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Composite Model Reference Adaptive Control under Finite Excitation with Unstructured Uncertainties

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Abstract— **This paper presents an online parameter update algorithm in the context of composite model reference adaptive control based on intermittent signal holding to improve convergence properties of the parameters representing the unstructured uncertainties in the absence of persistent excitation. The present study extends the algorithm which was previously developed by considering only the structured uncertainties for which the basis functions are known a priori. The proposed extension utilises the Gaussian radial basis function neural network as the model for the uncertainty assuming appropriate placement of the local basis functions in the state space. A notable distinction from the case with full knowledge of the features constituting the linearly-parameterised uncertainty model is that the extended algorithm introduces a robustifying modification in the earlier phase of operation to deal with the inevitable learning residual.**

I. Introduction

Online model learning endows an autonomous system with the capability to improve its state prediction accuracy over time, enabling more precise and efficient planning/control using the learned model. In general, a parametric control-oriented model for the system dynamics can be optimised either separately or jointly with a parametric controller for the system-level performance with respect to a given task, bearing a wide range of possibilities in the definition of objective. In this perspective, online model learning can be formulated as the problem of minimising the state prediction error described using the online acquired trajectory data. Recent studies clearly show that an accurate dynamic model is not necessary but sufficient for optimal downstream task performance [1], [2]. The optimality of predictive planning and control methods directly benefit from the improved accuracy of the learned model parameters.

Robustly stable and accurate online model learning requires a careful design for both exploration and exploitation aspects. In the context of adaptive control, online model learning is often referred to as long-term learning or slow adaptation to highlight the point that a system needs certain period of time to collect enough information for identifying the uncertain part of the model and to process the information. Regarding the exploration side of the task, the trajectory data should be rich enough to provide a large information gain. Regarding the exploitation side of the task, the loss function associated with the model learning objective should be constructed to enforce strict convexity with respect to the parameter, provided that the optimiser employed is capable of converging to a local minimum of the loss function supplied.

From the perspective that views parameter evolution over time according to an update algorithm as the result of an optimisation problem solver unrolled in time, the non-strict convexity of the loss function associated with the model learning task with respect to the parameter is detrimental to the stability and convergence characteristics of the overall closed-loop system. The non-uniqueness of the solution to the optimisation problem, i.e., the multitude of local minima of the loss function, which can essentially be the infinitude of feasible fitting solutions manifests itself as the parameter staying at an undesired point or even drifting along the connected set of feasible points possibly in an unbounded manner.

On one hand, explicit regularisation as a remedy can enforce strict convexity and enhance robustness, however, at the cost of possibly shifting the minimum point from the true parameter to an arbitrary value. On the other hand, assuring persistent excitation as a remedy does not change the loss function itself, however, relying on persistent excitation for convergence of parameters in classical estimation algorithms is unreasonable since stable control precedes accurate learning in most practices.

To overcome the difficulties arising from the stringent requirement of persistent excitation, various methods have been developed to ensure parameter convergence under a more relaxed condition such as the finite excitation in the initial interval in the context of composite adaptive control [3]–[5]. The effectiveness of composite adaptation for stable simultaneous learning and control in an adaptive control system has already been well understood in the earlier studies [6]–[8]. However, classical parameter estimation schemes for linearly-parameterised models such as the instantaneous gradient-based or the recursive least squares estimators depend on the persistency of excitation in order to guarantee parameter convergence. In the classical methods, the information matrix given by the rank-deficient outer product of the basis function at each instance is the main cause rendering the loss function to be non-strictly convex. The concurrent learning algorithm presented in [9]–[11] addressed the rank-deficiency by utilising a set of recorded historical data together with the current measurement to

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populate the information matrix to full rank over time. The composite adaptation algorithm presented in [12] utilised the Kreisselmeier's memory regressor extension scheme as reviewed in [4] together with an intermittent signal holding algorithm to avoid ill-conditioned information matrix. Exponential parameter convergence could be guaranteed in the structured uncertainty case assuming only finite excitation.

