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Robust Adaptive Prescribed Performance Control for Unknown Nonlinear Systems with Input Amplitude and Rate Constraints

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***Abstract*— A novel approximation-free prescribed performance control scheme for unknown nonlinear systems with amplitude and rate saturation on the control input signal is designed in this paper. The proposed control strategy employs nested smooth saturation functions and introduces a reconciling relaxation of the performance constraints based on the actuator limitations. The straightforward gain selection along with the low complexity of the control scheme, simplifies the practical implementation of our algorithm. The adoption of the adaptive prescribed performance control technique ensures the desired trade-off between input and output constraints for any input-to-state stable (ISS) system. Inevitably, for generic systems, the boundedness properties are guaranteed only locally; thus, we provide a sufficient boundedness condition for all closed-loop signals. Finally, an illustrative simulation study is conducted to demonstrate the efficacy of the proposed approach.**

***Index Terms*— Prescribed performance control, input saturation, unknown systems, adaptive control.**

I. INTRODUCTION

The development of effective controllers for input constrained nonlinear systems subjected to output performance specifications constitutes a significant step towards versatile control applications. In many practical applications the output of the system should be enforced to track a reference trajectory and meet predefined transient and steady-state performance specifications. Nevertheless, input saturation standing as an ubiquitous feature in control systems, since all actuators undergo limitations with respect to both the amplitude and the rate of the control signal they provide to the system, may yield performance degradation or even closed-loop instability. The significance of designing control schemes respecting constraints in both amplitude and rate of the control signal is justified by various tragic incidents, such as numerous aircraft accidents owing to unstable pilot-induced-oscillations as well as the disastrous meltdown of the Chernobyl nuclear station (see the survey paper [1] and the references therein). Jerk (i.e., the rate of acceleration) limits are often used in the design of autonomous vehicles, where rapid acceleration or deceleration can cause discomfort or even injury to passengers [2]. In robotics and manufacturing, jerk limits are critical to minimize mechanical stress on components, which may result in premature wear or failure [3]. Therefore, it is crucial to develop control strategies that can handle output in combination with input constraints to ensure reliable and safe operation of closed-loop systems.

The majority of research on input amplitude and rate saturation (ARS) has focused on linear systems, reflecting the inherent difficulty to compensate for this type of input nonlinearity. Regarding nonlinear systems subject to ARS, a constrained adaptive backstepping control scheme was proposed in [4]. The authors in [5] designed a control framework based on model reference adaptive control that robustifies the error dynamics against input constraints. A state feedback controller based on a disturbance observer utilizing a linear matrix inequality (LMI) approach was presented in [6]. Finally, a backstepping-based controller providing finite-time stabilization for a rigid spacecraft was proposed in [7], leveraging a second-order anti-windup model.

On the other hand, two main approaches dealing with transient and steady state performance characteristics for unknown nonlinear systems have been developed over the past few years. Funnel control (FC) [8] (which is an extension of adaptive high-gain control methodology) and Prescribed Performance Control (PPC) [9], [10] (which introduces a transformation of the constrained control problem into an unconstrained one) guarantee the evolution of the output tracking error strictly within any user-specified performance envelope via bounded control signals. Nevertheless, input constraints may lead both control methods to singularity when the limited control effort fails to maintain the output error within the performance funnel, (i.e., the control signal diverges to infinity as the tracking error approaches the performance bounds, leading in unbounded closed-loop signals). In this vein, input saturation constitutes a problem of pivotal importance within an output constrained control framework. Works based on PPC [11], [12] as well as on FC [13] propose control strategies exploiting adaptive performance boundaries that provide a balance between output constraints and the feasible control effort owing to saturation in the amplitude of the input signal. An alternative output constrained control strategy that encounters amplitude saturation for uncertain nonlinear systems was developed in [14], utilizing Control Zeroing Barrier Functions. Nevertheless, none of the aforementioned efforts considered rate constraints.

In this paper, we elaborate on the PPC technique to introduce a robust approximation-free control scheme with adaptive prescribed performance for unknown $n-$th order SISO nonlinear systems in the canonical form under input saturation regarding both the amplitude and the rate of the control signal. In particular, we develop a low-complexity control algorithm that efficiently adapts the performance

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boundaries based on the current status of the saturation, accomplishing a reconciliation between the desired performance specifications and the feasible control capabilities. Moreover, the implementation of our controller is facilitated by a simple gain selection. Notably, this is the first time a control scheme has been proposed to address transient and steady-state performance specifications for uncertain nonlinear systems subject to amplitude and rate input constraints.

II. Problem Formulation and Preliminaries

Consider an n-th order nonlinear dynamical system in the canonical form:

$$\begin{aligned} \dot{x}_i &= x_{i+1},\ i = 1,\dots,n-1 \\ \dot{x}_n &= f(x) + g(x)u + d(t), \\ y &= x_1 \end{aligned} \tag{1}$$

where $x = [x_1, x_2, ..., x_n]^T \in \mathbb{R}^n$ denotes the state vector, which is considered available for measurement, $y \in \mathbb{R}$ is the output, $f(x),\ g(x)$ are unknown, locally Lipschitz nonlinear scalar functions and $d(t)$ models bounded and piece-wise continuous external disturbance effects. Moreover, both the control input u and its rate $\dot{u}$ are constrained within the compact sets $\mathcal{U} := [-\bar{u}, \bar{u}],\ \mathcal{R} := [-\bar{r}, \bar{r}]$ for some known positive constants $\bar{u},\ \bar{r}$, respectively. Finally, consider a bounded reference trajectory $y_d(t)$ with known bounded up to $n-1$-th order derivatives as well as the state reference vector $x_d(t) = [y_d(t), \frac{d}{dt}y_d(t), \dots, (\frac{d}{dt})^{n-1}y_d(t)] \in \Omega_{x_d} \subseteq \mathbb{R}^n$. In this paper, assuming no prior knowledge on the system nonlinearities $f(x)$, $g(x)$ and the external disturbances $d(t)$, we aim at designing an approximation-free, state feedback controller for the input constrained system (1) that meets the following properties:

