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Synchronisation in electrical circuits with memristors and grounded capacitors

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Abstract— **Motivated by neuromorphic computing applications, this paper considers electrical circuits comprising memristors and grounded capacitors, connected to external sources. By using the flux-charge domain modelling approach, we will derive an initial value problem describing the dynamic behaviour of this circuit. Given an initial value and a fixed input, we will show that the fluxes in this circuit converge to an equilibrium. Furthermore, we show that when the fluxes reach this equilibrium, we achieve voltage synchronisation, i.e. no more currents are flowing through the circuit. These results are emphasised in an illustration.**

I. INTRODUCTION

In 1971, Chua introduced a new two-terminal element, called the memristor [1], to complement the classical resistor, capacitor, and inductor. This element postulates a nonlinear relation between the electric charge and magnetic flux, and behaves as a resistor with memory storage. In 2008, researchers at Hewlett-Packard were the first to claim memristive behaviour in a material [2], i.e., behaviour that shows similarities with that of memristors. This achievement has furthered research activities in memristor-based circuits, which are expected to have a broad range of applications. For example, they play an important role in the field of neuromorphic computing [3], [4]. Due to their ability to store memory and their dynamic nature, they have the potential to act as synaptic weights for the implementation of artificial neural networks (ANN) in hardware [5].

Advances in material science have shown memristive behaviour in many different materials [6]. Currently, the interest of material scientists is gradually shifting in the direction of *networks* of devices exhibiting memristive behaviour [7]. However, these materials can often not be regarded as pure networks of memristors; parts of the network might correspond to other circuit elements such as capacitors.

Motivated by these advances in material science, and driven by their potential for neuromorphic hardware, this paper studies the dynamics of electrical circuits comprising memristors and (grounded) capacitors.

The current literature on memristor-based circuits includes studies on simple interconnections of memristors [8], [9], studies based on simulations [10], and a study on memristive port-Hamiltonian systems [11]. The works [12], [13] proposed to analyse memristor-based circuits in the flux-charge rather than current-voltage domain, which has led to an impulse in the study of circuits of memristors. Examples range from the study of bifurcations [14] to their use in solving optimisation problems [15]. We note that electrical circuits with memristors are necessarily nonlinear and that the study of nonlinear circuits has a long history [16], see also [17] for a recent contribution.

In this paper, we focus on fundamental properties of memristor-capacitor networks, and, motivated by the potential of memristors to act as synaptic weights for the implementation of ANNs in hardware, we study synchronisation in such networks. Namely, for memristors to be utilised as synaptic weights, the network needs to enable tuning of their resistance values during the training phase of the ANN, and, their instantaneous resistance value needs to be preserved when the circuit is not forced by external stimuli. The latter requires the voltage across a memristor to be zero, which corresponds to the synchronisation of voltage potentials.

This paper considers a specific class of circuits comprising memristors and grounded capacitors. Circuits of this class have dynamics that can be related to the literature on consensus networks (including dynamics on the edges) for which synchronisation is studied regularly [18]–[20].

In particular, the present paper has the following two contributions. First, we present a model for memristor networks with grounded capacitors in the flux-charge domain, inspired by [12], [13], and assuming that the memristors have strongly monotone behaviour. This results in a highly structured model which can be interpreted as a consensus network with state-dependent weights, when viewed in terms of current and voltages rather than fluxes and charges. However, the model in the flux-charge domain has the advantage of only requiring the memristor fluxes as states, thus leading to a reduced representation. Even though this comes at the cost of an explicit dependence on the initial condition, this significantly simplifies analysis.

Second, we study the synchronisation properties of these models, where synchronisation is defined in terms of the capacitor voltages. We show that synchronisation can be analysed in terms of the stability properties of an equilibrium of the memristor fluxes, and we use this to prove synchronisation.

The remainder of this paper is organised as follows. In Section II, we characterise memristors and capacitors, and formulate our problem statement. Section III presents the framework for modelling memristor circuits with grounded capacitors. Section IV deals with the synchronisation prop-

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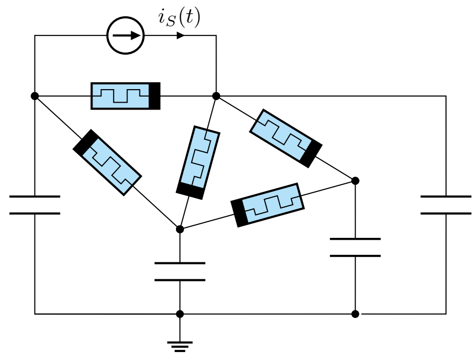

Fig. 1. Network of memristors and grounded capacitors attached to a source applying an input $i_S(t)$ to the circuit.

erties of such circuits, whereas Section V illustrates these results through an example. Section VI concludes the paper.

II. Problem formulation

In this section, we make the first steps in providing a mathematical framework for studying memristor-capacitor networks and we introduce the problem studied in this paper.

A. Memristors and capacitors networks

We consider electrical circuits with N nodes and $B = B_M + B_E + B_C$ branches consisting of memristors and grounded capacitors, connected to external sources. Here, B_M branches represent memristors which are connected to each other in an arbitrary manner to form a connected network of $N-1$ nodes. Then, B_E edges correspond to sources, each connecting two (distinct) nodes from the memristor network. Finally, a grounded capacitor is connected, to each of those $B_C = N-1$ nodes, with the ground node labelled as N. For an example, see Figure 1.

