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Integral Controlled Lagrangians for underactuated mechanical systems subject to position-dependent matched disturbances

Enrico Franco[1], and Fulvio Forni[2]

Abstract— **This work investigates the dynamic extension of the Controlled Lagrangians methodology for underactuated mechanical systems subject to matched disturbances that depend on the generalized position. A new passivity-preserving controller design procedure is presented for a class of underactuated mechanical systems. An interpretation of the dynamic extension as first-order low-pass filter is proposed. Simulations results on a inertia-wheel pendulum with various types of disturbances demonstrate the effectiveness of the new controller.**

I. Introduction

The control of underactuated mechanical systems has been approached with various methodologies [1], each having its respective merits. In particular, energy-shaping controllers have risen to prominence thanks to their interpretability in terms of mechanical energy, and to their ability to accommodate nonlinearities. The most notable energy-shaping methodologies include the *Interconnection and damping assignment Passivity based control* (IDA-PBC) [2] and the *Controlled Lagrangians* (CL) [3], [4], [5]. Both approaches hinge on designing the control action such that the closed-loop dynamics preserves the structure of a mechanical system and is characterized by a desired total energy. In spite of their different implementations, which rely on either the port-controlled Hamiltonian (PCH) or the Lagrangian formulation, there is a clear similarity between IDA-PBC and CL, see [6], and both approaches have found wide applicability in various domains. For simplicity, the original formulations of IDA-PBC and CL did not account for the effect of disturbances or model uncertainties. The effect of velocity-dependent forces was then investigated as part of the CL methodology, see [5], [7]. In parallel, a number of research works investigated the robustification of IDA-PBC *vis a vis* constant matched disturbances resulting in the so-called *integral IDA-PBC* methodology (iIDA-PBC) [8], [9], [10]. More recent works have extended the iIDA-PBC methodology to systems with non-constant matched disturbances [11], with state-dependent matched disturbances [12], and with constant unmatched disturbances [13], [14]. While there is a long tradition of employing integral actions for the control of mechanical systems, see [15], [16], [17], [18], to the best of the authors' knowledge, an exact equivalent of the iIDA-PBC for the CL methodology that preserves passivity and that is applicable to underactuated mechanical systems is not available. This limits the applicability of the CL methodology to real mechanical systems, where external disturbances and model uncertainties are ubiquitous.

The main contribution of this work is a new passivity-preserving dynamic extension of the CL methodology for underactuated mechanical systems that compensates the effect of matched additive disturbances dependent on the generalized position. This is a relevant problem in engineering practice, since position-dependent disturbances are representative of uncertain stiffness in mechanical components. Differently from prior works on iIDA-PBC [12], the proposed controller is expressed in a more general form, which is directly applicable to systems with non-constant input matrix. The simpler case of constant matched disturbances is also discussed, and an interpretation of the dynamic extension as first-order low-pass filter is proposed. The new controller design procedure is compared with the iIDA-PBC methodology and with the use of nonlinear observers. Finally, the effectiveness of the controller is demonstrated with simulations on a inertia-wheel pendulum with various matched disturbances.

Notation. Function arguments are specified on first use and subsequently omitted in equations for conciseness.

II. Overview of Controlled Lagrangians

Consider an underactuated mechanical system with n DOFs and the control input $u \in \mathbb{R}^m$ applied through the input matrix $G(q) \in \mathbb{R}^{n\times m}$, where $\operatorname{rank}(G) = m < n$ for all $q \in \mathbb{R}^n$, and subject to the disturbances $\delta(q) \in \mathbb{R}^n$. The system states are the position $q \in \mathbb{R}^n$ and the velocity $\dot q \in \mathbb{R}^n$, and the Lagrangian is defined as

$$L(q,\dot q) = \frac{1}{2}\dot q^\top M(q)\dot q - V(q), \tag{1}$$

where the inertia matrix is $M(q) = M(q)^\top \succ 0$, and the potential energy is $V(q)$. The system dynamics is defined as

$$\frac{\mathrm{d}}{\mathrm{d}t}\partial_{\dot q} L(q,\dot q) - \partial_q L(q,\dot q) = G(q)u - \delta(q). \tag{2}$$

The CL methodology [3] aims at stabilizing the prescribed equilibrium $(q,\dot q) = (q^\star, 0)$ by assigning a new Lagrangian $L_c(q,\dot q) = \frac{1}{2}\dot q^\top M_c(q)\dot q - V_c(q)$ such that in closed loop

$$\frac{\mathrm{d}}{\mathrm{d}t}\partial_{\dot q} L_c(q,\dot q) - \partial_q L_c(q,\dot q) = 0, \tag{3}$$

where the potential energy $V_c(q) > 0$ admits a strict minimizer in $q^\star$ hence verifying the conditions $\partial_q V_c(q^\star) = 0$ and $\partial_q^2 V_c(q^\star) \succ 0$. The other design parameter besides $V_c(q)$ is the inertia matrix $M_c(q) = M_c^\top(q) \succ 0$.

