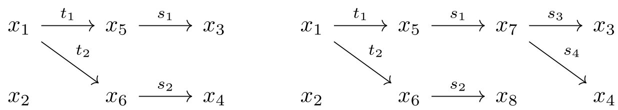

Fig. 4. Case 3 (left). Case 4 (right).

One can see the following proposition.

Proposition 4: Consider an LFSA $\mathcal{S}$ as in (1), its observer $\mathcal{S}_{obs}$ as in (2), its detector $\mathcal{S}_{det}$ as in (3), and its ε-extended self-composition $\mathrm{CC}_{\mathrm{A}}^{\leftarrow\epsilon}(\mathcal{S})$ as in (5). Assume states $x_1, x_2, x_3, x_4 \in X$ such that $x_1 \neq x_2$ and $x_3 \neq x_4$. The following hold.

(iii) For every transition $\{x_1, x_2\} \xrightarrow{\sigma} \{x_3, x_4\}$ in $\mathcal{S}_{det}$, there is an observable transition sequence $(x_1, x_2) \xrightarrow{s'} (x_3, x_4)$ or $(x_1, x_2) \xrightarrow{s'} (x_4, x_3)$ in $\mathrm{CC}_{\mathrm{A}}^{\leftarrow\epsilon}(\mathcal{S})$ such that $\ell(s') = \sigma$.

(iv) For every transition $\{x_1, x_2\} \xrightarrow{\sigma} \{x_3\}$ in $\mathcal{S}_{det}$, there is an observable transition sequence $(x_1, x_2) \xrightarrow{s'} (x_3, x_3)$ in $\mathrm{CC}_{\mathrm{A}}^{\leftarrow\epsilon}(\mathcal{S})$ such that $\ell(s') = \sigma$.

(v) For every transition $\{x_1\} \xrightarrow{\sigma} \{x_3, x_4\}$ in $\mathcal{S}_{det}$, there is an observable transition sequence $(x_1, x_1) \xrightarrow{s'} (x_3, x_4)$ in $\mathrm{CC}_{\mathrm{A}}^{\leftarrow\epsilon}(\mathcal{S})$ such that $\ell(s') = \sigma$.

(vi) For every transition $\{x_1\} \xrightarrow{\sigma} \{x_3\}$ in $\mathcal{S}_{det}$, there is an observable transition sequence $(x_1, x_1) \xrightarrow{s'} (x_3, x_3)$ in $\mathrm{CC}_{\mathrm{A}}^{\leftarrow\epsilon}(\mathcal{S})$ such that $\ell(s') = \sigma$.

(vii) In $\mathrm{CC}_{\mathrm{A}}^{\leftarrow\epsilon}(\mathcal{S})$, consider an arbitrary transition sequence $x'_0 \xrightarrow{s'_0} x'_1 \xrightarrow{t'_1} x'_2 \xrightarrow{s'_1} \dots \xrightarrow{t'_n} x'_{2n} \xrightarrow{s'_n} x'_{2n+1}$, where $x'_0 \in X'_0$, $x'_1, \dots, x'_{2n+1} \in X'$, $s'_0, \dots, s'_n \in (T'_{uo} \cup \{\varepsilon\})^*$, $t'_1, \dots, t'_n \in T'_o$. For every $i \in [\![0, n]\!]$, denote the union of all states of unobservable transition sequence $x_{2i} \xrightarrow{s'_i} x_{2i+1}$ by q_i, then we obtain a sequence $q_0 \xrightarrow{\ell(t'_1)} \dots \xrightarrow{\ell(t'_n)} q_n$. Then for every $i \in [\![1, n]\!]$, there exists $\bar{q}_i \supset q_i$ such that $\mathcal{M}(\mathcal{S}, \epsilon) \xrightarrow{\ell(t'_1)} \bar{q}_1 \xrightarrow{\ell(t'_2)} \dots \xrightarrow{\ell(t'_n)} \bar{q}_n$ is a transition sequence of $\mathcal{S}_{obs}$.

Proof: (iii) We need to consider four different cases of transition sequences in $\mathcal{S}$ (shown in Figs. 3, 4) that form the transition $\{x_1, x_2\} \xrightarrow{\sigma} \{x_3, x_4\}$ in $\mathcal{S}_{det}$, where in these figures, $t_1, t_2 \in T_o$, $\ell(t_1) = \ell(t_2) = \sigma$, $s_1, s_2, s_3, s_4 \in (T_{uo})^*$, $x_5, x_6, x_7, x_8 \in X$.

