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Stability Analysis for Multirate Interlaced Kalman Filter

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***Abstract*— Distributed systems are often chosen since centralized solutions are often impractical when dealing with state estimation of complex systems due to computational complexity. The Interlaced Extended Kalman Filter is a distributed state observer that enables each subsystem to predict a subset of the state space and communicate with other subsystems. However, the Interlaced Extended Kalman Filter requires precise synchronization between subsystems, which may be unfeasible when, for instance, the sampling rates of the subsystems vary.**

To address this issue, this paper suggests an Interlaced Extended Kalman Filter extension that enables each subsystem to use the most recent estimate when up-to-date information is unavailable. Adjusting the covariance matrix, which can be done using Age of Information metrics, increases the uncertainty in the approximation. Each subsystem's stability is investigated, showing that changes in the covariance matrix do not affect the analysis.

The suggested algorithm is validated in a scenario with four water tanks fed by two pumps, where the operating rates of the subsystems are different but fixed. The findings demonstrate that the proposed algorithm successfully handles the multi-rate problem while striking a reasonable balance between convergence rate and efficiency.

I. INTRODUCTION

The control of distributed systems aims to simplify complex problems by breaking them down into simpler subsystems while maintaining performance [1]. However, distributed systems introduce challenges and opportunities, particularly when dealing with subsystems operating at different sampling times [2].

An extension of the Extended Kalman Filter (EKF) known as the Interlaced Extended Kalman Filter (IEKF) was developed to reduce the computational load of estimating nonlinear system states [3]. IEKF functions as a distributed architecture for EKF, applicable to systems divisible into subsystems. Each subsystem estimates its state using predicted state and covariance matrix data from other subsystems.

IEKF assumes synchronized operation among subsystems, which may not always hold due to factors like varying sensor sampling rates or communication channel noise. This paper investigates the stability of IEKF when subsystems operate at different sampling rates.

Two prominent distributed estimation algorithms are the distributed particle filter [4] and distributed Kalman filter [5]. Distributed Particle Filters are suitable for addressing large-scale, nonlinear, and non-Gaussian estimation challenges in agent network applications [6]. Examples of these techniques in asynchronous environments can be found in [7] and [8].

The Distributed Kalman Filter is commonly used for solving linear and nonlinear distributed state estimation problems [9], [10]. In this approach, nodes in a network exchange information with their immediate neighbors to improve their state estimations. Consensus-based distributed filtering methods have also gained popularity due to their reduced communication resource utilization [11].

In a typical sensor network, a distributed Unscented Kalman Filter is proposed in [12]. It addresses consensus issues using a weighted average consensus approach.

The primary contribution of this paper lies in the analysis of IEKF convergence when subsystems operate at different rates. The approach involves using the most recent state estimate for other subsystems and increasing the covariance matrix to account for augmented uncertainty in state estimation. Data freshness is assessed using the Age-of-Information (AoI) metric, which measures receiver-centric delays. A similar strategy can be applied when dealing with unreliable communication channels. The paper demonstrates that this novel IEKF algorithm converges under the same assumptions as traditional IEKF.

II. MULTIRATE INTERLACED KALMAN FILTER

In this paper, we consider a physical process that evolves according to (1)

$$\begin{array}{rcl} x_{k+1} & = & f(x_k, u_k) + w_k \\ y_k & = & h(x_k, u_k) + v_k. \end{array} \tag{1}$$

where $x_k \in \mathbb{R}^{n_x}$ is the process state, $u_k \in \mathbb{R}^{n_u}$ is the input, $y_k \in \mathbb{R}^{n_y}$ is the output, and $w_k \in \mathbb{R}^{n_x}$, $v_k \in \mathbb{R}^{n_u}$ are the uncertainties that affect the process and the measurement, respectively. Both the noises w_k and v_k are modeled as a zero-mean Gaussian stochastic variable, having known constant covariance matrices (i.e., $w_k \sim \mathcal{N}(0, Q)$ and $v_k \sim \mathcal{N}(0, R)$). The state transition and observation are nonlinear and represented by $f(\cdot)$ and $h(\cdot)$, respectively, and we assume they are differentiable on $\mathbb{R}^{n_x}$.

To reduce the computational complexity of the EKF, [3] introduced the Interlaced Extended Kalman Filter (IEKF). The IEKF is composed of p parallel EKF implementations,

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each estimating only a portion of the state variables and considering the remainder as a deterministic parameter.

Assumption 1: The state vector

$$x_{k_i} = \left[\hat{x}^{(1)}_{k_i}; \ldots ; \hat{x}^{(p)}_{k_i}\right]^T \tag{2}$$

can be partitioned such that each state variable $x^{(I)}_{k_i}$ is estimated by one and only one subsystem.

We denote by $x^{(i)}_{k_i} \in \mathbb{R}^{n_i}$ the subset of state variables to be estimated by the generic filtering station i, with $n_x = \sum_{i=1}^{p} n_i$, where p is the number of stations.

Assumption 2: All the subsystems are directly connected.

These assumptions allow us to write the system equations as p subsystems. The $i-th$ subsystem is described by:

$$\begin{array}{rcl} x^{(i)}_{k_i+1} & = & f^{(i)}\left(x_{k_i}, u_{k_i}\right) + w^{(i)}_{k_i} \\ y^{(i)}_{k_i} & = & h^{(i)}\left(x_{k_i}, u_{k_i}\right) + v^{(i)}_{k_i} \end{array} \tag{3}$$

In this work, we assume that each subsystem operates at a fixed rate, denoted by S_i, where for every S_i, the station performs a prediction and correction cycle using the proposed paradigm. If two stations have different operating times, their synchronization occurs at the least common multiple of their operating times, denoted by S_{ij}. We define the hyper-period S [13] as the minimum interval after which all the subfilters are synchronized. We differentiate the indices of the instants at which prediction and correction cycles are performed by each filter, denoted by k_i.