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[1]Mechanical Engineering Department, Imperial College London, SW7 2AZ, London, UK.
[2]Department of Engineering, University of Cambridge, CB2 1PZ UK.
Corresponding author: Enrico Franco, ef1311@imperial.ac.uk

This *shaping* of the system energy is achieved, in the absence of disturbances (i.e., $\delta = 0$), by using the control law [6]

$$\begin{aligned} u_{CL} = G^{\dagger} \left(\partial_q(M\dot{q})\dot{q} - \partial_q L\right) \\ -G^{\dagger} M M_c^{-1} \left(\partial_q(M_c\dot{q})\dot{q} - \partial_q L_c\right), \end{aligned} \tag{4}$$

where $G^{\dagger} = \left(G^{\top} G\right)^{-1} G^{\top}$. The control law (4) is implementable provided that $M_c(q)$ and $V_c(q)$ verify for all $(q, \dot{q}) \in \mathbb{R}^{2n}$ the partial differential equations (PDEs)

$$G^{\perp} \left(-\partial_q \left(\frac{1}{2}\dot{q}^{\top} M \dot{q}\right) + M M_c^{-1} \partial_q \left(\frac{1}{2}\dot{q}^{\top} M_c \dot{q}\right)\right)$$
$$+G^{\perp} \left(\partial_q(M\dot{q})\dot{q} - M M_c^{-1} \partial_q(M_c\dot{q})\right) = 0, \tag{5}$$
$$G^{\perp} \left(\partial_q V - M M_c^{-1} \partial_q V_c\right) = 0, \tag{6}$$

where $G^{\perp}(q)$ is defined so that $G^{\perp} G = 0$ and $\text{rank}\left(G^{\perp}\right) = n - m, \forall q \in \mathbb{R}^n$. As demonstrated in [6], the CL control law (4) is identical to the IDA-PBC control law [2]

$$u_{IDA} = G^{\dagger} \left(\partial_q H - M_d M^{-1} \partial_q H_d + J_2 \partial_p H_d\right),$$

where $p = M\dot{q}$ are the momenta, $H(q,p) = \frac{1}{2} p^{\top} M^{-1} p + V$ and $H_d(q,p) = \frac{1}{2} p^{\top} M_d^{-1} p + V_d$ are the open-loop Hamiltonian and the desired Hamiltonian respectively, provided that

$$M_c(q) = M M_d^{-1} M,\ V_c(q) = V_d(q),$$
$$J_2 = M_d M^{-1} \left(\partial_q (M M_d^{-1} p)^{\top} - \partial_q (M M_d^{-1} p)\right) M^{-1} M_d.$$

The effect of physical damping can be accounted for in (2) by introducing a dissipation function $\mathcal{R}(\dot{q})$ such that $\dot{q}^{\top} \partial_{\dot{q}} \mathcal{R} > 0, \forall \dot{q} \neq 0$, see [5]. Similarly, damping injection can be achieved for the purpose of stabilizing the equilibrium $(q, \dot{q}) = (q^{\star}, 0)$ by introducing in (3) the closed-loop dissipation $\partial_{\dot{q}} \mathcal{R}_c = M_c M^{-1} \left(\partial_{\dot{q}} \mathcal{R} + G K_v G^{\top} M^{-1} M_c \dot{q}\right)$ such that $\dot{q}^{\top} \partial_{\dot{q}} \mathcal{R}_c > 0, \forall \dot{q} \neq 0$, where $K_v = K_v^{\top} \succ 0$ is a tuning parameter. This is achieved by including a so-called damping-injection term u_d in the control law (4), that is

$$u = u_{CL} + u_d,\ u_d = -K_v G^{\top} M^{-1} M_c \dot{q}. \tag{7}$$

Defining the storage function $W = V_c + \frac{1}{2}\dot{q}^{\top} M_c \dot{q}$ and computing its time-derivative along the trajectories of the closed-loop system (3) yields then

$$\dot{W} = -\dot{q}^{\top} \partial_{\dot{q}} \mathcal{R}_c \leq y_c^{\top} u_d \leq 0, \tag{8}$$

where $y_c = G^{\top} M^{-1} M_c \dot{q}$ is a passive output of (3). Invoking LaSalle's theorem it follows that the equilibrium $(q, \dot{q}) = (q^{\star}, 0)$ is asymptotically stable if $\dot{q}^{\top} \partial_{\dot{q}} \mathcal{R}_c > 0,\ \forall \dot{q} \neq 0$. In the absence of physical damping (i.e., $\mathcal{R} = 0$), asymptotic stability is concluded for all $K_v \succ 0$ provided that y_c is detectable, that is $y_c \to 0 \implies (q, \dot{q}) \to (q^{\star}, 0)$, see [6],[5].

III. Main Result

A. System class definition

A new dynamic extension of the CL methodology (i.e., integral CL) is outlined here for a class of underactuated mechanical systems defined by the following assumptions.

Assumption 1. The PDEs (5) and (6) are solvable analytically with the parameters M_c and V_c, where $q^{\star} = \text{argmin}\,(V_c)$, that is $\partial_q V_c(q^{\star}) = 0$ and $\partial_q^2 V_c(q^{\star}) \succ 0$. Physical damping is defined by the dissipation function $\mathcal{R}(\dot{q})$, with $\dot{q}^{\top} \partial_{\dot{q}} \mathcal{R} \geq 0, \forall \dot{q} \neq 0$. All model parameters are exactly known, and all system states are measurable.

Assumption 2. The disturbance is defined as $\delta(q) = \delta_1 G G^{\top} h(q)$, where $\delta_1 \in \mathbb{R}$ is an unknown scalar constant, while $h(q) \in \mathbb{R}^n$ is a known globally bounded and continuously differentiable function of q.