We need to prove for each case, there is an observable transition sequence $(x_1, x_2) \xrightarrow{s'} (x_3, x_4)$ in $\mathrm{CC}_{\mathrm{A}}^{\leftarrow\epsilon}(\mathcal{S})$ such that $\ell(s') = \sigma$. We only need to consider the most complex Case 4, all the other cases can be dealt with similarly. For Case 4, by definition, the corresponding observable transition sequence is $(x_1, x_2) \xrightarrow{\varepsilon} (x_1, x_1) \xrightarrow{(t_1,t_2)} (x_5, x_6) \xrightarrow{s'_1} (x_7, x_8) \xrightarrow{\varepsilon} (x_7, x_7) \xrightarrow{s'_2} (x_3, x_4)$, where $s'_1(L) = s_1$, $s'_1(R) = s_2$, $s'_2(L) = s_3$, $s'_2(R) = s_4$.

(iv), (v), and (vi) can be proved similarly.

(vii) directly follows from definition. ■

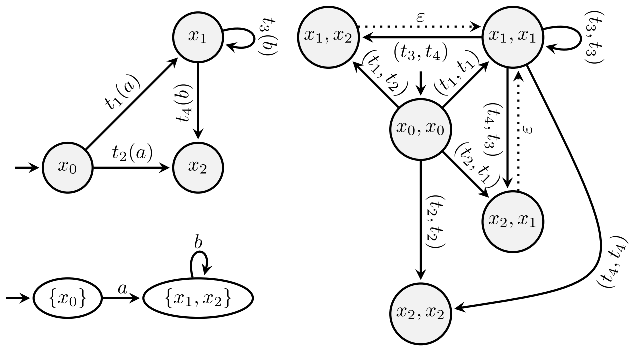

Fig. 5. LFSA $\mathcal{S}_3$ (upper left), its detector $\mathcal{S}_{3det}$ (lower left, the same as observer $\mathcal{S}_{3obs}$), its self-composition $\mathrm{CC}_{\mathrm{A}}(\mathcal{S}_3)$ (right, dotted transitions excluded), and its ε-extended self-composition $\mathrm{CC}_{\mathrm{A}}^{\leftarrow\epsilon}(\mathcal{S}_3)$ (right).

Example 2: Consider LFSA $\mathcal{S}_3$, its detector $\mathcal{S}_{3det}$, and its (ε-extended) self-composition $\mathrm{CC}_{\mathrm{A}}(\mathcal{S}_3)$ ($\mathrm{CC}_{\mathrm{A}}^{\leftarrow\epsilon}(\mathcal{S}_3)$) shown in Fig. 5. There is a transition $\{x_1, x_2\} \xrightarrow{b} \{x_1, x_2\}$ in $\mathcal{S}_{3det}$, but there is neither transition sequence $(x_1, x_2) \xrightarrow{s'} (x_1, x_2)$ nor $(x_1, x_2) \xrightarrow{s'} (x_2, x_1)$ such that $\ell(s') = b$ in $\mathrm{CC}_{\mathrm{A}}(\mathcal{S}_3)$. However, in $\mathrm{CC}_{\mathrm{A}}^{\leftarrow\epsilon}(\mathcal{S}_3)$, there is a transition sequence $(x_1, x_2) \xrightarrow{\varepsilon} (x_1, x_1) \xrightarrow{(t_3,t_4)} (x_1, x_2)$ such that $\ell(\varepsilon(t_3, t_4)) = b$.

With these properties, we are ready to give a new polynomial-time algorithm for verifying strong periodic detectability by using $\mathrm{CC}_{\mathrm{A}}^{\leftarrow\epsilon}(\mathcal{S})$.

Theorem 3: An LFSA $\mathcal{S}$ is not strongly periodically detectable if and only if in its ε-extended self-composition $\mathrm{CC}_{\mathrm{A}}^{\leftarrow\epsilon}(\mathcal{S})$ as in (5), at least one of the two following conditions holds.

(viii) There is a reachable state $(x, \bar{x})$ such that $x \neq \bar{x}$ and there is a transition sequence $x \xrightarrow{s_1} x' \xrightarrow{s_2} x'$ in $\mathcal{S}$ for some $s_1 \in (T_{uo})^*$, $s_2 \in (T_{uo})^+$, $x' \in X$.