When two stations are not synchronized, they have old data on a certain partition of the system state. To address this issue, we use the Age of Information (AoI) [14] as a metric representing the freshness of the information available at the receiver. The AoI metric of the station i with respect to the station j is given by

$$AoI^{(ij)}_{k_i} = k_i - m_{ij}. \tag{4}$$

where m_{ij} is the last time instant station i and station j have been synchronized. We use $AoI^{(ij)}_{k_i}$ to increase the covariance matrix $P^{(j)}_{k_i}$ when station i has old information about the subspace observed by station j. With $AoI^{(ij)}_{k_i}$ so defined, the Multirate Interlaced Kalman Filter is made up of the following equations.

$$\hat{x}^{(j)}_{k_i} = \hat{x}^{(j)}_{m_{ij}} \tag{5}$$

$$P^{(j)}_{k_i} = \alpha_{ij}^{AoI^{(ij)}_{k_i}} P^{(j)}_{m_{ij}} \tag{6}$$

Compute the prediction step:

$$\hat{x}^{(i)}_{k_i+1|k_i} = f^{(i)}(\hat{x}_{k_i}, u_{k_i}) \tag{7}$$

$$\tilde{Q}^{(i)}_{k_i} = Q^{(i)} + \sum_{j\in M, j\neq i} F^{(ij)}_{k_i} P^{(j)}_{k_i} F^{(ij)^T}_{k_i} \tag{8}$$

$$P^{(i)}_{k_i+1|k_i} = F^{(ii)}_{k_i} P^{(i)}_{k_i} F^{(ii)^T}_{k_i} + \tilde{Q}^{(i)}_{k_i} \tag{9}$$

Compute the correction step:

$$\hat{x}^{(i)}_{k_i+1} = \hat{x}^{(i)}_{k_i+1|k_i} + K^{(i)}_{k_i+1}\nu^{(i)}_{k_i+1} \tag{10}$$

$$\tilde{R}^{(i)}_{k_i+1} = R^{(i)} + \sum_{j\in M, j\neq i} H^{(ij)}_{k_i+1} P^{(j)}_{k_i} H^{(ij)^T}_{k_i+1} \tag{11}$$

$$K^{(i)}_{k_i+1} = P^{(i)}_{k_i+1|k_i} H^{(ii)^T}_{k_i+1} \cdot \left(H^{(ii)}_{k_i+1} P^{(i)}_{k_i+1|k_i} H^{(ii)^T}_{k_i+1} + \tilde{R}^{(i)}_{k_i+1}\right)^{-1} \tag{12}$$

$$P^{(i)}_{k_i+1} = \left(I - K^{(i)}_{k_i+1} H^{(ii)}_{k_i+1}\right) P^{(i)}_{k_i+1|k_i} \tag{13}$$

where $\alpha_{ij} > 1$ is an arbitrary parameter which can be selected based on how rapidly the dynamics of the system evolves,

$$\nu^{(i)}_{k_i+1} = y^{(i)}_{k_i+1} - h^{(i)}\left(\hat{x}_{k_i+1|k_i}, u_{k+1}\right)$$

$$H^{(ii)}_{k_i+1} = \left.\frac{\partial h^{(i)}}{\partial x^{(i)}_{k_i}}\right|_{x_{k_i} = \hat{x}_{k_i+1|k_i}}, \quad H^{(ij)}_{k_i+1} = \left.\frac{\partial h^{(i)}}{\partial x^{(j)}_{k_i}}\right|_{x_{k_i} = \hat{x}_{k_i+1|k_i}}$$

$$F^{(ii)}_{k_i} = \left.\frac{\partial f^{(i)}}{\partial x^{(i)}_{k_i}}\right|_{x_k = \hat{x}_k}, \quad F^{(ij)}_{k_i} = \left.\frac{\partial f^{(i)}}{\partial x^{(j)}_{k_i}}\right|_{x_{k_i} = \hat{x}_{k_i}}$$

Definition 2.1: Let $e_{k_i|k_i-1}$ and e_{k_i} denote the error in the predicted and filtered state respectively of the subsystem i, that is:

$$e^{(i)}_{k_i+1|k_i} = x^{(i)}_{k_i+1} - \hat{x}^{(i)}_{k_i+1|k_i} \tag{14}$$

$$e^{(i)}_{k_i+1} = x^{(i)}_{k_i+1} - \hat{x}^{(i)}_{k_i+1} \tag{15}$$

Because $F^{(i)} \in \mathcal{C}^1$ and $h^{(i)} \in \mathcal{C}^1$, it may be written as

$$h^{(i)}(x_k) - h^{(i)}(\hat{x}_{k|k-1}) = H^{(ii)}_k (x_k - \hat{x}_{k|k-1}) + \phi^{(i)}_h (x_k, \hat{x}_{k|k-1}) \tag{16}$$

$$f^{(i)}(x_k) - f^{(i)}(\hat{x}_k) = F^{(ii)}_k (x_k - \hat{x}_k) + \phi^{(i)}_f (x_k, \hat{x}_k) \tag{17}$$

where $\phi^{(i)}_h(x_k, \hat{x}_{k|k-1})$ and $\phi^{(i)}_f(x_k, \hat{x}_k)$ are the remainder terms of the functions h and f, respectively. They are denoted in the rest of the paper as $\phi^{(i)}_h(x, \hat{x})$ and $\phi^{(i)}_h(x, \hat{x})$, respectively.