- The boundedness of all closed-loop signals is guaranteed.
- The desired trajectory $y_d(t)$ is tracked with adaptive prescribed transient and steady state performance specifications.

Finally, to solve the aforementioned problem, we pose the following assumptions:

Assumption 1: The sign of the unknown function $g(x)$ is known and there exists a positive constant g^* such that $\inf_{x\in\mathbb{R}^n}\{|g(x)|\} = g^*$. Without loss of generality it is assumed that $g(x)$ is positive.

Assumption 2: There exists an unknown positive constant D such that $|d(t)| \leq D,\ \forall t \geq 0$ which implies the boundedness of the external disturbances d.

A. Preliminaries on PPC

In PPC, the prescribed performance characteristics concern the convergence of the output tracking error to a predefined and arbitrarily small residual set with predetermined minimum convergence rate. In particular, according to the approximation-free approach proposed in [10], given a reference trajectory $y_d(t)$ and any initial condition $y(0)$, we define the measurable tracking error $e(t) = y(t) - y_d(t)$. Subsequently, we select the performance function:

$$\rho(t) = (\rho(0) - \rho_\infty)\exp(-\lambda t) + \rho_\infty \tag{2}$$

where ρ_∞ denotes the maximum size of $e(t)$ at the steady state and λ determines its minimum convergence rate. Thus, prescribed performance on the tracking error is imposed if $e(t)$ evolves strictly within the performance envelope circumscribed by the performance functions $\rho(t)$ and $-\rho(t)$ for all $t \geq 0$. Note that the initial tracking error $e(0)$ should be included in the performance envelope at $t = 0$, (i.e. $\rho(0) > |e(0)|$).

To enforce prescribed performance on system (1), let us first define the linear filter:

$$s(\tilde{e}(t)) = q^T\tilde{e}(t) = \left[\prod_{i=1}^{n-1}\left(\frac{d}{dt} + r_i\right)\right] e(t) \tag{3}$$

where $\tilde{e}(t) = [\tilde{e}_1(t), \dots, \tilde{e}_n(t)]^T \in \mathbb{R}^n$ denotes the state tracking error, with $\tilde{e}_i(t) = x_i(t) - x_{d_i}(t),\ i = 1, \dots, n$, and $q = [q_1, \dots, q_{n-1}, 1]^T \in \mathbb{R}^n$ contains the coefficients of a Hurwitz polynomial $s^{n-1} + q_{n-1}s^{n-2} + \dots + q_2 s + q_1$ with real negative roots $-r_i < 0,\ i = 1, \dots, n-1$.

Proposition 1: [15] (pp. 277) The tracking problem of (1) is equivalent to driving the state error $\tilde{e}(t)$ on the invariant boundary layer $S := \{\tilde{e} \in \mathbb{R}^n : |s(\tilde{e})| \leq \bar{s}\}$ with $\bar{s}$ denoting a positive constant. If $s(\tilde{e}(0)) \in S$ then all $\tilde{e}_i(t),\ i = 1, \dots, n$ are bounded with respect to compact sets whose size is determined by $\bar{s}$.

According to this Proposition, bounds on (3) can be directly translated into bounds on the tracking error vector $\tilde{e}(t)$. Thus, $s(\tilde{e})$ represents an actual output performance metric of (1).

Proposition 2: [16] Consider the state tracking error $\tilde{e}(t)$ of system (1) as well as the metric $s(\tilde{e}(t))$ as defined in (3) and the conventional performance function (2) with $\lambda < \min\{r_i\},\ i \in \{1, \dots, n-1\}$. If $|s(\tilde{e}(t))| < \rho(t), \forall t \geq 0$, it follows that all $\tilde{e}_i(t)$ converge to the compact sets $E_i := \{\tilde{e}_i \in \mathbb{R} : |\tilde{e}_i| \leq \frac{2^{i-1}\rho_\infty}{\prod_{j=1}^{n-i} r_j}\}$ with minimum convergence rate $\exp(-\lambda t)$ for all $i = 1, \dots, n$.