While the solvability of PDEs remains a major challenge for the CL methodology, *Assumption 1* is verified by various systems with non-constant $G(q)$, see [19]. Thus the investigation of this aspect is beyond the scope of this work. The assumption on the disturbance is similar to [12] to facilitate the comparison with iIDA-PBC, see Section IV. However, the input matrix $G(q)$ is not required to be constant hence the new controller is applicable to a wider class of systems.

B. Controller design

Proposition 1. Consider the system (2) satisfying *Assumptions 1* and *2* and define the desired closed-loop dynamics

$$\frac{\mathrm{d}}{\mathrm{d}t} \partial_{\dot{q}} L_c(q, \dot{q}) - \partial_q L_c(q, \dot{q}) + \partial_{\dot{q}} \mathcal{R}_c(q, \dot{q}) + \partial_{\dot{q}} \Theta(q, \dot{q}, \zeta) = 0,$$
$$\partial_{\dot{q}} \mathcal{R}_c(q, \dot{q}) = M_c M^{-1} \left(\partial_{\dot{q}} \mathcal{R} + G K_v G^{\top} M^{-1} M_c \dot{q}\right),$$
$$\Theta = \frac{k_I}{2} \left(\zeta - \Psi - \frac{\delta_1}{k_I}\right)^2,\ L_c(q, \dot{q}) = \frac{1}{2}\dot{q}^{\top} M_c \dot{q} - V_c. \tag{9}$$

C1. The dynamics (9) is achieved with the control law

$$\begin{aligned} \Psi(q, \dot{q}) = h(q)^{\top} G G^{\top} M^{-1} M_c \dot{q}, \\ u = u_{CL} + u_d + G^{\top} h(q) k_I (\zeta - \Psi), \end{aligned} \tag{10}$$

where $k_I > 0$ is a scalar tuning parameter, and the time-derivative of ζ (i.e., the integral action) is given by

$$\begin{aligned} \dot{\zeta} = -\left(\partial_{\dot{q}} \Psi\right)^{\top} M_c^{-1} \left(\partial_q V_c + \partial_{\dot{q}} \mathcal{R}_c\right) \\ -\left(\partial_{\dot{q}} \Psi\right)^{\top} M_c^{-1} \left(\partial_q \left(M_c \dot{q}\right) \dot{q} - \partial_q \left(\frac{1}{2}\dot{q}^{\top} M_c \dot{q}\right)\right) \\ -\left(\partial_{\dot{q}} \Psi\right)^{\top} \dot{q} + \left(\partial_q \Psi\right)^{\top} \dot{q}. \end{aligned} \tag{11}$$

C2. If $\dot{q}^{\top} \partial_{\dot{q}} \mathcal{R}_c > 0, \forall \dot{q} \neq 0$, the equilibrium point $(q, \dot{q}) = (q^{\star}, 0)$ is locally asymptotically stable for all $k_I > 0, K_v \succ 0$.

C3. If in addition $|h(q)| > \epsilon > 0$ globally, then $\zeta \to \frac{\delta_1}{k_I}$ at the equilibrium.

C4. If $\mathcal{R} = 0, \forall \dot{q}$, the equilibrium is locally asymptotically stable for all $k_I > 0, K_v \succ 0$, provided that the passive output $y_c = G^{\top} M^{-1} M_c \dot{q}$ is detectable.

Proof. To construct the control law (10), substitute (1) into (2) while accounting for the dissipation function $\mathcal{R}(\dot{q})$, and compute the acceleration $\ddot{q}$, which yields

$$\begin{aligned} \ddot{q} = M^{-1} \left(G u - \delta_1 G G^{\top} h(q) - \partial_q V - \partial_{\dot{q}} \mathcal{R}\right) \\ -M^{-1} \partial_q \left(M \dot{q}\right) \dot{q} + M^{-1} \partial_q \left(\frac{1}{2}\dot{q}^{\top} M \dot{q}\right). \end{aligned} \tag{12}$$

Computing the acceleration $\ddot{q}$ from (9) yields instead

$$\begin{aligned} \ddot{q} = M_c^{-1} \left(-\partial_q V_c - \partial_{\dot{q}} \mathcal{R}_c - \partial_{\dot{q}} \Theta\right) \\ -M_c^{-1} \partial_q \left(M_c \dot{q}\right) \dot{q} + M_c^{-1} \partial_q \left(\frac{1}{2}\dot{q}^{\top} M_c \dot{q}\right). \end{aligned} \tag{13}$$

Equating (12) and (13) while substituting Θ and $\mathcal{R}_c$ yields

$$\begin{aligned} &M^{-1}\left(Gu-\delta_1 GG^\top h(q)-\partial_q V-\partial_{\dot q}\mathcal{R}\right)\\ &-M^{-1}\partial_q\left(M\dot q\right)\dot q+M^{-1}\partial_q\left(\frac{1}{2}\dot q^\top M\dot q\right)=\\ &-M_c^{-1}\partial_q V_c-M^{-1}\left(\partial_{\dot q}\mathcal{R}+GK_vG^\top M^{-1}M_c\dot q\right)\\ &+M^{-1}GG^\top h(q)k_I\left(\zeta-\Psi-\frac{\delta_1}{k_I}\right)\\ &-M_c^{-1}\partial_q\left(M_c\dot q\right)\dot q+M_c^{-1}\partial_q\left(\frac{1}{2}\dot q^\top M_c\dot q\right). \end{aligned} \tag{14}$$