Let $i_M(t) \in \mathbb{R}^{B_M}$ and $v_M(t) \in \mathbb{R}^{B_M}$ capture the currents through and the voltages across the memristors. Denoting the corresponding vectors of charges by $q_M(t)$ and fluxes by $\varphi_M(t)$, we have

$$\tfrac{\mathrm{d}}{\mathrm{d}t} q_M(t) = i_M(t), \qquad \tfrac{\mathrm{d}}{\mathrm{d}t}\varphi_M(t) = v_M(t). \tag{1}$$

We consider flux-controlled memristors described by

$$q_M(t) = g(\varphi_M(t)) \tag{2}$$

where $g : \mathbb{R}^{B_M} \to \mathbb{R}^{B_M}$ is a vector-valued function that collects the (scalar) constitutive relations g_j of the individual memristors. In addition, we assume the functions g_j to be strongly monotone, i.e.,

$$\begin{aligned}(g(\varphi_M) - g(\varphi_M'))^\top (\varphi_M - \varphi_M') \\ \geq (\varphi_M - \varphi_M')^\top \mathcal{A} (\varphi_M - \varphi_M')\end{aligned} \tag{3}$$

holds for all $\varphi_M, \varphi_M' \in \mathbb{R}^{B_M}$. Here, $\mathcal{A}$ is a diagonal positive definite matrix.

Remark 1: We observe that, using (1), (2) can be equivalently modeled as the dynamical system

$$i_M(t) = \frac{\partial g(\varphi_M(t))}{\partial \varphi_M} v_M(t), \qquad \tfrac{\mathrm{d}}{\mathrm{d}t}\varphi_M(t) = v_M(t). \tag{4}$$

This can be regarded as a model of state-dependent resistors with the entries of $\frac{\partial g(\varphi_M)}{\partial \varphi_M}$ representing state-dependent conductance values. Then, (3) implies that these conductance values are positive, hence, our memristors are guaranteed to be passive, see [1, Theorem 6].

Next, we define $i_C(t) \in \mathbb{R}^{B_C}$ and $v_C(t) \in \mathbb{R}^{B_C}$ as the vectors of currents through and voltages across the capacitors, respectively. Consecutively, we define $q_C(t)$ as the vector of charges at the capacitor branches, which satisfies

$$\tfrac{\mathrm{d}}{\mathrm{d}t} q_C(t) = i_C(t).$$

Now, as in [21, Chapter 2.3.1], we model the capacitors as

$$q_C(t) = C v_C(t). \tag{5}$$

Here, C is a diagonal matrix containing the capacitance values of the different capacitors and hence is a positive definite matrix. Time-differentiation of the above leads to

$$i_C(t) = C \tfrac{\mathrm{d}}{\mathrm{d}t} v_C(t). \tag{6}$$

Then, going back to the full circuit, we characterise our circuit through the incidence matrix $D \in \mathbb{R}^{N \times B}$ and the corresponding graph $\mathcal{G}(D)$. Here, every column of D corresponds to a branch in $\mathcal{G}(D)$ and reads $e_k - e_\ell$ for a branch $\{k, \ell\}$ oriented from k to ℓ. Consequently, we have that $\mathbb{1}^\top D = 0$. Now, we split the incidence matrix D into the incidence matrices $\bar{D}_M \in \mathbb{R}^{N \times B_M}$, $\bar{D}_C \in \mathbb{R}^{N \times B_C}$, and $\bar{D}_E \in \mathbb{R}^{N \times B_E}$ corresponding to the memristors, capacitors and external sources, respectively. We assume that the incidence matrix D is ordered as $D = \begin{pmatrix} \bar{D}_M & \bar{D}_C & \bar{D}_E \end{pmatrix}$. Then, denoting the currents associated with the external sources by $i_E(t) \in \mathbb{R}^{B_E}$, Kirchhoff's current law reads

$$\bar{D}_M i_M(t) + \bar{D}_C i_C(t) + \bar{D}_E i_E(t) = 0. \tag{KCL}$$

On the other hand, Kirchhoff's voltage law can be stated as

$$\exists p(t) \in \mathbb{R}^N \text{ s.t. } \begin{pmatrix} v_C(t) \\ v_M(t) \\ v_E(t) \end{pmatrix} = \begin{pmatrix} \bar{D}_C^\top \\ \bar{D}_M^\top \\ \bar{D}_E^\top \end{pmatrix} p(t), \tag{KVL}$$

and thus guarantees the existence of a vector of voltage potentials at the nodes $p(t) \in \mathbb{R}^N$. Here, $v_E(t) \in \mathbb{R}^{B_E}$ denotes the vector of voltages associated with the external sources. In fact, we will consider current sources providing a vector of currents $i_S(t)$ and denote the corresponding voltages as $v_S(t)$. It follows that

$$i_E(t) = -i_S(t), \qquad v_E(t) = v_S(t). \tag{7}$$

B. Problem statement

To obtain a general model for our class of circuits comprising memristors and grounded capacitors, we will combine (KCL), (KVL) with the dynamics of the memristors (2) and capacitors (5). By the assumption that all capacitors are connected to the ground node and the memristors and external sources are not, we can write the incidence matrices as

$$\bar{D}_M = \begin{pmatrix} D_M \\ \mathbb{0} \end{pmatrix}, \bar{D}_C = \begin{pmatrix} I \\ -\mathbb{1}^\top \end{pmatrix}, \bar{D}_E = \begin{pmatrix} D_E \\ \mathbb{0} \end{pmatrix} \tag{8}$$

where the graph $\mathcal{G}(D_M)$ is connected.