III. Adaptive PPC Design

In this section, we first present an adaptive PPC control scheme and subsequently we prove that it solves the problem of input-output constrained control. Initially, let us define the state tracking error vector $\tilde{e}(t) = x(t) - x_d(t)$ of (1) and the scalar metric $s(\tilde{e}(t))$ according to (3). Thenceforward, we shall proceed imposing performance constraints on $s(\tilde{e}(t))$ since it incorporates performance specifications of the output tracking error $e(t) = y(t) - y_d(t)$ as well, as dictated by Propositions 1 and 2. Subsequently, let us define the error transformation function $\mathcal{T}(\chi) = \frac{1}{2}\ln\left(\frac{1+\chi}{1-\chi}\right)$ as well as its derivative $\mathcal{D}(\chi) = \frac{1}{1-\chi^2}$. The input constraints are encapsulated by a saturation function $\text{sat}_\sigma : \mathbb{R} \to [-\sigma, \sigma]$, where $\sigma > 0$ denotes the known saturation level, (i.e., $\bar{u}$ and

$\bar{r}$ for the amplitude and rate saturation, respectively). In this work, we adopt a differentiable saturation function:

$$\mathrm{sat}_\sigma(\chi) = \begin{cases} \chi & \text{if } |\chi| < \sigma - \beta \\ p(\chi) & \text{if } |\chi| \in [\sigma-\beta, \sigma+\beta] \\ s_\chi \sigma & \text{if } |\chi| > \sigma + \beta \end{cases}$$

where $p(\chi) = -\frac{1}{4\beta}\left(\chi^2 - 2s_\chi(\sigma+\beta)\chi + (s_\chi\sigma - s_\chi\beta)^2\right)$, with s_χ denoting the sign of χ and $\beta = 10^{-6}$ a small smoothing parameter. The design procedure of the control scheme is described as follows:

I-a. Select the desired control signal as:

$$u_d(t) = -k_1 \mathcal{D}\left(\frac{s(\tilde{e}(t))}{\rho_1(t)}\right) \mathcal{T}\left(\frac{s(\tilde{e}(t))}{\rho_1(t)}\right),\ k_1 > 0 \tag{4}$$

where $\mathcal{D} : (-1, 1) \to \mathbb{R}_+^*$ acts as a scaling factor.

I-b. Design the adaptive performance function incorporating the output performance specifications and the saturation on the input amplitude as:

$$\dot{\rho}_1 = -\lambda_1(\rho_1(t) - \rho_{1,\infty}) + \gamma_1\left(\frac{\mathrm{sat}_{\bar{u}}(u_d(t)) - u_d(t)}{s(\tilde{e}(t))/\rho_1(t)}\right) \tag{5}$$

with $\lambda_1, \gamma_1, \rho_{1,\infty} > 0$ and $\rho_1(0) > |s(\tilde{e}(0))|$, where $\lambda_1, \rho_{1,\infty}$ denote the desired minimum exponential convergence rate and the maximum absolute value of steady state error, respectively and $\mathrm{sat}_{\bar{u}}(u_d)$ corresponds to the amplitude constrained control signal.

II-a. Design the intermediate control signal:

$$a(t) = -k_2 \mathcal{D}\left(\frac{e_2(t)}{\rho_2(t)}\right) \mathcal{T}\left(\frac{e_2(t)}{\rho_2(t)}\right),\ k_2 > 0 \tag{6}$$

with $e_2 := u(t) - \mathrm{sat}_{\bar{u}}(u_d(t))$. Then, the dynamical controller subjected to both amplitude and rate saturation, is given by:

$$\dot{u} = \mathrm{sat}_{\bar{r}}(a(t)),\ u(0) \in \mathcal{U}. \tag{7}$$

Therefore, the control effort applied by the actuators of the constrained system (1) is obtained by integrating (7) over time.

II-b. Finally, similarly to (5), design the second adaptive performance function incorporating the rate saturation on the control signal as:

$$\dot{\rho}_2 = -\lambda_2(\rho_2(t) - \rho_{2,\infty}) + \gamma_2\left(\frac{\mathrm{sat}_{\bar{r}}(a(t)) - a(t)}{e_2(t)/\rho_2(t)}\right) \tag{8}$$

with $\lambda_2, \gamma_2, \rho_{2,\infty} > 0$ and $\rho_2(0) > |u(0) - \mathrm{sat}_{\bar{u}}(u_d(0))|$.

Remark 1: The proposed actuator model, which incorporates two nested saturation constraints was inspired by [17]. More specifically, notice that the dynamical model of the actuator (7) represents a stable nonlinear system consolidating both amplitude and rate limitations. In particular, the variation of the control signal is limited owing to the nonlinear function $\mathrm{sat}_{\bar{r}}(\chi)$ directly affecting the control dynamics (7). Additionally, starting within the compact set $\mathcal{U}$, $|u(t)|$ will never exceed $\bar{u}$ for all $t \geq 0$ since $\dot{u} \to 0$ as $u(t) \to sat_{\bar{u}}(u_d(t))$. Consequently, the control input $u(t)$ is absolutely bounded by the amplitude saturation limit $\bar{u}$.

Remark 2: Note that the idea of dynamically adapting the performance functions first appeared in [18] for the decentralized control of multi-agent systems, when conflicting situations arise, and recently was extended into the input constrained problem within the FC [13] and PPC [12] approach. The first term of the adaptive performance functions (5) and (8) stands for the dynamics of the conventional exponential performance function (2), whereas the non-negative second term is activated when the corresponding input saturation occurs, (i.e., amplitude saturation for (5) and rate saturation for (8)), in order to reconcile the input and output constraints by properly relaxing the performance boundaries. Note that both adaptive performance functions are well defined, since the second term of (5), (8) vanishes as $s(\tilde{e}(t)) \to 0$ and $e_2(t) \to 0$, respectively, owing to the fact that the saturation is not active for a non-empty set around $s(\tilde{e}) = 0$ ans $e_2 = 0$. Thus, the trade-off between input and output constraints is eliminated when the saturation is not active and the performance functions are restored to their conventional form with exponential speed, determined by λ_1 and λ_2.