The predicted state error in Eq. (14) can be rewritten as

$$e^{(i)}_{k_i+1|k_i} = F^{(ii)}_{k_i} e_{k_i} + w^{(i)}_{k_i} + \phi^{(i)}_f(x_{k_i}, \hat{x}_{k_i}) \tag{18}$$

The filtered state error in Eq. (15) can be rewritten as

$$e^{(i)}_{k_i+1} = \tilde{F}^{(ii)}_{k_i+1} e^{(i)}_{k_i} + n^{(i)}_{k_i+1} + l^{(i)}_{k_i+1} \tag{19}$$

where

$$\begin{aligned}
\tilde{F}^{(ii)}_{k_i+1} &= [I - K^{(i)}_{k_i+1} H^{(i)}_{k_i+1}] F^{(ii)}_{k_i} \\
n^{(i)}_{k_i+1} &= [I - K^{(i)}_{k_i+1} H^{(ii)}_{k_i+1}] w^{(i)}_{k_i} + K^{(i)}_{k_i+1} v^{(i)}_{k_i+1} \\
l^{(i)}_{k_i+1} &= [I - K^{(i)}_{k_i+1} H^{(ii)}_{k_i+1}] \phi^{(i)}_f(x_{k_i}, \hat{x}_{k_i}) + K^{(i)}_{k_i+1} \phi^{(i)^+}_h(x, \hat{x}) \\
\phi^{(i)^+}_h(x, \hat{x}) &= \phi^{(i)}_h(x_{k_i+1}, \hat{x}_{k_i+1|k_i})
\end{aligned}$$

III. Stability Analysis

We assume that $k = k_i$ is the time when the subfilter i estimates its state.

Assumption 3: We assume the following upper and lower bounds:

$$\underline{p}^{(i)} I \leq P_{k+1}^{(i)} \leq \overline{p}^{(i)} I \tag{20}$$
$$\underline{q}^{(i)} I \leq P_{k+1|k}^{(i)} \leq \overline{q}^{(i)} I \tag{21}$$
$$r_1 I \leq R^{(i)}, \qquad q_1 I \qquad \leq Q^{(i)} \tag{22}$$
$$\underline{p}^{(j)} I \leq P_{k+1}^{(j)} \leq \overline{p}^{(j)} I \tag{23}$$
$$\underline{h}^{(i)} I \leq ||H_{k+1}^{(ii)}|| \leq \overline{h}^{(i)} I \tag{24}$$
$$||F_k^{(ii)}|| \leq \overline{f}^{(i)} \tag{25}$$
$$\underline{f}^{(j)} \leq ||F_k^{(ij)}|| \tag{26}$$
$$\underline{h}^{(j)} I \leq ||H_{k+1}^{(ij)}|| \tag{27}$$

Assumption 4: We assume that the noise processes are bounded in ∞-norm, i.e.

$$||w_k^{(i)}|| \leq \overline{w}^{(i)}, \qquad ||v_k^{(i)}|| \leq \overline{v}^{(i)} \tag{28}$$

Lemma 1: If the conditions given in Assumption 3 holds then an upper bound of the norm of the Kalman gain matrix is given by

$$||K_{k+1}^{(i)}|| \leq \frac{\overline{q}^{(i)} \overline{h}^{(i)}}{\underline{q}^{(i)} \underline{h}^{(i)^2} + r_1 + \sum_{j \in M, j \neq i} \underline{p}^{(j)} \underline{h}^{(j)^2}} = \tilde{\kappa} \tag{29}$$

Lemma 2: If the conditions given in Assumption 3 holds and assuming that F is nonsingular for all $k \geq 0$ then there exist a real number $0 < \gamma^{(i)} < 1$ such that

$$\tilde{F}_{k+1}^{(ii)^T} P_{k+1}^{(i)^{-1}} \tilde{F}_{k+1}^{(ii)} \leq \left(1 - \gamma^{(i)}\right) P_k^{(i)^{-1}} \tag{30}$$

Lemma 3: Let the assumptions 3 hold and there are positive numbers $\kappa_f, \kappa_h > 0$ such that the nonlinear functions $\phi_f^{(i)}, \phi_h^{(i)}$ in Eq. 19 are bounded via Eq. 34 and Eq. 33 for $\left|\left|e_k^{(i)}\right|\right| \leq \epsilon^{(i)'}$. Then there are positive real numbers $\epsilon^{(i)'}$, $\kappa_{nonl}^{(i)} > 0$ such that

$$l_{k+1}^{(i)^T} P_{k+1}^{(i)^{-1}} \left(2\tilde{F}_{k+1}^{(ii)} e_k^{(i)} + l_{k+1}^{(i)}\right) \leq \kappa_{nonl}^{(i)} \left|\left|e_k^{(i)}\right|\right|^3 + D(\tilde{\kappa}) \tag{31}$$

is fulfilled for $||e_k^{(i)}|| \leq \epsilon^{(i)'}$.

Theorem 1: Assume that the bounds given in Assumption 3 are fulfilled. Assume further that there exist an $\overline{\epsilon}$ such that

$$||e_{k|k}^{(i)}|| \leq \overline{\epsilon}^{(i)} \tag{32}$$

which implies $||x_k^{(i)} - \hat{x}_{k+1|k}^{(i)}|| \leq \epsilon_1^{(i)}(\overline{\epsilon}^{(i)})$ where $\epsilon_1^{(i)}(\overline{\epsilon}^{(i)}) = a\overline{\epsilon}^{(i)} + b$. Moreover, assume that

$$||\phi_h^{(i)}(x_{k+1}, \hat{x}_{k+1|k})|| \leq \varphi_h^{(i)} ||x_{k+1}^{(i)} - \hat{x}_{k+1|k}^{(i)}||^2 \tag{33}$$

and

$$||\phi_f^{(i)}(x_{k+1}, \hat{x}_{k+1})|| \leq \varphi_f^{(i)} ||x_{k+1}^{(i)} - \hat{x}_{k+1}^{(i)}||^2 \tag{34}$$

holds for $||x_{k+1}^{(i)} - \hat{x}_{k+1|k}^{(i)}|| \leq \epsilon_1^{(i)}(\overline{\epsilon}^{(i)}) = \epsilon_1^{(i)}$ and $||x_{k+1}^{(i)} - \hat{x}_{k+1|k}^{(i)}|| \leq \epsilon_1^{(i)}(\overline{\epsilon}^{(i)}) = \epsilon_1^{(i)}$, respectively.