After deriving the model description, we will study the behaviour of φ_M and v_C when no external source is applied to the circuit, i.e. when $i_S(t) = 0$ for all time $t \in \mathbb{R}$. More precisely, we will show that φ_M converges to an equilibrium implying that the circuit achieves voltage synchronisation.

Definition 1: Consider the electrical circuit defined through (KCL), (KVL), with memristors (2), capacitors (5), and sources (7). This circuit is said to achieve voltage synchronisation if, for all solutions corresponding to $i_S(t) = 0$ for all $t \in R$, we have that

$$\lim_{t\to\infty} v_C(t) = \alpha \mathbb{1} \text{ for some } \alpha \in \mathbb{R}.$$

III. Flux-charge analysis of memristor capacitor networks

Inspired by [12], [13], we will derive a modelling and analysis framework for our memristor-capacitor circuit in the domain of charges and fluxes. As memristors are described by a relation between charges and fluxes, this modelling approach will turn out to be more convenient than the usual approach on the basis of voltages and currents. Differences between the two approaches are highlighted in Remark 4.

As a first step, we lift Kirchhoff's laws to the domain of charges and fluxes. For any $t \in \mathbb{R}$, time integration of (KCL) over the interval $[0, t]$ leads to

$$\begin{aligned}\bar{D}_M(q_M(t) - q_M(0)) + \bar{D}_C(q_C(t) - q_C(0))& \\ = \bar{D}_E(q_S(t) - q_S(0)).& \qquad (9)\end{aligned}$$

To derive this result, we used (7) and we defined

$$q_S(t) - q_S(0) = \int_0^t i_S(\tau) d\tau$$

as the supplied charge. By substitution of (8) in (9) we obtain

$$\begin{aligned}D_M(q_M(t) - q_M(0)) + (q_C(t) - q_C(0))& \\ = D_E(q_S(t) - q_S(0))& \qquad (10)\end{aligned}$$

and

$$\mathbb{1}^\top (q_C(t) - q_C(0)) = 0$$

which should hold for all $t \in \mathbb{R}$. We note that the second equation is nothing more than Kirchhoff's current law at the ground node which is always satisfied when (10) holds. This can be observed by multiplying (10) from the left by $\mathbb{1}^\top$ and by recalling that $\mathbb{1}^\top D = 0$. By utilising the definitions of the memristors and capacitors in the circuit, i.e. by using (2) and (5), we can rewrite (10) as

$$\begin{aligned}D_M\big(g(\varphi_M(t)) - g(\varphi_M(0))\big) + C(v_C(t) - v_C(0))& \\ = D_E(q_S(t) - q_S(0)).& \qquad (11)\end{aligned}$$

We will simplify notation by omitting the arguments and write $\varphi = \varphi_M(t)$, $\varphi_0 = \varphi_M(0)$ for the fluxes across the memristors. Following similar notation for the other variables, (11) leads to

$$\begin{aligned}D_M(g(\varphi) - g(\varphi_0)) + C(v_C - v_{C,0})& \\ = D_E(q_S - q_{S,0}).& \qquad (12)\end{aligned}$$

Next, we will turn our attention to (KVL). We recall that $D^\top \mathbb{1} = 0$, hence the vector of voltage potentials p satisfying (KVL) is not unique, i.e. $\tilde{p}(t) = p(t) + \alpha \mathbb{1}$ satisfies (KVL) for any $\alpha \in \mathbb{R}$. Without loss of generality, we assume the potential at the ground node to equal 0, giving the vector

$$p(t) = \begin{pmatrix} \bar{p}(t) \\ 0 \end{pmatrix}.$$

By substitution of (7), (8), and the above in (KVL), we obtain

$$\exists \bar{p} \in \mathbb{R}^{N-1} \text{ s.t. } \begin{pmatrix} v_C \\ v_M \\ v_S \end{pmatrix} = \begin{pmatrix} I \\ D_M^\top \\ D_E^\top \end{pmatrix} \bar{p}, \qquad (13)$$

where, for sake of simplicity, we omitted the arguments. We note that the first row of this equation can be used to rewrite the second and third row as

$$\begin{pmatrix} v_M \\ v_S \end{pmatrix} = \begin{pmatrix} D_M^\top \\ D_E^\top \end{pmatrix} v_C. \qquad (14)$$

By combining the above with (1) we obtain

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\varphi = D_M^\top v_C. \qquad (15)$$

Then, by substitution of (12) in (15), we obtain that a memristor-capacitor network can be described by

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}\varphi = &-D_M^\top C^{-1} D_M (g(\varphi) - g(\varphi_0)) + D_M^\top v_{C,0} \\ &+ D_M^\top C^{-1} D_E (q_S - q_{S,0}), \qquad \varphi(0) = \varphi_0. \quad (16)\end{aligned}$$

Remark 2: We note that the vector field in (16) depends on the initial condition φ_0. In fact, (16) represents an initial value problem of the form

$$\tfrac{\mathrm{d}}{\mathrm{d}t} x = f(x, x_0), \qquad x(0) = x_0.$$

Because of this structure, we will see later that properties of (16) depend on the initial condition φ_0.