The following theorem summarizes the main results.

Theorem 1: Consider system (1) initializing within a compact set Ω_x and a reference trajectory $x_d(t) \in \Omega_{x_d} \subset \Omega_x$, as well as the error metric (3) with $w_1 = \prod_{j=1}^{n-1} r_j^{-1}$. For sufficiently large $\bar{u}$ and $\bar{r}$, there exists an upper bound $\bar{\rho}_1$ of the performance function $\rho_1(t)$ such that the proposed adaptive control scheme (4)-(8) guarantees:

$$-w_1\rho_1(t) < y(t) - y_d(t) < w_1\rho_1(t)$$

with $\rho_1(t) \leq \bar{\rho}_1$ and $x(t) \in \Omega_x,\ \forall t \geq 0$.

Proof: Let us first define the normalized tracking errors $\xi_1 := \frac{s(\tilde{e}(t))}{\rho_1(t)}$, $\xi_2 := \frac{e_2(t)}{\rho_2(t)}$ and the constants $w_i = \frac{2^{i-1}}{\prod_{j=1}^{n-i} r_j}$, $i = 1, \ldots, n$. Differentiating ξ_1 with respect to time and substituting (1), (3), (4), (5) as well as adding and subtracting $g(x)\mathrm{sat}_{\bar{u}}(k_1\mathcal{D}(\xi_1)\mathcal{T}(\xi_1))$ and defining $l_y := \left(\frac{d}{dt}\right)^n y_d(t) + \sum_{i=1}^{n-1} q_i \tilde{e}_{i+1}$ we obtain:

$$\begin{aligned}\dot{\xi}_1 =& \tfrac{1}{\rho_1(t)}(h(x, x_d, t) + \xi_1(t)\lambda_1(\rho_1(t) - \rho_{1,\infty}) \\ &+ (\gamma_1 - g(x))\mathrm{sat}_{\bar{u}}(k_1\mathcal{D}(\xi_1)\mathcal{T}(\xi_1)) - \gamma_1 k_1 \mathcal{D}(\xi_1)\mathcal{T}(\xi_1))\end{aligned} \tag{9}$$

where:

$$h(x, x_d, t) = f(x) + g(x)e_2(t) + d(t) - l_y(t). \tag{10}$$

Following the same reasoning and substituting (1),(6),(7),(8), the dynamics of ξ_2 can be written as:

$$\begin{aligned}\dot{\xi}_2 =& \tfrac{1}{\rho_2(t)}(-k_1 S(\xi_1)\dot{\xi}_1 + \xi_2(t)\lambda_2(\rho_2(t) - \rho_{2,\infty}) \\ &+ (\gamma_2 - 1)\mathrm{sat}_{\bar{r}}(k_2\mathcal{D}(\xi_2)\mathcal{T}(\xi_2)) - \gamma_2 k_2 \mathcal{D}(\xi_2)\mathcal{T}(\xi_2))\end{aligned} \tag{11}$$

where $S(\xi_1) := \mathrm{sat}_{\bar{u}}'(-k_1\mathcal{D}(\xi_1)\mathcal{T}(\xi_1))(\mathcal{D}'(\xi_1)\mathcal{T}(\xi_1) + \mathcal{D}^2(\xi_1))$. To proceed, let us define the augmented state vector $\zeta = [x^T, \xi_1, \xi_2, \rho_1, \rho_2]^T$. Differentiating ζ with respect to time the closed-loop dynamical system of ζ can be written in compact form:

$$\dot{\zeta} = \phi(t, \zeta(t)). \tag{12}$$

Next, let us define the open set $\Omega := \Omega_x \times (-1, 1) \times (-1, 1) \times (0, \bar{\rho_1}) \times (0, \bar{\rho_2})$. Thenceforward, the proof proceeds in three phases. First the existence of a unique maximal solution $\zeta : [0, \tau_{\max}) \to \Omega$ of (12) is ensured, i.e., $\zeta(t) \in \Omega,\ \forall t \in [0, \tau_{\max})$. Next, we establish a sufficient condition regarding the amplitude and rate saturation level $\bar{u}$ and $\bar{r}$, such that the proposed control scheme guarantees the boundedness of all closed-loop signals of (12) for all $t \in [0, \tau_{\infty})$. Finally, we prove that ζ remains strictly within a compact subset of Ω, which leads to $\tau_{max} = \infty$ by contradiction.

Phase A. Consider the closed-loop dynamical system (12). By construction, it holds that $|\xi_i(0)| < 1, \rho_i(0) \in (0, \bar{\rho}_i),\ i \in \{1, 2\}$ (see Section III - Steps *I-b, II-b*) and therefore $\zeta(0) \in \Omega$. Moreover, $\phi : \Omega \to \mathbb{R}^{n+4}$ obeys the piece-wise continuity and locally integrability on t as well as locally Lipschitz on ζ properties over the open set Ω. Thus, invoking Theorem 54 in [19] (pp. 476) we conclude the existence and uniqueness of a maximal solution $\zeta : [0, \tau_{\max}) \to \Omega,\ \forall t \in [0, \tau_{\max})$.