Then there exists an $\epsilon^{(i)} > 0$ such that the solution of the error model in Eq. 18 is:

1) Locally exponential stable if the initial error satisfies $||e_0^{(i)}|| \leq \epsilon^{(i)}$ and $\overline{w}^{(i)} = \overline{v}^{(i)} = 0$.
2) Bounded by

$$\left|\left|e_{k+1}^{(i)}\right|\right|^2 \leq \frac{\overline{p}^{(i)}}{\underline{p}^{(i)}} \left(1 + \xi^{(i)}\right)^{k+1} \left|\left|e_0^{(i)}\right|\right|^2 - \frac{\overline{p}^{(i)}}{\xi^{(i)}} \rho^{(i)}(\tilde{\kappa}) \tag{35}$$

if the initial error satisfies $||e_0^{(i)}|| \leq \epsilon^{(i)}$, and $\tilde{\kappa}$ is the upper bound of the Kalman gain.

This theorem demonstrates the stability property of the IEKF even at different sampling times. The result of this theorem demonstrates how the increase in the covariance matrix $P_{k_i}^{(j)}$ exploiting the concept of the Age-of-Information does not affect the stability analysis. However, as described in [15], the increase of $P_{k_i}^{(j)}$ leads to a decrease in the Kalman gain matrix $K_{k_i+1}^{(i)}$, causing a slow convergence rate.

Furthermore, the steady state value of the error depends on the second part of the equation and therefore on $\rho^{(i)}(\tilde{\kappa})$ and $\xi^{(i)} = \xi^{(i)}(\tilde{\kappa})$.

IV. Case Study and Results

Let us consider the discretization of the four tanks model whose dynamical evolution is described in [16]. Differently from [16], we consider the output of the system as

$$\begin{bmatrix} \dot{x}_{1,k+1} \\ \dot{x}_{2,k+1} \\ \dot{x}_{3,k+1} \\ \dot{x}_{4,k+1} \end{bmatrix} = \begin{bmatrix} c_1(\sqrt{2gx_{k,3}} - \sqrt{2gx_{k,1}}) + c_2 u_{k,1} + w_{k,1} \\ c_3(\sqrt{2gx_{k,4}} - \sqrt{2gx_{k,2}}) + c_4 u_{k,2} + w_{k,2} \\ -c_5\sqrt{2gx_{k,3}} + c_6 u_{k,2} + w_{k,3} \\ -c_7\sqrt{2gx_{k,4}} + c_8 u_{k,1} + w_{k,4} \end{bmatrix}$$

$$\begin{bmatrix} y_{1,k} \\ y_{2,k} \end{bmatrix} = \begin{bmatrix} x_{1,k} - x_{3,k} + v_{1,k} \\ x_{2,k} - x_{4,k} + v_{2,k} \end{bmatrix}.$$

The system is shown in Fig. 1. We have two different subsystems, where the first has the task of estimating state variables $x_k^{(1)} = [x_{1,k}; x_{4,k}]$, corresponding respectively to the heights of Tank 1 and Tank 4 and can measure $u_{1,k}$ and $y_k^{(1)} = [y_{1,k}]$, while the second has the task of estimate $x_k^{(2)} = [x_{2,k}; x_{3,k}]$, corresponding respectively to the heights of Tank 2 and Tank 3 and can measure $u_{2,k}$ and $y_k^{(2)} = [y_{2,k}]$.

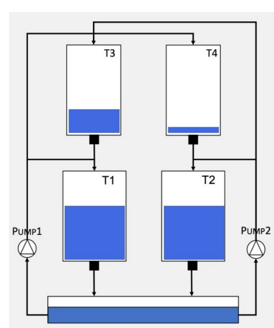

Fig. 1. Interconnection of the four tanks

We choose the desired output as $y_{d,1,k} = 20 + 0.2 \cdot \int \text{sgn}(\sin(k))dk$ and $y_{d,2,k} = 20 + 0.01 \cdot k$.

A straightforward proportional-integrative rule with the formula

$$u_k = K_p(y_d - y_k) + \frac{K_i(y_d - y_k)}{s},$$

has been chosen as the control law for this system, where K_p is the proportional controller gain and K_i the integrative control gain.

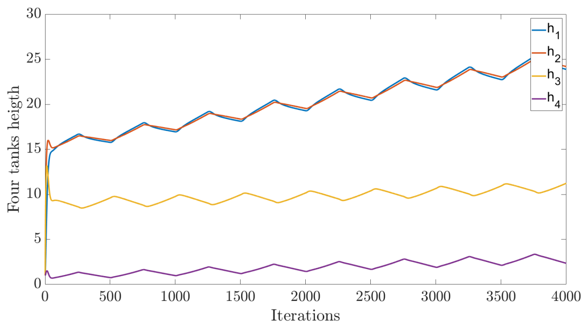

Fig. 2. Temporal evolution of the height of the four tanks

As shown in Fig. 2, the tank height oscillates due to the triangular input: this fact forces the linearisation point to change each instant, proving its effectiveness.

In this work two different simulations have been considered. All the simulations have as parameters $R_k^{(i)} = 10^{-3} \cdot I_{n_{x_i}}$, $R_k^{(j)} = 10^{-3} \cdot I_{n_{x_j}}$, $Q_k^{(i)} = 10^{-3} \cdot I_{n_{y_i}}$ and $Q_k^{(j)} = 10^{-3} \cdot I_{n_{y_j}}$.

In Fig. 3 and in Fig. 4, we consider the two subsystems that have $S_1 = 0.2$ sec and $S_2 = 0.5$ sec as sampling rates, but the time interval of the simulation is 800 seconds. In Fig. 3, the norm of error is rapidly decreasing and after around 200 seconds the two subsystems are steady-state. In Fig 4 the norm of the Kalman gain for the two subsystems is depicted. Both sub-filters exhibit an oscillating nature due to the small sampling time and the difference in the sampling frequencies between the two systems.