Remark 3: We note that a modelling approach on the basis of voltages and currents, instead of an initial value problem in φ, leads to a differential equation in *both* v_C and φ. Recall that the memristors described by (2) can be equivalently modelled as (4). Then, substitution of (14) in (4) gives

$$i_M = \frac{\partial g(\varphi)}{\partial \varphi} D_M^\top v_C, \qquad \tfrac{\mathrm{d}}{\mathrm{d}t}\varphi = D_M^\top v_C. \qquad (17)$$

Next, by combining (KCL) with (7) and (8), we observe that

$$D_M i_M + i_C - D_E i_S = 0$$

Substitution of the above in (17), together with the dynamics of the capacitors given in (6), leads to

$$\begin{aligned} C\tfrac{\mathrm{d}}{\mathrm{d}t} v_C &= -D_M \frac{\partial g(\varphi)}{\partial \varphi} D_M^\top v_C + D_E i_S, \\ \tfrac{\mathrm{d}}{\mathrm{d}t}\varphi &= D_M^\top v_C. \end{aligned} \qquad (18)$$

Note that the matrix $D_M \frac{\partial g(\varphi)}{\partial \varphi} D_M^\top$ can be viewed as a state-dependent Laplacian, such that (18) can be regarded as a consensus network with state-dependent weights. In other words, (18) represents a network dynamical system with dynamics on both the nodes (capacitors) and edges (memristors), as studied in, e.g., [19], [20].

Remark 4: We observe that (16) and (18) give equivalent model descriptions of networks of memristors with grounded capacitors, each with their advantages and disadvantages. Even though (18) has a clear interpretation as a consensus network with state-dependent Laplacian, it turns out that the asymptotic behaviour of φ is difficult to analyse. We will show that the latter can conveniently be done using (16), which has the additional advantage that it is a *reduced* description as it only has the memristor fluxes as state variables. The observation that (16) provides a reduced-order model compared with (18) is in agreement with similar results shown in [12].

IV. SYNCHRONISATION

In this section, we will study under which conditions our memristor-capacitor circuit achieves voltage synchronisation. To do so, we will show that, for an initial condition φ_0, the solution φ to the initial value problem (16) converges to an equilibrium, which in turn implies voltage synchronisation of the circuit. Throughout this section we will assume the graph $\mathcal{G}(D_M)$ to be connected, hence $\ker(D_M^\top) = \operatorname{span}\mathbb{1}$.

Lemma 1: Consider (16) for some given $\varphi_0 \in \mathbb{R}^{B_M}$, $q_S - q_{S,0} = \bar{q}_S \in \mathbb{R}^{B_E}$, and $v_{C,0} \in \mathbb{R}^{B_C}$. Assume that there exist $\bar{\varphi} \in \mathbb{R}^{B_M}$ such that

1) $\bar{\varphi}$ is an equilibrium point of (16), i.e.,

$$\begin{aligned} &- D_M^\top C^{-1} D_M (g(\bar{\varphi}) - g(\varphi_0)) \\ &\qquad + D_M^\top C^{-1} D_E \bar{q}_S + D_M^\top v_{C,0} = 0. \end{aligned} \tag{19}$$

2) the solution φ to the initial value problem (16) satisfies

$$\lim_{t\to\infty} \varphi(t) = \bar{\varphi}. \tag{20}$$

Then, we have that

$$\lim_{t\to\infty} v_C(t) = \alpha\mathbb{1} \text{ with } \alpha = \frac{\mathbb{1}^\top C v_{C,0}}{\mathbb{1}^\top C \mathbb{1}}. \tag{21}$$

Proof: Let $\bar{\varphi}$ be such that (19) and (20) are satisfied. Then, by substitution of (19) in (16), we can write the dynamics of φ in terms of $\bar{\varphi}$ as

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\varphi = -D_M^\top C^{-1} D_M (g(\varphi) - g(\bar{\varphi})). \tag{22}$$

Next, by combining the above with (15) we obtain

$$D_M^\top v_C = -D_M^\top C^{-1} D_M (g(\varphi) - g(\bar{\varphi})).$$

Taking the limit on both sides of the equation leads to $\lim_{t\to\infty} D_M^\top v_C(t) = 0$, due to our assumption that (20) holds. Since $\ker D_M^\top = \operatorname{span}\mathbb{1}$, we have that $\lim_{t\to\infty} v_C(t) = \alpha\mathbb{1}$ for some $\alpha \in \mathbb{R}$. Then, since all capacitors are connected to the ground node, Kirchhoff's current law gives that $\mathbb{1}^\top i_C(t) = 0$ for all $t \in \mathbb{R}$, and hence, by (6), $\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{1}^\top C v_C(t) = 0$ for all $t \in \mathbb{R}$. It follows that $\mathbb{1}^\top C v_C(t) = \mathbb{1}^\top C v_{C,0}$ for all $t \in \mathbb{R}$, consequently when the circuit achieves voltage synchronisation we have that $\alpha\mathbb{1}^\top C\mathbb{1} = \mathbb{1}^\top C v_{C,0}$ implying the desired result (21). ■