C1. Refactoring (14) cancels the parameter δ_1 and the physical damping $\partial_{\dot q}\mathcal{R}$. Premultiplying (14) with $G^\perp M$ yields the sum of the PDEs (5) and (6), which are verified by design. Premultiplying (14) with $G^\dagger M$ to compute u yields (10). To construct the dynamic extension (11), define

$$W(q,\dot q,\zeta)=V_c+\frac{1}{2}\dot q^\top M_c\dot q+\Theta>0, \tag{15}$$

and compute its time-derivative along the trajectories of system (9), which yields

$$\begin{aligned} \dot W=&\,\partial_q\left(\frac{1}{2}\dot q^\top M_c\dot q\right)^\top\dot q+\partial_{\dot q}\left(\frac{1}{2}\dot q^\top M_c\dot q\right)^\top\ddot q\\ &+\left(\partial_q V_c\right)^\top\dot q+\left(\partial_q\Theta\right)^\top\dot q+\left(\partial_{\dot q}\Theta\right)^\top\ddot q+\left(\partial_\zeta\Theta\right)^\top\dot\zeta. \end{aligned} \tag{16}$$

Substituting $\ddot q$ from (13), Θ from (9), and $\dot\zeta$ from (11) while noting that $\partial_q\Theta=-\partial_\zeta\Theta\partial_q\Psi$ and $\partial_{\dot q}\Theta=-\partial_\zeta\Theta\partial_{\dot q}\Psi$ yields

$$\begin{aligned} \dot W=&\,\partial_q\left(\frac{1}{2}\dot q^\top M_c\dot q\right)^\top\dot q+\left(\partial_q V_c\right)^\top\dot q-\left(\partial_\zeta\Theta\partial_q\Psi\right)^\top\dot q\\ &+\dot q^\top M_cM_c^{-1}\left(-\partial_q V_c-\partial_{\dot q}\mathcal{R}_c+\partial_\zeta\Theta\partial_{\dot q}\Psi\right)\\ &+\dot q^\top M_c\left(M_c^{-1}\partial_q\left(\frac{1}{2}\dot q^\top M_c\dot q\right)-M_c^{-1}\partial_q\left(M_c\dot q\right)\dot q\right)\\ &-\left(\partial_\zeta\Theta\partial_{\dot q}\Psi\right)^\top M_c^{-1}\left(-\partial_q V_c-\partial_{\dot q}\mathcal{R}_c+\partial_\zeta\Theta\partial_{\dot q}\Psi\right)\\ &+\left(\partial_\zeta\Theta\partial_{\dot q}\Psi\right)^\top M_c^{-1}\left(\partial_q\left(M_c\dot q\right)\dot q-\partial_q\left(\frac{1}{2}\dot q^\top M_c\dot q\right)\right)\\ &+\left(\partial_\zeta\Theta\right)^\top\left(\partial_{\dot q}\Psi\right)^\top M_c^{-1}\left(-\partial_q V_c-\partial_{\dot q}\mathcal{R}_c\right)\\ &-\left(\partial_\zeta\Theta\right)^\top\left(\left(\partial_{\dot q}\Psi\right)^\top\dot q-\left(\partial_q\Psi\right)^\top\dot q\right)\\ &-\left(\partial_\zeta\Theta\partial_{\dot q}\Psi\right)^\top M_c^{-1}\left(\partial_q\left(M_c\dot q\right)\dot q-\partial_q\left(\frac{1}{2}\dot q^\top M_c\dot q\right)\right). \end{aligned}$$

Refactoring terms yields finally

$$\dot W=-\dot q^\top\partial_{\dot q}\mathcal{R}_c-\left(\partial_{\dot q}\Theta\right)^\top M_c^{-1}(\partial_{\dot q}\Theta)\le 0. \tag{17}$$

It follows from (17) that the equilibrium is stable and all states are bounded for all $K_v\succ 0$.

C2. If $\dot q^\top\partial_{\dot q}\mathcal{R}_c>0,\forall\dot q\neq 0$, invoking LaSalle's theorem (see Theorem 3.4 in [20]) proves that the trajectories of the closed-loop system (9) converge asymptotically to the set $\dot q=0\cap\partial_{\dot q}\Theta=0$. Evaluating (13) within this set yields $\partial_q V_c=0$ thus it follows from *Assumption 1* that the equilibrium $(q,\dot q)=(q^\star,0)$ is locally asymptotically stable. Stability is global if (15) is radially unbounded, see [20].

C3. If in addition $|h(q)|>\epsilon>0$ globally, then $\dot q=0\cap\partial_{\dot q}\Theta=0\implies\zeta=\frac{\delta_1}{k_I}$.