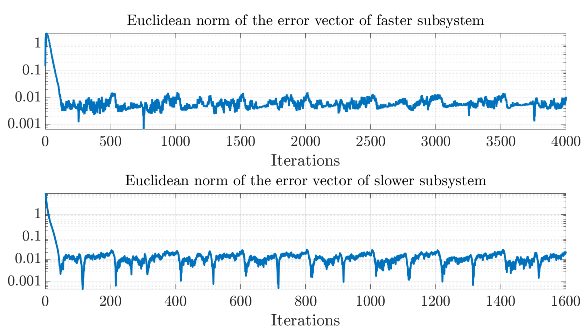

Fig. 3. Temporal evolution of the error of the two subsystems when $S_1 = 0.2$ sec, $S_2 = 0.5$ sec and $\alpha_{ij} = 1.2$

In Fig. 5 and Fig. 4, the simulation has parameters $S_1 = 0.2$ sec, $S_2 = 5$ sec and $\alpha_{ij} = 1.2$. In this case, the absolute value of the error (in Fig. 5) is much larger than in the previous case. The same observation is true for steady-state values. For the Kalman gains in Fig. 6, the second subsystem (the slower one) has the same trends as in the previous case, with greater values and fewer oscillations.

V. Conclusions and Future Works

The paper describes stability analysis for the IEKF algorithms when each subsystem has a different but fixed

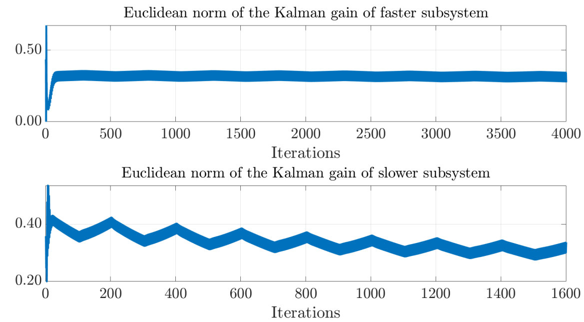

Fig. 4. Temporal evolution of the Kalman gains for the subsystems when $S_1 = 0.2$ sec, $S_2 = 0.5$ sec and $\alpha_{ij} = 1.2$

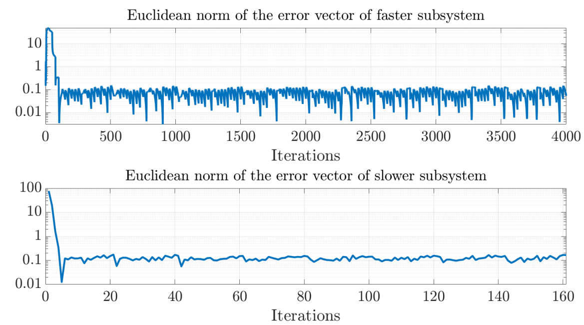

Fig. 5. Temporal evolution of the error of the two subsystems when $S_1 = 0.2$ sec, $S_2 = 5$ sec and $\alpha_{ij} = 1.2$

sampling rate. Each subsystem estimates a partition of state space considering the information exchanged (i.e., states and covariance matrices) with the other subsystems. The stability analysis demonstrates that the error is bound for each subsystem, even when the covariance matrix is increased, considering the oldness of the corresponding data.

Appendix I
Proof of Lemma 1

Proof: The symbols $\overline{\sigma}(\cdot)$ and $\underline{\sigma}(\cdot)$ denote the largest and smallest singular value, respectively.

$$\overline{\sigma}(K_{k+1}^{(i)}) \leq \overline{\sigma}\left(P_{k+1|k}^{(i)} H_{k+1}^{(ii)^T}\right) \left(\underline{\sigma}\left(H_{k+1}^{(ii)} P_{k+1|k}^{(i)} H_{k+1}^{(ii)^T} + \tilde{R}_{k+1}^{(i)}\right)\right)^{-1}$$

The matrices in the second factor are positive definite, so the singular values are the eigenvalues. Applying the Rayleigh-Ritz characterization [17]

$$\overline{\sigma}(K_{k+1}^{(i)}) \leq \overline{\sigma}\left(P_{k+1|k}^{(i)} H_{k+1}^{(ii)^T}\right) \left(\underline{\sigma}\left(H_{k+1}^{(ii)} P_{k+1|k}^{(i)} H_{k+1}^{(ii)^T}\right) + \underline{\sigma}\left(\tilde{R}_{k+1}^{(i)}\right)\right)^{-1}$$

Thus,

$$||K_{k+1}^{(i)}|| \leq \frac{\overline{q}^{(i)} \overline{\sigma}(H_{k+1}^{(ii)^T})}{\underline{q}^{(i)} \underline{\sigma}^2(H_{k+1}^{(ii)^T}) + \underline{\sigma}(\tilde{R}_{k+1}^{(i)})} \leq \frac{\overline{q}^{(i)} \overline{h}^{(i)}}{\underline{q}^{(i)} \underline{h}^{(i)^2} + r_1 + \sum_{j \in M, j \neq i} \underline{p}^{(j)} \underline{h}^{(j)^2}} \quad (36)$$

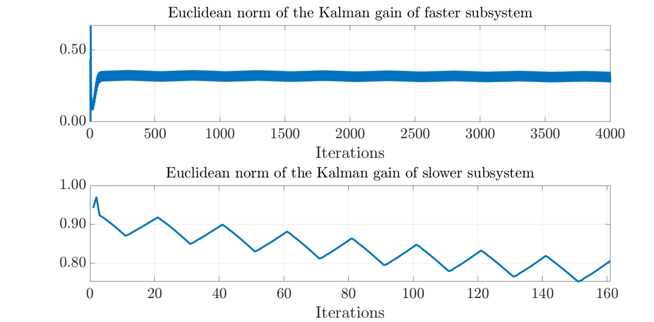

Fig. 6. Temporal evolution of the Kalman gains for the subsystems when $S_1 = 0.2$ sec, $S_2 = 5$ sec and $\alpha_{ij} = 1.2$