Phase B. In Phase A, we showed that $\zeta(t) \in \Omega,\ \forall t \in [0, \tau_{\max})$, which implies that the transformed errors $\epsilon_i := \mathcal{T}(\xi_i(t)),\ i \in \{1, 2\}$ are well defined for all $t \in [0, \tau_{\max})$ since $\xi_i(t) \in (-1, 1),\ i \in \{1, 2\}$. Utilizing the latter, we conclude that $s(\tilde{e}(t))$ is absolutely bounded by a positive constant $\rho^* < \bar{\rho}_1,\ \forall t \in [0, \tau_{\max})$. Hence, employing Proposition 1, we conclude that $s(\tilde{e}(t)) \in \Omega_{\tilde{e}},\ \forall t \in [0, \tau_{\max})$ with $\Omega_{\tilde{e}} := \{\tilde{e}(t) \in \mathbb{R}^n : |\tilde{e}_i| \le w_i \rho^*,\ \forall i = 1, \ldots, n\}$. Note that $x_i(t) = \tilde{e}_i(t) + (d/dt)^{i-1} y_d(t),\ i = 1, \ldots, n$. Owing to the boundedness of the reference signal and the fact that $s(\tilde{e}(t)) \in \Omega_{\tilde{e}} \subset \mathbb{R}^n,\ \forall t \in [0, \tau_{\max})$, we conclude that there exists a compact set $\Omega'_x \subset \Omega_x$ such that $x(t) \in \Omega'_x,\ \forall [0, \tau_{\max})$. Next consider the positive definite and radially unbounded Lyapunov function candidate $V_1 = \frac{1}{2}\epsilon_1^2$. Exploiting the fact that $s(\tilde{e}(t)) \in \Omega_{\tilde{e}},\ x(t) \in \Omega_x,\ \rho_2(t) \in (0, \bar{\rho}_2),\ \xi_2(t) \in (-1, 1),\ \forall t \in [0, \tau_{\max})$ as well as the locally Lipschitz property of $f(x),\ g(x)$ and the boundedness of $d(t)$, the fact that $x_d(t) \in \Omega_{x_d}$ and invoking the Extreme Value Theorem we conclude that $\|h\|_\infty \le \bar{H},\ \forall t \in [0, \tau_{\max})$. Additionally, since $|\mathrm{sat}_{\bar{u}}(k_1 \mathcal{D}(\xi_1)\epsilon_1)| \le \bar{u}$, by differentiating V_1 with respect to time and substituting (9), we obtain $\dot{V}_1 \le \frac{|\epsilon_1(t)|}{(1-\xi_1^2(t))\rho_1(t)}(\bar{H} + |\xi_1(t)|\lambda_1(\bar{\rho}_1 - \rho_{1,\infty}) + (\gamma_1 - g^*)\bar{u} - \gamma_1 k_1 \mathcal{D}(\xi_1)|\epsilon_1(t)|)$.

Moreover, owing to the fact that $\xi_1(t) \in (-1, 1), \forall t \in [0, \tau_{\max})$ it holds $\frac{1}{(1-\xi_1^2)} > 1$, whereas $\rho_1(t) \ge \rho_{1,\infty} > 0$ by construction. Therefore, $\epsilon_1(t)$ is ultimately bounded with respect to a compact set $\mathcal{E}_1 := \left\{\epsilon_1 : |\epsilon_1| \le \bar{\epsilon}_1 = \max_{\zeta \in \Omega}\{|\epsilon_1(0)|, C_1\}\right\}$, with $C_1 = \frac{\bar{H} + \bar{\xi}_1 \lambda_1(\bar{\rho}_1 - \rho_{1,\infty}) + (\gamma_1 - g^*)\bar{u}}{k_1 \mathcal{D}(\bar{\xi}_1)\gamma_1}$. Note that the existence of a performance bound $\bar{\rho}_1$ such that the right hand side of (5) becomes negative when $\rho_1(t) = \bar{\rho}_1$ and $|\epsilon_1(t)| = \epsilon_{1,\bar{\rho}} < \bar{\epsilon}_1$, implies the existence of a small positive constant δ_1 such that $\rho_1(t) \in [\rho_{1,\infty}, \bar{\rho}_1 - \delta_1], \forall t \in [0, \tau_{\max})$. In particular, exploiting (5), we conclude that $\rho_1(t) \in \Omega_{\rho_1}$, with $\Omega_{\rho_1} := [\rho_{1,\infty}, \bar{\rho}_1 - \delta_1]$, when the amplitude saturation level $\bar{u}$ satisfies the following inequality:

$$\bar{u} > \frac{1}{\gamma_1}\left(k_1 \gamma_1 \mathcal{D}(\xi_{1,\bar{\rho}})\epsilon_{1,\bar{\rho}} - \xi_{1,\bar{\rho}}\lambda_1(\bar{\rho}_1 - \rho_{1,\infty})\right) \qquad (13)$$