Lemma 1 shows that convergence of the solution φ of (16) towards an equilibrium $\bar{\varphi}$ implies voltage synchronisation of the circuit. We are left to show that such an equilibrium $\bar{\varphi}$ of (16) exists. Note that, by the structure of (16), $\frac{\mathrm{d}}{\mathrm{d}t}\varphi \in \operatorname{im} D_M^\top$, implying that, for all $t \geq 0$, φ satisfies

$$\varphi(t) \in \{\varphi \,|\, \varphi - \varphi_0 \in \operatorname{im} D_M^\top\}. \tag{23}$$

This motivates for searching an equilibrium $\bar{\varphi}$ in the above set. To do so, we will introduce some new notation.

Assume $q_S - q_{S,0} = \bar{q}_S \in \mathbb{R}^{B_E}$, $\varphi_0 \in \mathbb{R}^{B_M}$, and $v_{C,0} \in \mathbb{R}^{B_C}$ are given. Then, an equilibrium $\bar{\varphi}$ of (16) is such that (19) is satisfied. We observe that (19) can be rewritten as

$$\begin{aligned} &- C^{-1} D_M (g(\bar{\varphi}) - g(\varphi_0)) \\ &\qquad + C^{-1} D_E \bar{q}_S + v_{C,0} \in \operatorname{span}\mathbb{1}, \end{aligned} \tag{24}$$

since $\ker(D_M) = \operatorname{span}\mathbb{1}$. Furthermore, since any solution φ of (16) satisfies (23), for some $\xi \in \mathbb{R}^{B_C}$, we can write

$$\bar{\varphi} = D_M^\top C^{-1}\xi + \varphi_0. \tag{25}$$

By substitution of the above in (24), we obtain

$$\begin{aligned} &- C^{-1} D_M (g(D_M^\top C^{-1}\xi + \varphi_0) - g(\varphi_0)) \\ &\qquad + C^{-1} D_E \bar{q}_S + v_{C,0} \in \operatorname{span}\mathbb{1}. \end{aligned} \tag{26}$$

Now, we define

$$f(\xi) := C^{-1} D_M \big(g(D_M^\top C^{-1}\xi + \varphi_0) - g(\varphi_0)\big)$$

and

$$\gamma := C^{-1} D_E \bar{q}_S + v_{C,0}. \tag{27}$$

Using this new notation (26) reads

$$-f(\xi) + \gamma \in \operatorname{span}\mathbb{1}. \tag{28}$$

We can now state the following Lemma.

Lemma 2: For each $\gamma \in \mathbb{R}^{B_C}$ there exists a unique $\xi \in \ker\mathbb{1}^\top$ satisfying (28).

Proof: Consider the set $S = \ker\mathbb{1}^\top$. We can show, following [22, Chapter 2], that f is strongly monotone on S. Let $\xi, \xi' \in S$, then we have that

$$\begin{aligned} &(f(\xi) - f(\xi'))^\top(\xi - \xi') \\ &\qquad = \big(C^{-1} D_M (g(\bar{\varphi}) - g(\bar{\varphi}'))\big)^\top (\xi - \xi'). \end{aligned} \tag{29}$$

Here, $\bar{\varphi}$ and $\bar{\varphi}'$ are such that (25) and

$$\bar{\varphi}' = D_M^\top C^{-1}\xi' + \varphi_0 \tag{30}$$

are satisfied, respectively. By working out the first bracket in the right-hand side of (29), we obtain

$$(f(\xi) - f(\xi'))^\top(\xi - \xi') = (g(\bar{\varphi}) - g(\bar{\varphi}'))^\top(\bar{\varphi} - \bar{\varphi}'),$$

whose right-hand side can be bounded using (3) to give

$$(f(\xi) - f(\xi'))^\top(\xi - \xi') \geq (\bar{\varphi} - \bar{\varphi}')^\top \mathcal{A} (\bar{\varphi} - \bar{\varphi}').$$

Then, substitution of (25) and (30) in the above leads to

$$\begin{aligned} &(f(\xi) - f(\xi'))^\top(\xi - \xi') \\ &\qquad \geq (\xi - \xi')^\top C^{-1} D_M \mathcal{A} D_M^\top C^{-1} (\xi - \xi'). \end{aligned} \tag{31}$$

Since $\mathcal{A}$ is a positive definite matrix the right-hand side of the above is positive if $\xi - \xi'$ is such that $D_M^\top C^{-1}(\xi - \xi') \neq 0$. By using a contradiction argument, we will show that this is the case for all $\xi, \xi' \in S$. To establish a contradiction, let $\xi, \xi' \in S$ with $\xi \neq \xi'$ be such that $D_M^\top C^{-1}(\xi - \xi') = 0$, then $C^{-1}(\xi - \xi') = \alpha \mathbb{1}$ for some $\alpha \in \mathbb{R}$ since $\ker D_M^\top = \mathbb{1}$. It follows that $(\xi - \xi') = \alpha C \mathbb{1}$. However, as $\xi, \xi' \in S$, we have that $0 = \mathbb{1}^\top(\xi - \xi') = \alpha \mathbb{1}^\top C \mathbb{1}$ implying that $\alpha = 0$. We conclude that the right-hand side of (31) is positive for all $\xi, \xi' \in S$ with $\xi \neq \xi'$. In fact, one can show that there exists $\beta > 0$ such that, for all $\xi, \xi' \in S$,

$$(f(\xi) - f(\xi'))^\top (\xi - \xi') \geq \beta |\xi - \xi'|^2,$$

i.e., f is strongly monotone on S. The value of β can be computed using quadratic programming, e.g. [23, Chapter 16].