■

APPENDIX II
PROOF OF LEMMA 2

Proof: We start considering Eq. 13 that can be rewritten using Eq. 12

$$P^{(i)}_{k+1} = \left(I - K^{(i)}_{k+1} H^{(ii)}_{k+1}\right) P^{(i)}_{k+1|k} \left(I - K^{(i)}_{k+1} H^{(ii)}_{k+1}\right)^T + \\ + K^{(i)}_{k+1} \tilde{R}^{(i)}_{k+1} K^{(i)^T}_{k+1} \tag{37}$$

The equation can be re-written exploiting Eq. 9

$$P^{(i)}_{k+1} = \tilde{F}^{(ii)}_{k+1} P^{(i)}_k \tilde{F}^{(ii)^T}_{k+1} + K^{(i)}_{k+1} \tilde{R}^{(i)}_{k+1} K^{(i)^T}_{k+1} + \\ + \left(I - K^{(i)}_{k+1} H^{(ii)}_{k+1}\right) \tilde{Q}^{(i)}_k \left(I - K^{(i)}_{k+1} H^{(ii)}_{k+1}\right)^T$$

After some arrangements of terms, and multiplying from left and right with $\tilde{F}^{(ii)^{-1}}_{k+1}$ and $\tilde{F}^{(ii)^{-T}}_{k+1}$, it gives

$$\tilde{F}^{(ii)^{-1}}_{k+1} P^{(i)}_{k+1} \tilde{F}^{(ii)^{-T}}_{k+1} = P^{(i)}_k + F^{(ii)^{-1}}_k \tilde{Q}^{(i)}_k F^{(ii)^{-T}}_k + \\ + \tilde{F}^{(ii)^{-1}}_{k+1} K^{(i)}_{k+1} \tilde{R}^{(i)}_{k+1} K^{(i)^T}_{k+1} \tilde{F}^{(ii)^{-T}}_{k+1}$$

Taking the inverse of both sides, and since $\tilde{R}^{(i)}_{k+1} > 0$ the following inequality holds

$$\tilde{F}^{(ii)^T}_{k+1} P^{(i)}_{k+1} \tilde{F}^{(ii)}_{k+1} \leq \left(1 + \frac{q_1 + \sum_{j \in M, j \neq i} \underline{p}^{(j)} \underline{f}^{(j)^2}}{\overline{p}^{(i)} \overline{f}^{(i)}}\right)^{-1} P^{(i)^{-1}}_k \tag{38}$$

Setting $1 - \gamma^{(i)} = \left(1 + \frac{q_1 + \sum_{j \in M, j \neq i} \underline{p}^{(j)} \underline{f}^{(j)^2}}{\overline{p}^{(i)} \overline{f}^{(i)}}\right)^{-1}$ demonstrates the lemma. ■

APPENDIX III
PROOF OF LEMMA 3

Proof: From Eq. 13, we can write

$$\left|\left|\left(I - K^{(i)}_{k_i+1} H^{(ii)}_{k_i+1}\right)\right|\right| \leq \left|\left|P^{(i)}_{k_i+1|k_i} P^{(i)^{-1}}_{k_i+1}\right|\right| \leq \frac{\overline{p}^{(i)}}{\underline{q}^{(i)}} \tag{39}$$

$$\left|\left|\tilde{F}^{(ii)}_{k+1}\right|\right| \leq \left|\left|[I - K^{(i)}_{k+1} H^{(i)}_{k+1}] F^{(ii)}_k\right|\right| \leq \frac{\overline{p}^{(i)} \overline{f}^{(i)}}{\underline{q}^{(i)}} \tag{40}$$

Therefore, from Eq. 19 we yield

$$\left|\left|l^{(i)}_{k+1}\right|\right| \leq \frac{\overline{p}^{(i)}}{\underline{q}^{(i)}} \left|\left|\phi^{(i)}_f(x_k, \hat{x}_k)\right|\right| + \tilde{\kappa} \left|\left|\phi^{(i)^+}_h(x, \hat{x})\right|\right| \tag{41}$$

Using Eq. 18, we obtain

$$\left|\left|e^{(i)}_{k+1|k}\right|\right| \leq \overline{f}^{(i)} \left|\left|e^{(i)}_k\right|\right| + \overline{w}^{(i)} + \varphi^{(i)}_f \left|\left|e^{(i)}_k\right|\right|^2 \tag{42}$$

Choosing $\epsilon^{(i)'}$ for $\left|\left|e^{(i)}_k\right|\right| \leq \epsilon^{(i)'}$, and after some algebraic manipulations, we obtain

$$\left|\left|l^{(i)^T}_{k+1} P^{(i)^{-1}}_{k+1} \left(2\tilde{F}^{(ii)}_{k+1} e^{(i)}_k + l^{(i)}_{k+1}\right)\right|\right| \leq \kappa^{(i)}_{nonl} \left|\left|e^{(i)}_k\right|\right|^3 + D_1(\tilde{\kappa}) \tag{43}$$

where $\kappa^{(i)}_{nonl} = \left(A_1 \tilde{\kappa}^2 + B_1 \tilde{\kappa} + C_1\right)$,