with $\xi_{1,\bar{\rho}} = \mathcal{T}^{-1}(\epsilon_{1,\bar{\rho}})$. Note that the state vector of the initial system (1) can be written as $x(t) = \tilde{q}\rho_1(t)\xi_1(t) + x_d(t)$ with $\tilde{q} = \frac{q}{qq^T}$. Based on the aforementioned analysis for a sufficiently large saturation level $\bar{u}$ there exists a compact set $\Omega_s := \{s \in \mathbb{R}^n : s \le \tilde{q}_i(\bar{\rho}_1 - \delta)\bar{\xi}_1,\ i = 1, \ldots n\}$ such that the closed-loop signals of system (1) remain bounded within the Minkowski sum $\Omega_s + \Omega_{x_d} \subset \Omega_x,\ \forall t \in [0, \tau_{\max})$. Note that the size of Ω_s is heavily affected by $\bar{H}$, (i.e., the coupling among the unknown dynamics of the system as well as the disturbances upper bound D and the size of $e_2(t)$, which is influenced by the rate saturation level $\bar{r}$). Similarly to V_1, we consider the Lyapunov function candidate $V_2 = \frac{1}{2}\epsilon_2^2$. Differentiating with respect to time, substituting (11), exploiting $\rho_2(t) \in (0, \bar{\rho}_2), \xi_2(t) \in (-1, 1), \forall t \in [0, \tau_{\max})$ as well as the fact that $|\mathrm{sat}_{\bar{r}}(k_2 \mathcal{D}(\xi_2)\epsilon_2)| \le \bar{r}$ and $|k_1 S(\xi_1)\dot{\xi}_1| \le F$, which derives from the boundedness of ϵ_1, we get $\dot{V}_2 \le \frac{|\epsilon_2(t)|}{(1-\xi_2^2(t))\rho_2(t)}(F + |\xi_2(t)|\lambda_2(\bar{\rho}_2 - \rho_{2,\infty}) + (\gamma_2 - 1)\bar{r} - \gamma_2 k_2 \mathcal{D}(\bar{\xi}_2)|\epsilon_2(t)|)$. Thence, we conclude that $\epsilon_2(t)$ is ultimately bounded with respect to a compact set $\mathcal{E}_2 := \left\{\epsilon_2 : |\epsilon_2| \le \bar{\epsilon}_2 = \max_{\zeta \in \Omega}\{|\epsilon_2(0)|, C_2\}\right\}$, with $C_2 = \frac{F + \bar{\xi}_2 \lambda_2(\bar{\rho}_2 - \rho_{2,\infty}) + \gamma_2 \bar{r}}{k_2 \mathcal{D}(\bar{\xi}_2)\gamma_2}$. Note that the tracking error $e_2(t) \in [-2\bar{u}, 2\bar{u}]$ is uniformely bounded owing to the amplitude saturation. By letting $\bar{e}_2 = 2\bar{u}$ and exploiting (8), we obtain:

$$\dot{\rho}_2 \le -\lambda_2(\rho_2(t) - \rho_{2,\infty}) + \gamma_2 \frac{k_2 \mathcal{D}\left(\frac{\bar{e}_2}{\rho_2(t)}\right)\mathcal{T}\left(\frac{\bar{e}_2}{\rho_2(t)}\right) - \bar{r}}{\xi_2(t)}. \quad (14)$$

Notice that the second term of (14) vanishes as $|\xi_2(t)| \to 0$ or $\rho_2(t) \to \bar{\rho}_2$, yielding the right-hand side of (14) negative when the rate saturation is not active or $\rho_2(t) = \bar{\rho}_2$. Thus $\rho_2(t) \in \Omega_{\rho_2},\ \forall t \in [0, \tau_{\max})$ with $\Omega_{\rho_2} := [\rho_{2,\infty}, \bar{\rho}_2 - \delta_2]$, $\delta_2 > 0$ for any rate limit $\bar{r}$. Nevertheless, $\bar{r}$ affects the size of the compact set $\Omega_s + \Omega_{x_d}$ within which the closed-loop signals remain bounded.

Phase C. In Phase B, we proved that $x(t) \in \Omega'_x, \rho_i(t) \in \Omega_{\rho_i},\ i \in \{1, 2\}$ as well as the boundedness of the transformed errors $\epsilon_i(t) = \mathcal{T}(\xi_i(t)),\ i \in \{1, 2\}$ for all $t \in [0, \tau_{\max})$ within the compact sets $\mathcal{E}_i\ i \in \{1, 2\}$ for sufficiently large saturation levels $\bar{u}, \bar{r}$. Combining this with the inverse of mapping $\mathcal{T}$, we conclude the boundedness of the normalized errors $\xi_i(t)\ i \in \{1, 2\}$ within the compact sets $\Omega_{\xi_i} := [\underline{\xi}_i, \bar{\xi}_i] \subset (-1, 1),\ i \in \{1, 2\}$. Note that $\zeta(t) \in \Omega' \subset \Omega, \forall t \in [0, \tau_{\max})$, with $\Omega' := \Omega'_x \times \Omega_{\xi_1} \times \Omega_{\xi_2} \times \Omega_{\rho_1} \times \Omega_{\rho_2}$. Thus, assuming that $\tau_{\max} < \infty$, then according to Proposition C.3.6 in [19] (pp. 481) there exists a time instant τ' such that $\zeta(\tau') \notin \Omega'$, which is a clear contradiction. As a consequence, $\tau_{\max} = \infty$, which implies the state boundedness of the initial system (1). Finally, invoking Proposition 1 and the fact that $\xi_1(t) \in \Omega_{\xi_1},\ \forall t \ge 0$ we conclude that $-w_i\bar{\rho}_1 < -w_i\rho_1(t) < x_i(t) - x_{d_i}(t) < w_i\rho_1(t) < w_i\bar{\rho}_1$ for all $t \ge 0$ and $i = 1, \ldots, n$ within a compact set $\Omega_s + \Omega_{x_d} \subset \Omega_x$. Note that $x_1(t) - x_{d_1}(t)$ denotes the output tracking error $e(t) = y(t) - y_d(t)$, which evolves strictly within the adaptive performance envelope, thus completing the proof. ■