C4. If $\mathcal{R}=0,\forall\dot q$ (i.e., absence of physical damping), the trajectories of the closed-loop system (9) converge asymptotically to the set $y_c=G^\top M^{-1}M_c\dot q=0\cap\partial_{\dot q}\Theta=0$. Comparing (8) and (17) it follows that $\dot W\le-\dot q^\top\partial_{\dot q}\mathcal{R}_c\le y_c^\top u_d$, that is y_c is also a passive output of (9). If y_c is detectable, then evaluating (13) within the set $y_c=0\cap\partial_{\dot q}\Theta=0$ yields again $\partial_q V_c=0$, hence the equilibrium $(q,\dot q)=(q^\star,0)$ is locally asymptotically stable □

Remark 1. If $h(q)=\kappa,\kappa\neq 0,\kappa\in\mathbb{R}^n$ the condition $|h(q)|>\epsilon>0$ is always verified hence ζ converges asymptotically to δ_1/k_I at the equilibrium. If in addition G and M are constant, then the PDE (5) is verified by any constant M_c. In this case $\partial_q\Psi=0$ hence the controller design is considerably simplified, so that (10) and (11) yield

$$\begin{aligned} u&=u_{CL}+u_d+G^\top\kappa k_I(\zeta-\Psi),\\ \Psi(\dot q)&=\kappa^\top GG^\top M^{-1}M_c\dot q, \end{aligned} \tag{18}$$

$$\dot\zeta=-\left(\partial_{\dot q}\Psi\right)^\top M_c^{-1}\left(\partial_q V_c+\partial_{\dot q}\mathcal{R}_c\right)-\left(\partial_{\dot q}\Psi\right)^\top\dot q. \tag{19}$$

Remark 2. A general expression of the dissipation function $\mathcal{R}$ has been employed in *Assumption 1* to include various types of friction. In case of viscous friction defined by the matrix $D=D^\top\succeq 0$ we have $\mathcal{R}(\dot q)=\frac{1}{2}\dot q^\top D\dot q\ge 0$. Thus verifying the inequality $\dot q^\top\partial_{\dot q}\mathcal{R}_c>0,\forall\dot q\neq 0$ requires

$$\Gamma=M_cM^{-1}D+M_cM^{-1}GK_vG^\top M^{-1}M_c\succ 0,$$

where the product $GK_vG^\top$ is rank deficient. If $D\succ 0$, whether or not this condition is verified depends also on the matrices M and M_c. Substituting $M_c=MM_d^{-1}M$ in the above expression yields

$$MM_d^{-1}D+MM_d^{-1}GK_vG^\top M_d^{-1}M\succ 0,$$

which is equivalent to the corresponding condition for IDA-PBC in the presence of physical damping, that is $DM^{-1}M_d+GK_vG^\top\succ 0$, see [21].

IV. Comparative Analysis

A. Interpretation of the dynamic extension as low-pass filter

Recall that a first-order low-pass filter with input x, output z, time constant $\tau>0$, and bias $\Delta(t)$, is defined as

$$\dot z=\frac{1}{\tau}(x-z)+\Delta(t).$$

Defining $\eta=(\zeta-\Psi)$ and computing its time-derivative along the trajectories of the closed-loop system (9) yields

$$\dot\eta=\dot\zeta-\left(\partial_q\Psi\right)^\top\dot q-\left(\partial_{\dot q}\Psi\right)^\top\ddot q.$$

Substituting $\dot\zeta$ from (11) and $\ddot q$ from (13) in the above expression and refactoring terms yields

$$\underbrace{\dot\eta}_{\dot z}=\underbrace{\left(\partial_{\dot q}\Psi\right)^\top M_c^{-1}\left(\partial_{\dot q}\Psi\right)k_I}_{\frac{1}{\tau}}\underbrace{\left(\frac{\delta_1}{k_I}-\eta\right)}_{x-z}-\underbrace{\left(\partial_{\dot q}\Psi\right)^\top\dot q}_{\Delta},$$

which is clearly similar to the first-order low-pass filter: the output η follows the input δ_1/k_I, while the bias Δ depends on the position q and on the velocity $\dot q$.

If $|h(q)| > \epsilon > 0$ globally, then

$$\frac{1}{\tau} = (\partial_{\dot{q}}\Psi)^\top M_c^{-1} (\partial_{\dot{q}}\Psi) k_I > 0, \ \forall q.$$

This is the case if, for instance, $h(q), M, M_c$ and G are all constant, see *Remark 1*. Increasing k_I scales down δ_1 and increases the corner frequency of the low-pass filter yielding a faster response of the output η. The bias Δ ensures that the output η continues updating until the system reaches the equilibrium. The interpretation of the dynamic extension as first-order low-pass filter with input δ_1/k_I suggests that (11) has some ability to accommodate a time-varying parameter $\delta_1(t)$. This aspect will be investigated in our future work.

B. Comparison with iIDA-PBC

The iIDA-PBC design for underactuated mechanical systems with constant input matrix and position-dependent matched disturbances proposed in [12] yields

$$\begin{aligned} u_{iIDA} = u_{IDA} + u_D + v, \ u_D = -K_v G^\top M_d^{-1} p \\ v = (K_v G^\top G + \gamma_1 I) G^\top h(q) k_I \left(\zeta - p^\top G G^\top h(q)\right), \end{aligned} \quad (20)$$

$$\begin{aligned} \dot{\zeta} &= -h(q)^\top G(\gamma_1 I G^\top - G^\top J_2) M_d^{-1} p \\ &-h(q)^\top G G^\top M_d M^{-1} \left(\partial_q \Omega_d + \frac{1}{2}\partial_q \left(p^\top M_d^{-1} p\right)\right) \\ &+ \left(G G^\top \partial_q h(q) p\right)^\top M^{-1} p. \end{aligned} \quad (21)$$