This study mainly aims to extend the design and analysis of the adaptation algorithm developed in [12] only for the case of structured uncertainties to deal with unstructured uncertainties. The extended algorithm considers a shallow learning model known as the Gaussian radial basis function neural network for the approximate representation of the uncertainty. The key difference from the structured uncertainty case lies at the presence of the inevitable learning residual, leading to the necessity of robustifying modification in the design as well as the changes in the stability analysis concluding uniform ultimate boundedness.

The rest of the paper is organised as follows: The preliminaries and problem formulation are given in Sec. II. In Sec. III, the extended adaptation law is designed to deal with unstructured uncertainty and its stability is analysed assuming finite excitation. Concluding remarks are provided in Sec. IV.

II. Preliminaries and Problem Formulation

This section describes the definitions of excitation conditions and the formulation of a state feedback Model Reference Adaptive Control (MRAC) problem.

A. Preliminaries

In the followings, let $\|\cdot\|$, $\|\cdot\|_{\max}$, and $\|\cdot\|_F$ denote the induced 2-norm, the elementwise max norm, and the Frobenius norm, respectively. Also, let $\vec{(\cdot)}$, $\lambda_{\min}(\cdot)$, and $\lambda_{\max}(\cdot)$ denote the columnwise vectorisation, the minimum eigenvalue, and the maximum eigenvalue of a matrix, respectively. Then, for example, $V(t) := \frac{1}{2}\mathbf{e}^T\mathbf{P}\mathbf{e} + \frac{1}{2}\operatorname{tr}\left(\tilde{\mathbf{W}}^T\mathbf{\Gamma}_w{}^{-1}\tilde{\mathbf{W}}\right)$ satisfies

$$\begin{aligned} \frac{1}{2}\lambda_{\min}(\mathbf{P})\|\mathbf{e}(t)\|^2 \leq \frac{1}{2}\mathbf{e}^T\mathbf{P}\mathbf{e} \leq V(t) \\ \frac{1}{2}\lambda_{\min}\left(\mathbf{\Gamma}_w{}^{-1}\right)\left\|\tilde{\mathbf{W}}(t)\right\|_F^2 \leq \frac{1}{2}\operatorname{tr}\left(\tilde{\mathbf{W}}^T\mathbf{\Gamma}_w{}^{-1}\tilde{\mathbf{W}}\right) \leq V(t) \end{aligned} \quad (1)$$

Note that $\lambda_{\min}(\mathbf{P}) > 0$ for $\mathbf{P} > 0$, and $\lambda_{\min}\left(\mathbf{\Gamma}_w{}^{-1}\right) > 0$ for $\mathbf{\Gamma}_w > 0$.

Lemma 1 (Bounded-Input Bounded-State Stability of Linear Time-Invariant System)**.** *For a linear time-invariant system* $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ *where* $\mathbf{x} \in \mathbb{R}^{n\times 1}$ *and* $\mathbf{u} \in \mathbb{R}^{m\times 1}$*, if* $\mathbf{A}$ *is diagonal and Hurwitz,* $\mathbf{x}(t_0) = \mathbf{0}$*, and* $\sup_{\tau\in[t_0,t]}\|\mathbf{u}(\tau)\| = \bar{u}$*, then*

$$\|\mathbf{x}(t)\| \leq \frac{\|\mathbf{B}\|\,\bar{u}}{|\lambda_{\max}(\mathbf{A})|} \quad (2)$$

The main objective of this study is to extend the parameter estimation algorithm proposed in [12] assuming structured uncertainty to unstructured uncertainty for improved parameter convergence characteristics under a limited degree of excitation. The excitation conditions are formally defined as below.