Remark 3: Theorem 1 guarantees the boundedness of the output tracking error introducing a trade-off between output specifications and input limitations. However, when saturation is not active the system progressively retrieves the predefined performance characteristics exponentially fast. Hence, by selecting $\lambda_1 < \min\{r_i\},\ i \in \{1,\ldots,n-1\}$ and invoking Proposition 2 we conclude that when the amplitude saturation is not active, (i.e., $\dot{\rho}_1 = -\lambda_1(\rho_1(t) - \rho_{1,\infty})$) the output tracking error $e(t)$ converges to the compact set $E_1 \coloneqq \{e \in \mathbb{R} : |e| \leq \prod_{j=1}^{n-1} r_j^{-1}\rho_{1,\infty}\}$ with exponential rate at least $\exp(-\lambda_1(t))$. In that case, the performance envelope of $e(t)$ can be exactly defined by appropriately selecting the constants $\rho_{1,\infty}$ and $r_j,\ j = \{1,\ldots,n-1\}$.

Remark 4: Theorem 1 establishes conditions for local closed-loop signal boundedness of generic nonlinear systems subject to input constraints. However, the proposed controller (4)-(8) guarantees semi-global boundedness for systems that satisfy input-to-state stability (ISS). Note that the control input $u(t)$ applied to the system (1) is essentially bounded owing to amplitude limitation. Thus, the states $x(t)$ as well as the filtered error $s(\tilde{e}(t))$ are bounded. Therefore, following the analysis below (14) of the proof of Theorem 1, regarding the rate saturation level, we conclude that all closed-loop signals remain bounded for all $t \geq 0$, ensuring adaptive performance tracking for any amplitude and rate saturation level $\bar{u},\ \bar{r}$.

Remark 5: Note that the desired performance specifications of the closed-loop system are solely determined by the evolution of $\rho_1(t)$, (i.e., via the selection $\rho_1(0),\ \lambda_1, \rho_{1,\infty}$). However, in presence of ARS the performance constraints are relaxed to ensure that all signals remain bounded. The relaxation degree depends on the gains k_1, γ_1. Furthermore, the fluctuation of the performance envelope depends on the tracking error $e_2(t)$ as stated in Theorem 1. Thus, a fast convergence rate of $e_2(t)$ is desired and can be achieved by selecting a relatively large value for λ_2. However, selecting large values for the gains $k_i, \gamma_i,\ i = \{1,2\}$ may lead to over-relaxation of the performance functions (5), (8) resulting in unnecessary degradation of the tracking performance. That is owing to the fact that the update laws (4), (6) reach saturation faster as $k_i,\ i = \{1,2\}$ increase, as well as that the second term of the adaptive performance functions (5), (8) increases with $\gamma_i,\ i = \{1,2\}$, respectively.

IV. Simulation Results

In this section, we demonstrate the effectiveness of the proposed algorithm via controlling the wing-rock motion of a delta wing aircraft. Wing-rock is an oscillatory rolling motion of an aircraft with increasing amplitude. The mathematical model [20] for a 80° delta wing aircraft is given by:

$$\ddot{\phi} + c_0\phi + c_1\dot{\phi} + c_2|\dot{\phi}|\dot{\phi} + c_3\phi^3 + c_4\phi^2\dot{\phi} = u$$

where ϕ denotes the roll angle in rad, u is the constrained control input and the coefficients $c_i,\ i = 0,1,2,3,4$ are nonlinear functions of the angle of attack (AOA). A typical set of c_i for AOA 35° at Reynolds number 636000 is: $c_0 = 0.008, c_1 = -0.03, c_2 = 0.4, c_3 = -0.01, c_4 = 0.06$. The reference trajectory is set to $y_d = \frac{1}{2}\sin(2t)$ for $t \in [0,20]$ seconds. Moreover, in order to clarify the functionality and demonstrate the robustness of the proposed control scheme, we induce external disturbances $d(t) = 0.35\cos\left(\frac{13}{2}(t-5)\right),\ \forall t \in [5,15]$. Note that no information about the system is employed and the gains of the controller are selected based on Remark 5 as $k_i = 1, \gamma_i = 1,\ i = \{1,2\}$. The performance specifications are determined by $\lambda_1 = 1, \lambda_2 = 5, \rho_{1,\infty} = 0.005, \rho_{2,\infty} = 0.001$ with $r_1 = 2$ denoting the coefficient of filter (3). The initial conditions are $\phi(0) = 1.2, \dot{\phi}(0) = 0.5, u(0) = 0, \rho_1(0) = |s(\tilde{e}(0))| + 0.1, \rho_2(0) = |e_2(0)| + 0.1$. The simulation is conducted under two different scenarios. We first consider that the control signal is rate constrained only, with $\bar{r} = 25$. Subsequently, we present the capability of the proposed controller in handling more strict amplitude and rate constraints, with $\bar{u} = 2.1,\ \bar{r} = 15$.