Note that (20) and (21) have a very similar structure to (10) and (11) respectively: i) in both cases, the dynamic extension is modular with respect to the baseline u_{CL} and u_{IDA}; ii) the original PDEs (5) and (6) are preserved, while the same applies to u_{IDA}, see [12]; iii) it follows from (17) that the projected dynamics of the extended system (9) is the same as that of the baseline CL design resulting in (8), that is

$$\dot{W} = -\dot{q}^\top \partial_{\dot{q}} \mathcal{R}_c - (\partial_{\dot{q}}\Theta)^\top M_c^{-1} (\partial_{\dot{q}}\Theta) \le -\dot{q}^\top \partial_{\dot{q}} \mathcal{R}_c \le 0,$$

and the same holds for u_{IDA}. There are however three notable differences between (10) and (20): i) the controller (20) requires the input matrix G to be constant, see [12], thus (21) contains $\partial_q h(q)$, while (11) is expressed in terms of Ψ and its partial derivatives, which is more general; ii) the dynamic extension in (10) contains the inertia matrix M_c within Ψ, while v in (20) does not depend on M_d resulting in a different parameterization; iii) the iIDA-PBC controller (20) contains the additional parameter γ_1, which is not present in (10) and affects the closed-loop dissipation (i.e., increasing γ_1 yields slower response and reduced control action – see [12]). Instead, increasing k_I in both (10) and (20) yields a faster response and a more aggressive control.

A key difference between the new controller (10) and the iIDA-PBC (20) is the design procedure employed for the dynamic extension (11). In particular, ζ is included in the system dynamics (9) through the scalar function Θ. Drawing a parallel to the dissipation function $\mathcal{R}_c$, the function Θ can also be interpreted in terms of dissipation due to the residual $z = ((\zeta - \Psi) - \delta_1/k_I)$. This is motivated by the fact that the disturbance δ typically represents non-conservative forces. The time-derivative of ζ in (11) is then defined from (16) to ensure that $\dot{W} \le 0$, resulting in (17). Instead, with the iIDA-PBC design procedure, (21) results from enforcing a PCH structure of the closed-loop dynamics.

C. Comparison with nonlinear observer

The proposed dynamic extension (11) bears a close similarity with a nonlinear observer designed according to the *Immersion and Invariance (I&I)* methodology [22]. To demonstrate this point, define the *I&I* estimation error for the unknown scalar constant δ_1 as $z = \zeta' - \Psi' - \delta_1$, where

$$\begin{aligned} \Psi'(q,\dot{q}) &= k_I h(q)^\top G G^\top \dot{q}, \\ \dot{\zeta}' &= -\left(\partial_{\dot{q}}\Psi'\right)^\top M^{-1} \left(\partial_q V + \partial_{\dot{q}}\mathcal{R} - Gu\right) \\ &- \left(\partial_{\dot{q}}\Psi'\right)^\top M^{-1} \left((\zeta' - \Psi') G G^\top h(q)\right) + \left(\partial_q \Psi'\right)^\top \dot{q} \\ &- \left(\partial_{\dot{q}}\Psi'\right)^\top M^{-1} \left(\partial_q \left(M\dot{q}\right)\dot{q} - \partial_q \left(\frac{1}{2}\dot{q}^\top M \dot{q}\right)\right). \end{aligned} \quad (22)$$

Computing the time-derivative $\dot{z}$ along the trajectories of (2) while accounting for the dissipation function $\mathcal{R}(\dot{q})$ and substituting $\dot{\zeta}'$ from (22) yields

$$\dot{z} = -\frac{1}{k_I}\left(\partial_{\dot{q}}\Psi'\right)^\top M^{-1} \left(\partial_{\dot{q}}\Psi'\right) z.$$

Thus, employing the *I&I* methodology, z converges to zero for all $k_I > 0$ provided that $|h(q)| > \epsilon > 0$.
Taking inspiration from (10), define an adaptive control law by augmenting (7) with the observer (22), that is

$$u_{I\&I} = u_{CL} + u_d + G^\top h(q)(\zeta' - \Psi'). \quad (23)$$

Define the storage function $W'(q,\dot{q},\zeta') = V_c + \frac{1}{2}\dot{q}^\top M_c \dot{q} + \frac{1}{2}z^2$. Computing its the time-derivative along the trajectories of (2) in closed loop with (23) while substituting (22) yields

$$\begin{aligned} \dot{W}' = -\dot{q}^\top \partial_{\dot{q}}\mathcal{R}_c - \frac{1}{k_I}\left(\partial_{\dot{q}}\Psi'\right)^\top M^{-1}\left(\partial_{\dot{q}}\Psi'\right) z^2 \\ + \frac{1}{k_I}\dot{q}^\top M^{-1}\left(\partial_{\dot{q}}\Psi'\right) z. \end{aligned} \quad (24)$$

Substituting $\partial_{\dot{q}}\mathcal{R}_c$ from (9) into (24), assuming $\partial_{\dot{q}}\mathcal{R} = D\dot{q}$ with $D \ge 0$ as in *Remark 2*, and refactoring terms yields

$$\begin{aligned} \dot{W}' &= -\overline{x}^\top \begin{bmatrix} \Gamma & -\frac{1}{2k_I}M^{-1} \\ -\frac{1}{2k_I}M^{-1} & \frac{1}{k_I}M^{-1} \end{bmatrix} \overline{x}, \\ \overline{x}^\top &= \begin{bmatrix} \dot{q}^\top & z\left(\partial_{\dot{q}}\Psi'\right)^\top \end{bmatrix} \\ \Gamma &= M_c M^{-1} D + M_c M^{-1} G K_v G^\top M^{-1} M_c. \end{aligned} \quad (25)$$