$$\begin{aligned} A_1 = \frac{\varphi_h^{(i)2}}{\underline{p}^{(i)}} \Big(& \varphi_f^{(i)4} \epsilon^{(i)'5} + 4\varphi_f^{(i)3} \overline{f}^{(i)} \epsilon^{(i)'4} + 2\varphi_f^{(i)2} \epsilon^{(i)'3} \times \\ & \left(2\varphi_f^{(i)} \overline{w}^{(i)} + 3\overline{f}^{(i)2}\right) + 4\varphi_f^{(i)} \overline{f}^{(i)} \epsilon^{(i)'2} \Big(3\varphi_f^{(i)} \overline{w}^{(i)} + \\ & + \overline{f}^{(i)2}\Big) + \varphi_h^{(i)2} \epsilon^{(i)'} \Big(6\varphi_f^{(i)2} \overline{w}^{(i)} + 12\varphi_f^{(i)} \overline{f}^{(i)2} \overline{w}^{(i)} + \\ & + \overline{f}^{(i)4}\Big) + +4\varphi_h^{(i)2} \overline{f}^{(i)} \overline{w}^{(i)} \left(3\varphi_f^{(i)} \overline{w}^{(i)} + \overline{f}^{(i)2}\right)\Big), \\ B_1 = 2\varphi_f^{(i)} \varphi_h^{(i)} \frac{\overline{p}^{(i)}}{\underline{p}^{(i)} \underline{q}^{(i)}} \Big[& \varphi_f^{(i)2} \epsilon^{(i)'3} + 2\varphi_f^{(i)} \overline{f}^{(i)} \epsilon^{(i)'2} \times \\ & \left(1 + \varphi_f^{(i)}\right) + \epsilon^{(i)'} \left(2\varphi_f^{(i)} \overline{f}^{(i)2} + \overline{f}^{(i)2} + 3\varphi_f^{(i)} \overline{w}^{(i)}\right) + \\ & + \overline{f}^{(i)} \left(\overline{f}^{(i)2} + 3\overline{w}^{(i)}\right)\Big], \\ C_1 = 2\varphi_f^{(i)} & \frac{\overline{p}^{(i)2}}{\underline{p}^{(i)} \underline{q}^{(i)2}} \left(\overline{f}^{(i)} + \epsilon^{(i)'}\right), \end{aligned}$$

and

$$\begin{aligned} D = \tilde{\kappa}^2 \varphi_h^{(i)2} \overline{w}^{(i)2} \frac{1}{\underline{p}^{(i)}} \Big(& \overline{w}^{(i)} \left(4\varphi_f^{(i)} \epsilon^{(i)'2} + \overline{w}^{(i)}\right) + \\ & + 2\overline{f}^{(i)} \epsilon^{(i)'} \left(3\overline{f}^{(i)} \epsilon^{(i)'} + 2\right)\Big) + \tilde{\kappa} \varphi_f^{(i)} \varphi_h^{(i)} \overline{w}^{(i)} \frac{\overline{p}^{(i)}}{\underline{p}^{(i)} \underline{q}^{(i)}} \epsilon^{(i)'} \times \\ & \left(\overline{f}^{(i)2} \epsilon^{(i)'} + \overline{w}^{(i)} \epsilon^{(i)'} + 2\overline{f}^{(i)} \overline{w}^{(i)}\right) \end{aligned}$$

■

APPENDIX IV
PROOF OF THEOREM 1

Proof: Let $V^{(i)} : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ be a positive function defined by

$$V^{(i)}(e^{(i)}_k) = e^{(i)^T}_k P^{(i)^{-1}}_k e^{(i)}_k \tag{44}$$

such that from Eq. 20

$$\frac{1}{\overline{p}^{(i)}} ||e^{(i)}_k||^2 \leq V^{(i)}(e^{(i)}_k) \leq \frac{1}{\underline{p}^{(i)}} ||e^{(i)}_k||^2 \tag{45}$$

$$\begin{aligned} \Delta V := {} & e^{(i)^T}_{k+1} P^{(i)^{-1}}_{k+1} e^{(i)}_{k+1} - e^{(i)^T}_k P^{(i)^{-1}}_k e^{(i)}_k \\ = {} & e^{(i)^T}_k \left(\tilde{F}^{(ii)^T}_{k+1} P^{(i)^{-1}}_{k+1} \tilde{F}^{(ii)}_{k+1} - P^{(i)^{-1}}_k\right) e^{(i)}_k + \\ & + n^{(i)^T}_{k+1} P^{(i)^{-1}}_{k+1} n^{(i)}_{k+1} + l^{(i)^T}_{k+1} P^{(i)^{-1}}_{k+1} \left(2\tilde{F}^{(ii)}_{k+1} e^{(i)}_k + l^{(i)}_{k+1}\right) \\ & + 2n^{(i)^T}_{k+1} P^{(i)^{-1}}_{k+1} \left(\tilde{F}^{(ii)}_{k+1} e^{(i)}_k + l^{(i)}_{k+1}\right) \end{aligned} \tag{46}$$

Applying Lemma 2 and Lemma 3, ΔV can be written as

$$\Delta V \leq -\gamma^{(i)} V(e_k^{(i)}) + \kappa_{nonl}^{(i)} \left|\left|e_k^{(i)}\right|\right|^3 + D(\tilde{\kappa}) \\ + n_{k+1}^{(i)^T} P_{k+1}^{(i)^{-1}} n_{k+1}^{(i)} + 2 n_{k+1}^{(i)^T} P_{k+1}^{(i)^{-1}} \left(\tilde{F}_{k+1}^{(ii)} e_k^{(i)} + l_{k+1}^{(i)}\right) \tag{47}$$

for $\left|\left|e_k^{(i)}\right|\right| \leq \epsilon^{(i)'}$.