The simulation results are depicted in Fig. 1 for 20 seconds. Fig. 1(a) depicts the evolution of the output tracking error $e(t) = y(t) - y_d(t)$ and Fig. 1(c) shows the evolution of input tracking error $e_2(t) = u(t) - u_d(t)$ under rate constrained control input. The high relaxation of the performance boundary $\rho_2(t)$ at $t = 15$ seconds is attributed to the abrupt vanishing of the external disturbance $d(t)$. Notice that the performance specifications of the output remain unaltered despite the heavily rate saturated control signal as illustrated in Fig. 1(g). Moreover, the rate limitation results in a smooth control signal $u(t)$ that avoids the chattering of the desired control input $u_d(t)$ as depicted in Fig. 1(e). Regarding the control under both amplitude and rate constraints, the evolution of the errors $e(t), e_2(t)$ are illustrated in Fig. 1(b) and 1(d), respectively, whereas Fig. 1(f) shows the actual control input $u(t)$ versus the desired control signal $u_d(t)$. Finally, Fig. 1(h) depicts the evolution of the rate of the control input $\dot{u}(t)$ constrained by the limit $\bar{r}$ along with the ideal rate $a(t)$. Notice that the proposed control scheme, balances effectively the output performance constraints in accordance with the input limitations, re-establishing the predefined specifications, when it is feasible, while securing the boundedness of the closed-loop signals. The incorporation of adaptive performance functions in the proposed control scheme is crucial, since the application of the conventional PPC design [10] resulted in system instability in both simulation scenarios because the limited control effort was insufficient to keep the tracking errors $s(\tilde{e}(t)),\ e_2(t)$ within the conventional performance envelopes.

V. Conclusions

In this paper, the PPC problem for a class of unknown nonlinear systems subjected to both amplitude and rate input saturation was considered. A dynamical control law, incorporating input limitations, in combination with performance functions that adapt depending on the saturation status, guarantee the trade-off between input and output constraints. Contrary to the related literature, no approximation mechanisms nor any auxiliary systems are required, leading to a low-complexity robust controller with simple gain selection.

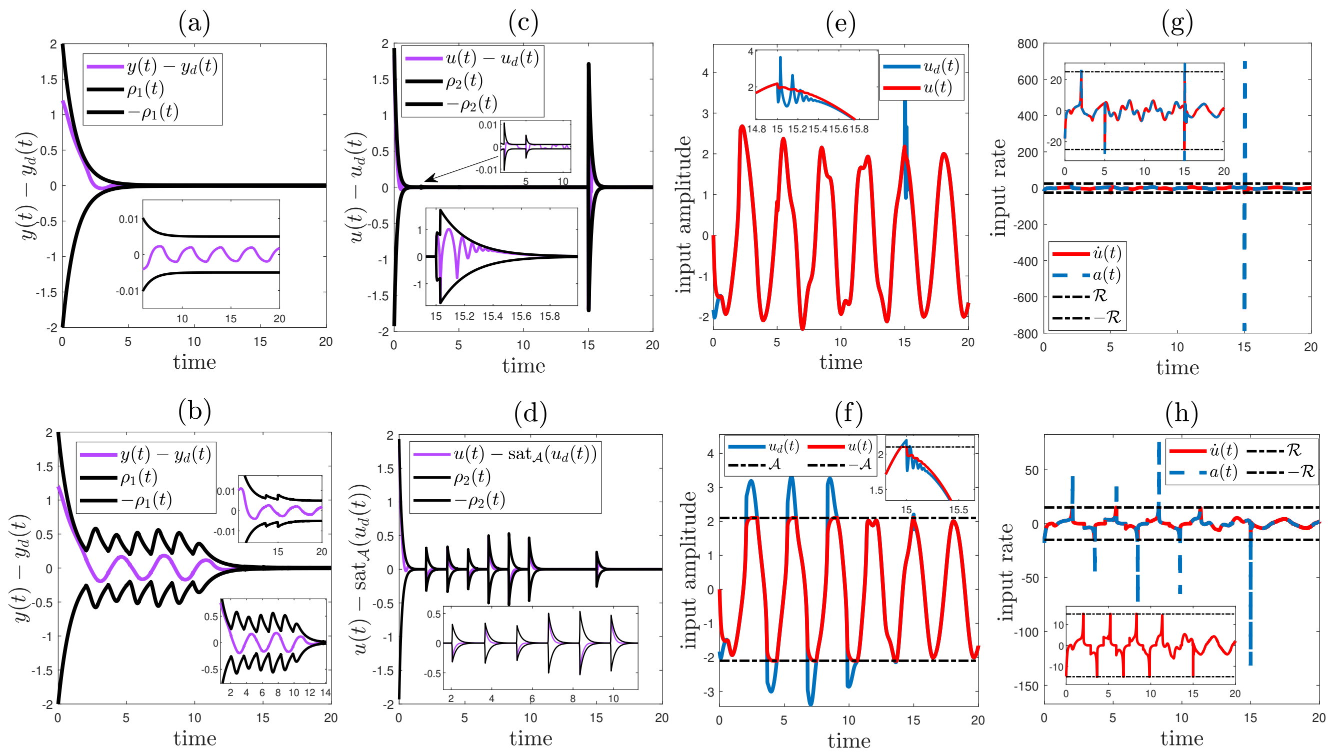

Fig. 1. Simulation results; (a),(b): The evolution of output tracking error $e(t)$ for the rate saturation and ARS scenario, respectively; (c),(d): The evolution of input tracking error $e_2(t)$ for the rate saturation and ARS scenario, respectively; (e),(f): The evolution of the actual control input $u(t)$ and the desired control signal $u_d(t)$ for the rate saturation and ARS scenario, respectively; (g),(h): The evolution of control input rate $\dot{u}(t)$ and the intermediate signal $a(t)$ for the rate saturation and ARS scenario, respectively.

Future research efforts will be devoted towards considering full state constraints.

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