Now, strong monotonicity of f on S and the fact that $\operatorname{span} \mathbb{1} = S^\perp$ imply that (28) has a unique (on S) solution ξ, see [22, Theorem 2F.9 on p. 111]. ■

Lemma 2 now enables us to state the following result.

Theorem 3: Consider (16) for some given $\varphi_0 \in \mathbb{R}^{B_M}$, $q_S - q_{S,0} = \bar{q}_S$, and $v_{C,0} \in \mathbb{R}^{B_C}$. Then, there exists a unique $\bar{\varphi}$ of the form (25) such that (19) holds.

Proof: By (27), we can compute γ. Then, by Lemma 2 there exists a unique $\xi \in \ker \mathbb{1}^\top$ satisfying (28). Then, $\bar{\varphi}$ given as (25) has the desired properties, as follows from the definition of f. ■

The above result shows the existence of an equilibrium $\bar{\varphi}$ of (16). Now, we discuss the asymptotic behaviour of the solution φ to (16).

Lemma 4: Let $\varphi_0 \in \mathbb{R}^{B_M}$, $q_S - q_{S,0} = \bar{q}_S \in \mathbb{R}^{B_E}$, and $v_{C,0} \in \mathbb{R}^{B_C}$ be given and assume $\bar{\varphi}$ such that (19) holds. Then, the solution φ to the initial value problem (16) satisfies

$$\lim_{t \to \infty} \varphi(t) = \bar{\varphi}.$$

Proof: We define the Lyapunov function candidate $V : \mathbb{R}^{B_M} \to \mathbb{R}$ as

$$V(\varphi) = \sum_{j=1}^{B_M} \int_{\bar{\varphi}_j}^{\varphi_j} g_j(\theta_j) - g_j(\bar{\varphi}_j) d\theta_j. \tag{32}$$

It is easy to see that $V(\bar{\varphi}) = 0$ and it follows from (3) that $V(\varphi) > 0$ for all $\varphi \in \mathbb{R}^{B_M} \backslash \{\bar{\varphi}\}$. In addition, we can show that it is a strongly convex function. We first observe that, by definition, (32) is continuously differentiable and its structure implies that the row vector of partial derivatives reads

$$\frac{\partial V(\varphi)}{\partial \varphi} = (g(\varphi) - g(\bar{\varphi}))^\top. \tag{33}$$

Then, for all $\varphi, \varphi' \in \mathbb{R}^{B_M}$, we obtain

$$\left(\frac{\partial V(\varphi)}{\partial \varphi} - \frac{\partial V(\varphi')}{\partial \varphi} \right) (\varphi - \varphi') \geq (\varphi - \varphi')^\top \mathcal{A} (\varphi - \varphi').$$

Here, we have used (3) to obtain the inequality. As $\mathcal{A}$ is positive definite, it follows from [24, Theorem 2.1.9] that V is strongly convex. This, together with $V(\bar{\varphi}) = 0$ and the fact that $V(\varphi) > 0$ on $\mathbb{R}^{B_M} \backslash \{\bar{\varphi}\}$ shows that V is positive definite and radially unbounded. Next, we will evaluate V along the solution φ to the initial value problem (16). To do so, note that a direct computation, using (33) and (22), gives

$$\tfrac{\mathrm{d}}{\mathrm{d}t} V(\varphi) = -\big(g(\varphi) - g(\bar{\varphi})\big)^\top D_M^\top C^{-1} D_M (g(\varphi) - g(\bar{\varphi})).$$

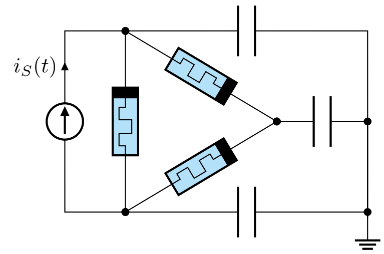

Fig. 2. Circuit consisting of three memristors and three grounded capacitors, attached to a source applying an input $i_S(t)$ to the circuit.

We observe that $\frac{\mathrm{d}}{\mathrm{d}t} V(\varphi)$ is negative definite relative to $\varphi \in \mathbb{R}^{B_M} \backslash \{\bar{\varphi}\}$. To show this, we note that $\frac{\mathrm{d}}{\mathrm{d}t} V(\varphi) = 0$ implies

$$D_M (g(\varphi) - g(\bar{\varphi})) = 0, \tag{34}$$

since C is a positive definite matrix by definition. Substitution of (34) in (22) leads to $\frac{\mathrm{d}}{\mathrm{d}t}\varphi = 0$, hence φ is an equilibrium. However, by Theorem 3 there only exists a unique equilibrium $\bar{\varphi}$ of (16) satisfying (23), hence $\varphi = \bar{\varphi}$. Finally, Lyapunov's stability theorem [25, Theorem 4.2 on p. 124] implies that the trajectory φ converges to $\bar{\varphi}$. ■

Now, we can derive the final result.