(ix) There is a reachable transition cycle $(x_1, \bar{x}_1) \xrightarrow{s'_1} \dots \xrightarrow{s'_n} (x_{n+1}, \bar{x}_{n+1})$ for some positive integer n such that $(x_1, \bar{x}_1) = (x_{n+1}, \bar{x}_{n+1})$, $x_i \neq \bar{x}_i$, and $\ell(s'_i) \in \Sigma$ for all $i \in [\![1, n]\!]$.

Proof: We use Theorem 1 and Propositions 3 and 4 to prove this result.

We first check (viii) is equivalent to (i) of Theorem 1.

"⇒": Assume (viii) holds. By (vii) of Proposition 4 and Proposition 3, for every reachable state (x, x') of $\mathrm{CC}_{\mathrm{A}}^{\leftarrow\epsilon}(\mathcal{S})$ such that $x \neq x'$, either $\{x, x'\} \subset \mathcal{M}(\mathcal{S}, \epsilon)$ or $\{x, x'\}$ is reachable in $\mathcal{S}_{det}$. Hence (i) holds.

"⇐": Assume (i) holds. If $q' = \mathcal{M}(\mathcal{S}, \epsilon)$, then (viii) holds. Otherwise (i.e., in case $|q'| = 2$ and $q' \neq \mathcal{M}(\mathcal{S}, \epsilon)$), by (iii), (iv), (v), (vi) of Proposition 4, one has (viii) holds.

We second check (ix) is equivalent to (ii) of Theorem 1.

"⇒": Assume (ix) holds. By (vii) of Proposition 4 and the Pigeonhole Principle, there is a reachable transition cycle in $\mathcal{S}_{obs}$ none of whose states is a singleton. The by Proposition 3, (ii) holds.

"⇐": Assume (ii) holds. By (iii) of Proposition 4, (ix) holds. ■

Similar to the case that Theorem 1 implies Theorem 2, Theorem 3 also implies an NL upper bound for strong periodic detectability of LFSAs without any assumption.

Example 3: We next use one example to compare Theorem 1 with Theorem 3. Reconsider $\mathcal{S}_3$ in Example 2 (shown in Fig. 5, upper left). The existence of reachable transition cycle $\{x_1, x_2\} \xrightarrow{b} \{x_1, x_2\}$ in $\mathcal{S}_{3det}$ implies that $\mathcal{S}_3$ is not strongly periodically detectable by Theorem 1 (satisfying (ii)). The existence of reachable transition cycle $(x_1, x_2) \xrightarrow{\varepsilon} (x_1, x_1) \xrightarrow{(t_3,t_4)} (x_1, x_2)$ such that $\ell(\varepsilon(t_3, t_4)) = b \in \Sigma$ in $\mathrm{CC}_{\mathrm{A}}^{\leftarrow\epsilon}(\mathcal{S})$ also implies that $\mathcal{S}_3$ is not strongly periodically detectable, by Theorem 3 (satisfying (ix)).

C. Verifying Strong Periodic D-Detectability

We also first characterize the negation of T_{spec}-strong periodic D-detectability. The following result directly follows from Definition 2.

Proposition 5: An LFSA $\mathcal{S}$ is not strongly periodically D-detectable with respect to T_{spec} if and only if for each positive integer k, there exist $s_k \in L^{\omega}(\mathcal{S})$ and $s' \sqsubset s_k$ such that for every $s'' \in T^*$, $|\ell(s'')| < k$ and $s's'' \sqsubset s$ imply $(\mathcal{M}(\mathcal{S}, \ell(s's'')) \times \mathcal{M}(\mathcal{S}, \ell(s's''))) \cap T_{\text{spec}} \neq \emptyset$.

Theorem 4: An LFSA $\mathcal{S}$ is not strongly periodically D-detectable with respect to T_{spec} if and only if in its observer $\mathcal{S}_{obs}$ as in (2), at least one of the two following conditions holds.

(x) There is a reachable state $q \in 2^X$ in $\mathcal{S}_{obs}$ and $x \in q$ such that $(q \times q) \cap T_{\text{spec}} \neq \emptyset$ and there is a transition sequence $x \xrightarrow{s_1} x' \xrightarrow{s_2} x'$ in $\mathcal{S}$ for some $s_1 \in (T_{uo})^*$, $s_2 \in (T_{uo})^+$, $x' \in X$.

(xi) There is a reachable transition cycle such that each state q of the cycle satisfies $(q \times q) \cap T_{\text{spec}} \neq \emptyset$.