Employing a Schur complement argument yields $\dot{W}' \le 0$ provided that $\Gamma k_I - \frac{1}{4}M^{-1} \succ 0$. If $D = 0$ the former inequality may not be verified by any $K_v \succ 0$ since Γ becomes rank-deficient. In summary, employing the nonlinear observer yields a dynamic extension $\dot{\zeta}'$ that is similar but not identical to (11), and which does not preserve passivity. This occurs since $\dot{\zeta}'$ is lacking the term $(\partial_{\dot{q}}\Psi)^\top \dot{q}$, which correspond to the bias Δ in the low-pass filter interpretation.

V. Simulation Results

The inertia-wheel pendulum (IWP) consists of an unactuated planar inverted pendulum with an actuated wheel at the tip, see Fig. 1. The system has two degrees-of-freedom: the angular position of the pendulum q_1, and the angular position of the wheel q_2. The system dynamics is given by (2) where

$$V = a_3 \left(\cos(q_1) + 1\right),$$
$$G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; M = \begin{bmatrix} a_1 + a_2 & a_2 \\ a_2 & a_2 \end{bmatrix},$$

and a_1, a_2, a_3 are constant parameters that depend on the size of the pendulum and of the wheel [2]. The control goal consists in reaching the prescribed position $(q_1, q_2) = (0, q_2^\star)$. The CL controller (7) is computed with the parameters

$$V_c = -a_3\gamma_1 \cos(q_1) + \frac{1}{2}k_p \left(q_2 + \gamma_1 \epsilon (a_1 + a_2) q_1\right)^2,$$
$$\gamma_1 = \frac{1}{a_2(m_2 - m_1)}, \; M_c = M M_d^{-1} M,$$
$$M_d = a_1 a_2 \begin{bmatrix} m_1 & \frac{m_1 a_2}{a_1 + a_2} + \epsilon \\ \frac{m_1 a_2}{a_1 + a_2} + \epsilon & m_3 \end{bmatrix}.$$

The tuning parameters are $k_p, K_v, m_1, m_3, \epsilon$. Although M, M_c and G are constant matrices, various disturbances have been considered, including the case $G^\top h(q) = q_2$ and $G^\top h(q) = q_2^2$, to illustrate the applicability and effectiveness of the new controller (10). This results in a non-constant $\Psi(q_2)$ even though G, M and M_c are constant.

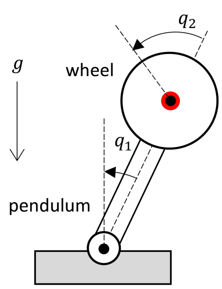

Fig. 1. Schematic of inertia-wheel pendulum system.

Simulations have been performed in MATLAB using an ODE23 solver with the model parameters $a_1 = 0.0124, a_2 = 0.0025, a_3 = 0.4446$ for illustrative purposes. The tuning parameters for the controllers (7), (10), (18), (20), and (23) have been set as $k_p = 1, K_v = 0.0005, m_1 = 0.4, m_3 = 5, \epsilon = 1.08$ and $k_I = 0.1$. The initial conditions are $(q_1, q_2, \dot{q}_1, \dot{q}_2, \zeta) = (0.1, 0.2, 0, 0, 0)$, and the prescribed position is $q_2^\star = 0.2$.

Fig. 2 shows the system response with the constant matched disturbance $\delta_1 = 0.03$ and $G^\top h(q) = 1$ resulting in $\delta = 0.03G$. Employing the controller (10) with the dynamic extension (11), the position reaches the prescribed equilibrium $(q_1^\star, q_2^\star) = (0, 0.2)$. The iIDA-PBC controller (20) with the same parameters and $\gamma_1 = 0.1$ yields a slower response in this case. This results from the different parameterization employed in (20), see Section IV.B. Responsiveness can be improved by increasing k_I. Finally, the baseline CL controller (7) yields a large error on the wheel position.

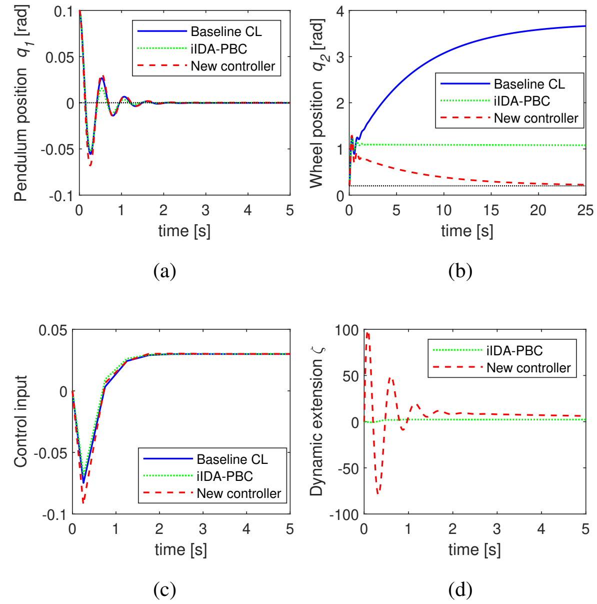

Fig. 2. Simulation results for IWP with constant matched disturbance $\delta_1 = 0.03$ and $G^\top h(q) = 1$: (a) pendulum position; (b) wheel position; (c) control input; (d) dynamic extension ζ.