Theorem 5: Consider (16) for some given $\varphi_0 \in \mathbb{R}^{B_M}$, $q_S - q_{S,0} = \bar{q}_S \in \mathbb{R}^{B_E}$, and $v_{C,0} \in \mathbb{R}^{B_C}$, then

$$\lim_{t \to \infty} v_C(t) = \left(\frac{\mathbb{1}^\top C v_{C,0}}{\mathbb{1}^\top C \mathbb{1}} \right) \mathbb{1}. \tag{35}$$

Proof: By Theorem 3 there exists $\bar{\varphi}$ such that (19) holds. Lemma 4 implies that the solution φ to (16) satisfies (20). Finally, Lemma 1 gives the desired result (35). ■

Theorem 5 shows that, for each initial value, our circuit achieves voltage synchronisation.

V. Illustration

In the previous section, we saw that, for each initial value, the fluxes in our circuit converge to an equilibrium, implying voltage synchronisation. In this section, we will demonstrate these results by providing simulation results for the simple circuit depicted in Figure 2.

We assume that the functions g_j in (2) are given by

$$g_j(\varphi_j) = 2\varphi_j - \tfrac{1}{2}\log(\varphi_j^2 + 1) + \varphi_j \arctan(\varphi_j)$$

for each $j \in \{1, 2, 3\}$, and we let $C = 2I$. We assume the input $\bar{q}_S = 0$ and the initial condition $v_{C,0}^\top = \begin{pmatrix} 0 & 5 & 1 \end{pmatrix}$. Then, the solution φ to (16) and accompanying v_C from (11) are plotted for two different initial conditions $\varphi_0^\top = \begin{pmatrix} -1 & -1 & -1 \end{pmatrix}$ and $\varphi_0^\top = \begin{pmatrix} 0 & 3 & -2 \end{pmatrix}$ in Figure 3 and 4, respectively. We observe that for both initial conditions φ_0 we achieve voltage synchronisation, and although the fluxes converge to different equilibria, in both cases the voltages converge to the value two. This is in agreement with the result in Theorem 5. In addition, as a consequence, we note that, different initial conditions lead to different resistance values of the memristors. This has relevance when considering memristors to act as synaptic weights in ANNs as it shows a way to tune the resistance values of the memristors.

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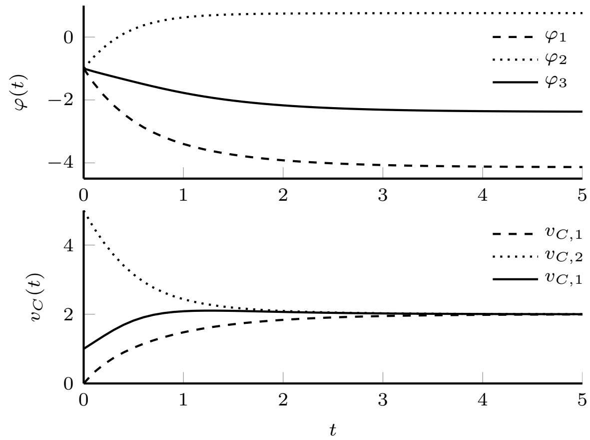

Fig. 3. Curves of $\varphi(t)$ and $v_C(t)$ for the circuit depicted in Figure 2 corresponding to the initial conditions $\varphi_0 = \begin{pmatrix}-1 & -1 & -1\end{pmatrix}^\top$, $v_{C,0} = \begin{pmatrix}0 & 5 & 1\end{pmatrix}^\top$, and the input $\bar{q}_s = 0$.

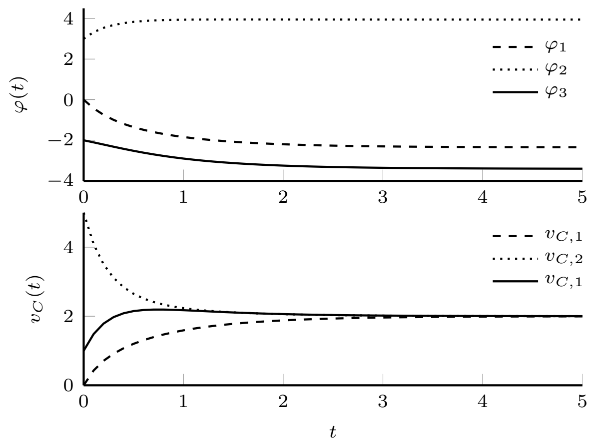

Fig. 4. Curves of $\varphi(t)$ and $v_C(t)$ for the circuit depicted in Figure 2 corresponding to the initial condition $\varphi_0 = \begin{pmatrix}0 & 3 & -2\end{pmatrix}^\top$, $v_{C,0} = \begin{pmatrix}0 & 5 & 1\end{pmatrix}^\top$, and the input $\bar{q}_s = 0$.