As done in [18], choosing $\overline{\epsilon}^{(i)} = \min(\epsilon^{(i)'}, \frac{\gamma^{(i)}}{\psi^{(i)} \overline{p}^{(i)} \kappa_{nonl}})$ gives for $\left|\left|e_k^{(i)}\right|\right| \leq \overline{\epsilon}^{(i)}$ with $\psi^{(i)} > 1$

$$\kappa_{nonl}^{(i)} \left|\left|e_k^{(i)}\right|\right|^3 \leq \frac{\gamma^{(i)}}{\psi^{(i)}} V(e_k^{(i)}) \tag{48}$$

Considering the term $n_{k+1}^{(i)^T} P_{k+1}^{(i)^{-1}} n_{k+1}^{(i)}$ (see also [18]), the Eq. 39 and that $||w_k^{(i)}|| \leq \overline{w}^{(i)}$ and $||v_k^{(i)}|| \leq \overline{v}^{(i)}$, the following can be stated

$$\left|\left|n_{k+1}^{(i)^T} P_{k+1}^{(i)^{-1}} n_{k+1}^{(i)}\right|\right| \leq \frac{1}{\underline{p}^{(i)}} \left(\frac{\overline{p}^{(i)}}{\underline{q}^{(i)}} \overline{w}^{(i)} + \tilde{\kappa} \overline{v}^{(i)}\right)^2 \tag{49}$$

The same approach can be exploited for

$$\left|\left|2 n_{k+1}^{(i)^T} P_{k+1}^{(i)^{-1}} \left(\tilde{F}_{k+1}^{(ii)} e_k^{(i)} + l_{k+1}^{(i)}\right)\right|\right| \leq \frac{1}{\underline{p}^{(i)}} \left[2 \frac{\overline{p}^{(i)}}{\underline{q}^{(i)}} \overline{w}^{(i)} + \right. \tag{50}$$

$$\left. + \tilde{\kappa} \overline{v}\right] \left[\frac{\overline{p}^{(i)}}{\underline{q}^{(i)}} \overline{f}^{(i)} \left|\left|e_k^{(i)}\right|\right| + \frac{\overline{p}^{(i)}}{\underline{q}^{(i)}} \varphi_f^{(i)} \left|\left|e_k^{(i)}\right|\right|^2 + \tilde{\kappa} \varphi_h^{(i)} \left|\left|e_{k+1|k}^{(i)}\right|\right|^2\right]$$

Adding the two terms, we obtain

$$\left|\left|n_{k+1}^{(i)^T} P_{k+1}^{(i)^{-1}} n_{k+1}^{(i)}\right|\right| + \left|\left|2 n_{k+1}^{(i)^T} P_{k+1}^{(i)^{-1}} \left(\tilde{F}_{k+1}^{(ii)} e_k^{(i)} + l_{k+1}^{(i)}\right)\right|\right| \\ \leq A_2 \tilde{\kappa}^2 + B_2 \tilde{\kappa} + C_2 \tag{51}$$

where

$$A_2 = \frac{\overline{v}^{(i)}}{\underline{p}^{(i)}} \left(\varphi_f^{(i)2} \varphi_h^{(i)} \epsilon^{(i)'^4} + 2 \varphi_f^{(i)} \varphi_h^{(i)} \overline{f}^{(i)} \epsilon^{(i)'^3} + \right. \\ + \varphi_h^{(i)} \overline{f}^{(i)2} \epsilon^{(i)'^2} + \varphi_f^{(i)} \varphi_h^{(i)} \overline{w}^{(i)} \epsilon^{(i)'^2} + \\ \left. + 2 \varphi_h^{(i)} \overline{f}^{(i)} \overline{w}^{(i)} \epsilon^{(i)'} + \overline{v}^{(i)} + \varphi_h^{(i)} \overline{w}^{(i)}\right)$$

$$B_2 = \frac{\overline{p}^{(i)}}{\underline{p}^{(i)} \underline{q}^{(i)}} \left(2 \varphi_f^{(i)2} \varphi_h^{(i)} \epsilon^{(i)'^4} + 4 \varphi_f^{(i)} \varphi_h^{(i)} \overline{f}^{(i)} \epsilon^{(i)'^3} + \right. \\ + \varphi_h^{(i)} \overline{v}^{(i)} \epsilon^{(i)'^2} + 2 \varphi_h^{(i)} \overline{f}^{(i)2} \epsilon^{(i)'^2} + 2 \varphi_f^{(i)} \varphi_h^{(i)} \overline{w}^{(i)} \epsilon^{(i)'^2} + \\ \left. + \overline{f}^{(i)} \overline{v}^{(i)} \epsilon^{(i)'} + 4 \varphi_f^{(i)} \overline{f}^{(i)} \overline{w}^{(i)} \epsilon^{(i)'} + \varphi_f^{(i)} \overline{w}^{(i)2} + 2 \overline{w}^{(i)} \overline{v}^{(i)}\right)$$

$$C_2 = \frac{\overline{p}^{(i)2}}{\underline{p}^{(i)} \underline{q}^{(i)2}} \left(\varphi_f^{(i)} \epsilon^{(i)'^2} + 2 \overline{f}^{(i)} \epsilon^{(i)'} + \overline{w}^{(i)2}\right)$$

Therefore,

$$\Delta V \leq \frac{\gamma^{(i)}(1 - \psi^{(i)})}{\psi^{(i)}} V(e_k^{(i)}) + \rho^{(i)}(\tilde{\kappa}) \tag{52}$$

where

$$\rho^{(i)}(\tilde{\kappa}) = A_2 \tilde{\kappa}^2 + B_2 \tilde{\kappa} + C_2 + D_1$$

Since $0 < \gamma^{(i)} < 1$ and $\psi^{(i)} > 1$,

$$\xi^{(i)} = \frac{\gamma^{(i)}(1 - \psi^{(i)})}{\psi^{(i)}} \in (-1, 0) \tag{53}$$

it implies

$$\left|\left|e_{k+1}^{(i)}\right|\right|^2 \leq \frac{\overline{p}^{(i)}}{\underline{p}^{(i)}} \left(1 + \xi^{(i)}\right)^{k+1} \left|\left|e_0^{(i)}\right|\right|^2 - \frac{\overline{p}^{(i)}}{\xi^{(i)}} \rho^{(i)}(\tilde{\kappa}) \tag{54}$$

■

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