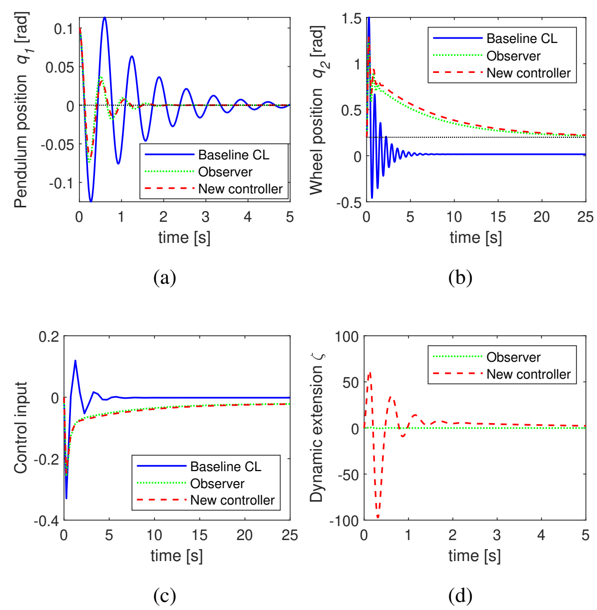

Fig. 3. Simulation results for IWP with position-dependent matched disturbance $\delta_1 = -0.1$ and $G^\top h(q) = q_2$: (a) pendulum position; (b) wheel position; (c) control input; (d) dynamic extension ζ.

Fig. 3 shows the system response with the position-dependent matched disturbance $\delta_1 = -0.1$ and $G^\top h(q) = q_2$ resulting in $\delta = -0.1Gq_2$. Employing the controller (10) with the dynamic extension (11), the position reaches the prescribed equilibrium $(q_1^\star, q_2^\star) = (0, 0.2)$. The controller with *I&I* observer (23) yields a similar response. The baseline CL controller (7) with the same parameters results in a noticeable error on the wheel position and in a larger control effort.

Fig. 4 shows the system response with $\delta_1 = -10$ and

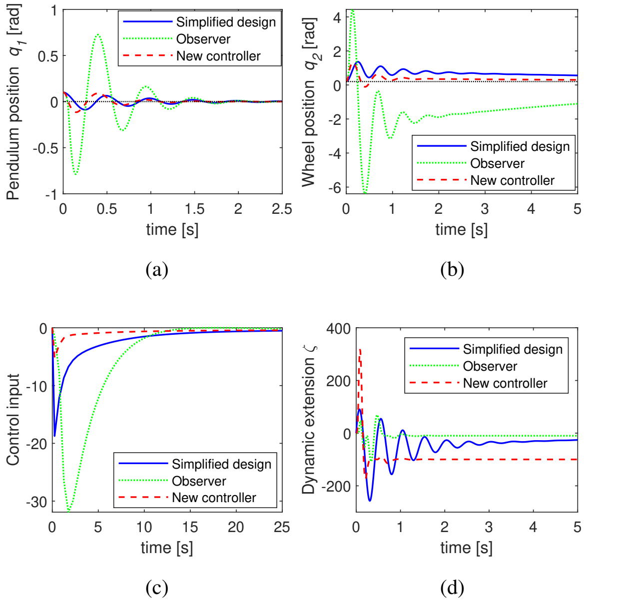

Fig. 4. Simulation results for IWP with position-dependent matched disturbance $\delta_1 = -10$ and $G^\top h(q) = q_2^2$: (a) pendulum position; (b) wheel position; (c) control input; (d) dynamic extension ζ. "Simplified design" refers to the controller (18) which assumes $G^\top h(q) = 1$ and $\partial_q \Psi = 0$.

$G^\top h(q) = q_2^2$. Note that the dynamic extension ζ computed with (11) converges to the correct value, that is $\zeta = \delta_1/k_I = -100$, at the equilibrium. Employing the simplified controller (18) that treats the disturbance as constant for simplicity (i.e., assuming $G^\top h(q) = 1$ hence $\partial_q \Psi = 0$) and using the same parameters as in (10), the system still reaches the prescribed equilibrium. This is expected since, as the system approaches equilibrium, the disturbance converges to a constant value. However, the convergence is slower and the control effort is considerably higher (i.e., see "Simplified design" in Fig. 4). In this case, the controller with *I&I* observer (23) and the same parameters yields degraded transient performance and higher control effort, which can be improved by increasing k_I, see Section IV.C. Finally, the baseline CL controller (7) fails to stabilize the prescribed equilibrium.

VI. Conclusion

In this work we have presented a new passivity-preserving dynamic extension of the CL methodology for underactuated mechanical systems that compensates the effect of position-dependent matched disturbances. An interpretation of the dynamic extension as a first-order low-pass filter has been proposed. Although the new controller bears similarities with the iIDA-PBC methodology and with a *I&I* nonlinear observer, it is not identical to these, and it is derived using a different design procedure. This can result in different performance if the same parameters are used. Simulation results on a inertia-wheel pendulum demonstrate the effectiveness of the new controller and indicate that explicitly accounting for position-dependent disturbances can benefit performance. Future work will aim to extend this result to a broader class of disturbances, including time-varying parameters.

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