VI. Conclusions

In this paper, we have applied the flux-charge domain modelling approach to derive a modelling framework describing circuits of memristors connected to grounded capacitors. This framework was utilised to show voltage synchronisation in this type of circuit. These results provide a first step in creating a better understanding of the dynamical behaviour of memristor-capacitor circuits.

Future research will be devoted to studying how the instantaneous resistance value of the memristors can be influenced by external stimili. In addition, we want to study the (synchronisation) properties of a broader class of memristor-based circuits and their use in neuromorphic computing applications.

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References

[1] L. Chua, "Memristor-the missing circuit element," *IEEE Transactions on Circuit Theory*, vol. 18, no. 5, pp. 507–519, 1971.
[2] D. B. Strukov, G. S. Snider, D. R. Stewart, and R. S. Williams, "The missing memristor found," *Nature*, vol. 453, no. 7191, pp. 80–83, 2008.
[3] C. Mead, "Neuromorphic electronic systems," *Proceedings of the IEEE*, vol. 78, no. 10, pp. 1629–1636, 1990.
[4] C. D. James, J. B. Aimone, N. E. Miner, C. M. Vineyard, F. H. Rothganger, K. D. Carlson, S. A. Mulder, T. J. Draelos, A. Faust, and M. J. Marinella, "A historical survey of algorithms and hardware architectures for neural-inspired and neuromorphic computing applications," *Biologically Inspired Cognitive Architectures*, vol. 19, pp. 49–64, 2017.
[5] C. Zamarreño-Ramos, L. A. Camuñas-Mesa, J. A. Pérez-Carrasco, T. Masquelier, T. Serrano-Gotarredona, and B. Linares-Barranco, "On spike-timing-dependent-plasticity, memristive devices, and building a self-learning visual cortex," *Frontiers in Neuroscience*, vol. 5, 2011.
[6] T. Chang, Y. Yang, and W. Lu, "Building neuromorphic circuits with memristive devices," *IEEE Circuits and Systems Magazine*, vol. 13, no. 2, pp. 56–73, 2013.
[7] J. L. Rieck, D. Cipollini, M. Salverda, C. P. Quinteros, L. R. B. Schomaker, and B. Noheda, "Ferroelastic domain walls in $BiFeO_3$ as memristive networks," *Advanced Intelligent Systems*, p. 2200292, 2022.
[8] D. Yu, H. H.-C. Iu, Y. Liang, T. Fernando, and L. O. Chua, "Dynamic behavior of coupled memristor circuits," *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 62, no. 6, pp. 1607–1616, 2015.
[9] Q. Xia and J. J. Yang, "Memristive crossbar arrays for brain-inspired computing," *Nature Materials*, vol. 18, no. 4, pp. 309–323, mar 2019.
[10] I. Vourkas and G. C. Sirakoulis, Eds., *Memristor-based nanoelectronic computing circuits and architectures*. Springer, 2016, vol. 19.
[11] D. Jeltsema and A. J. van der Schaft, "Memristive port-hamiltonian systems," *Mathematical and Computer Modelling of Dynamical Systems*, vol. 16, no. 2, pp. 75–93, 2010.
[12] F. Corinto and M. Forti, "Memristor circuits: Flux-charge analysis method," *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 63, no. 11, pp. 1997–2009, 2016.
[13] F. Corinto, M. Forti, and L. O. Chua, *Nonlinear Circuits and Systems with Memristors*. Springer International Publishing, 2021.
[14] F. Corinto and M. Forti, "Memristor circuits: Bifurcations without parameters," *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 64, no. 6, pp. 1540–1551, 2017.
[15] M. Di Marco, M. Forti, L. Pancioni, G. Innocenti, and A. Tesi, "Memristor neural networks for linear and quadratic programming problems," *IEEE Transactions on Cybernetics*, vol. 52, no. 3, pp. 1822–1835, 2022.
[16] L. O. Chua, C. A. Desoer, and E. S. Kuh, *Linear and Nonlinear Circuits*. McGraw-Hill, 1987.
[17] T. Chaffey and R. Sepulchre, "Monotone one-port circuits," arXiv:2012.11533, 2021.
[18] M. Mesbani and M. Egerstedt, *Graph Theoretic Methods in Multiagent Networks*. Princeton University Press, 2010.
[19] M. Bürger, D. Zelazo, and F. Allgöwer, "Duality and network theory in passivity-based cooperative control," *Automatica*, vol. 50, no. 8, pp. 2051–2061, 2014.
[20] H. Chen, D. Zelazo, X. Wang, and L. Shen, "Convergence analysis of signed nonlinear networks," *IEEE Transactions on Control of Network Systems*, vol. 7, no. 1, pp. 189–200, 2020.
[21] C. A. Desoer and E. S. Kuh, *Basic Circuit Theory*. McGraw-Hill, 1969.
[22] A. L. Dontchev and R. T. Rockafellar, *Implicit Functions and Solution Mappings*. Springer New York, 2009.
[23] J. Nocedal and S. J. Wright, *Numerical Optimization*. Springer, 2006.
[24] Y. Nesterov, *Introductory Lectures on Convex Optimization: A basic course*. Kluwer Academic Publishers, 2004.
[25] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Prentice-Hall, 2002.