Theorem 5: The problem of verifying strong periodic D-detectability with respect to T_{spec} belongs to PSPACE.

Proof: Condition (x) can be checked by guessing $q \in 2^X$, $x \in q$, and $x' \in X$ and doing the corresponding checks by nondeterministic research. Since each state q of $\mathcal{S}_{obs}$ is bounded by the number of states of $\mathcal{S}$, and $(q \times q) \cap T_{\text{spec}} \neq \emptyset$ can be checked in time quadratic in the number of states of $\mathcal{S}$, (x) can be checked in NPSPACE.

Condition (xi) can be checked by nondeterministically guessing a sequence of label sequence and checking whether the sequence leads $\mathcal{S}_{obs}$ to such a transition cycle. Hence, (x) can also be checked in NPSPACE.

Hence by Theorem 4, the problem of verifying strong periodic D-detectability with respect to T_{spec} belongs to coNPSPACE, i.e., PSPACE. ■

Remark 2: One directly sees that the necessary and sufficient condition for strong periodic D-detectability of LFSAs under Assumption 1 given in [2, Th. 9] (collected in Proposition 1) is exactly the negation of (xi) in Theorem 4. So the algorithm induced from [2, Th. 9] usually does not work correctly without Assumption 1. See the following example.

Reconsider $\mathcal{S}_2$ (shown in Fig. 1, right) and its observer $\mathcal{S}_{2obs}$ (shown in Fig. 2, right). As shown in Example 1, $\mathcal{S}_2$ violates Assumption 1. Now choose $T_{\text{spec}} = \{(x_1, x_2)\}$. For every positive integer k, choose $s_k = t_2(t_4)^{\omega} \in L^{\omega}(\mathcal{S}_2)$, then for all $(t_4)^n$, where $n \geq 0$, one has $t_2(t_4)^n \sqsubset s_k$, $|\ell((t_4)^n)| = 0 < k$, $\ell(t_2(t_4)^n) = a$, and $(\mathcal{M}(\mathcal{S}_2, a) \times \mathcal{M}(\mathcal{S}_2, a)) \cap T_{\text{spec}} = T_{\text{spec}} \neq \emptyset$. That is, $\mathcal{S}_2$ is not strongly periodically D-detectable with respect to T_{spec} by definition. However, since there is no cycle in $\mathcal{S}_{2obs}$, the condition "every reachable transition cycle contains at least one state q such that $(q \times q) \cap T_{\text{spec}} = \emptyset$" in Proposition 1 is satisfied vacuously. Thus, $\mathcal{S}_{2obs}$ is strongly periodically D-detectable with respect to T_{spec} by Proposition 1, which is incorrect.

Example 4: We next illustrate Theorem 4. Reconsider LFSA $\mathcal{S}_3$ in Fig. 5 (upper left) and its observer $\mathcal{S}_{3obs}$ in Fig. 5 (lower left). If we choose $T^1_{\text{spec}} = \{(x_1, x_2)\}$, the existence of self-loop $\{x_1, x_2\} \xrightarrow{b} \{x_1, x_2\}$ in $\mathcal{S}_{3obs}$ satisfies (xi) of Theorem 4 (i.e., $(\{x_1, x_2\} \times \{x_1, x_2\}) \cap T^1_{\text{spec}} = T^1_{\text{spec}} \neq \emptyset$), hence $\mathcal{S}_3$ is not strongly periodically D-detectable with respect to T^1_{spec}. If we choose $T^2_{\text{spec}} = \{(x_0, x_2)\}$, then by $\mathcal{S}_{3obs}$, one sees neither (x) nor (xi) is satisfied, hence $\mathcal{S}_3$ is strongly periodically D-detectable with respect to T^2_{spec}.

IV. Conclusion

In this letter, we obtained an NL upper bound for verifying strong periodic detectability of LFSAs without any assumption, strengthening the related results given in [2], [6] under two assumptions of deadlock-freeness and divergence-freeness. We also obtained a PSPACE upper bound for verifying strong periodic D-detectability of LFSAs without any assumption, strengthening the related result given in [2], [7] also under the two assumptions.

As shown in our previous paper [3], the self-composition method can be used to verify (delayed) strong detectability of LFSAs without any assumption, but the detector method cannot. In this letter, we showed that both the detector method and a variant of the self-composition method can be used to verify strong periodic detectability of LFSAs without any assumption. It is an interesting future topic to study the intrinsic relationships between the two methods.

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