Definition 1 (Finite Excitation)**.** A bounded vector signal $\mathbf{v}(t)$ verifies Finite Excitation (FE) condition over a finite time interval $[t_s, t_s + T]$, if there exist $T > 0$, $t_s \geq t_0$, and $\gamma > 0$ such that

$$\int_{t_s}^{t_s+T} \mathbf{v}(\tau)\mathbf{v}^T(\tau)\,d\tau \geq \gamma\mathbf{I} > \mathbf{0} \quad (3)$$

Definition 2 (Persistent Excitation)**.** A bounded vector signal $\mathbf{v}(t)$ verifies Persistent Excitation (PE) condition, if there exist $T > 0$ and $\gamma > 0$ such that

$$\int_{t}^{t+T} \mathbf{v}(\tau)\mathbf{v}^T(\tau)\,d\tau \geq \gamma\mathbf{I} \quad \text{for } \forall t \geq t_0 \quad (4)$$

B. Problem Formulation

1) System Dynamics: Consider a class of uncertain linear Multi-Input Multi-Output (MIMO) dynamic system given by

$$\begin{aligned} \dot{\mathbf{x}}_p(t) &= \mathbf{A}_p\mathbf{x}_p(t) + \mathbf{B}_p\left(\mathbf{u}(t) + \mathbf{\Delta}(\mathbf{x}_p(t))\right) \\ \mathbf{z}(t) &= \mathbf{H}_p\mathbf{x}_p(t) \end{aligned} \quad (5)$$

where $\mathbf{x}_p(t) \in \mathbb{R}^{n_p\times 1}$ is the state which is assumed to be fully measurable, $\mathbf{u}(t) \in \mathbb{R}^{m\times 1}$ is the control input, $\mathbf{z}(t) \in \mathbb{R}^{m\times 1}$ is the performance output, and $\mathbf{\Delta}(\mathbf{x}_p(t)) \in \mathbb{R}^{m\times 1}$ is the state-dependent uncertainty. $\mathbf{A}_p \in \mathbb{R}^{n_p\times n_p}$, $\mathbf{B}_p \in \mathbb{R}^{n_p\times m}$, and $\mathbf{H}_p \in \mathbb{R}^{m\times n_p}$ in Eq. (5) are known constant matrices which satisfies i) controllability of the pair $(\mathbf{A}_p, \mathbf{B}_p)$, and ii) the linear independence of the columns in $\mathbf{B}_p$.

The objective is to achieve tracking of a given bounded piecewise continuous command $\mathbf{z}_{\text{cmd}}(t) \in \mathbb{R}^{m\times 1}$ with the performance output $\mathbf{z}(t)$. Augmenting Eq. (5) with the integrated output tracking error $\mathbf{e}_{z_I}(t) \triangleq \int_{t_0}^{t}(\mathbf{z}(\tau) - \mathbf{z}_{\text{cmd}}(\tau))\,d\tau$ yields the extended system as follows

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\left(\mathbf{u}(t) + \mathbf{\Delta}(\mathbf{x}_p(t))\right) + \mathbf{B}_r\mathbf{z}_{\text{cmd}}(t) \\ \mathbf{z}(t) &= \mathbf{H}\mathbf{x}(t) \end{aligned} \quad (6)$$

where $\mathbf{x} \triangleq \begin{bmatrix}\mathbf{x}_p \\ \mathbf{e}_{z_I}\end{bmatrix} \in \mathbb{R}^{n\times 1}$ with $n = n_p + m$ is the extended state vector and

$$\begin{aligned} \mathbf{A} &\triangleq \begin{bmatrix}\mathbf{A}_p & \mathbf{0}_{n_p\times m} \\ \mathbf{H}_p & \mathbf{0}_{m\times m}\end{bmatrix} \in \mathbb{R}^{n\times n}, \quad \mathbf{B} \triangleq \begin{bmatrix}\mathbf{B}_p \\ \mathbf{0}_{m\times m}\end{bmatrix} \in \mathbb{R}^{n\times m} \\ \mathbf{B}_r &\triangleq \begin{bmatrix}\mathbf{0}_{n_p\times m} \\ -\mathbf{I}_{m\times m}\end{bmatrix} \in \mathbb{R}^{n\times m}, \qquad \mathbf{H} \triangleq \begin{bmatrix}\mathbf{H}_p & \mathbf{0}_{m\times m}\end{bmatrix} \in \mathbb{R}^{m\times n} \end{aligned} \quad (7)$$

Note that $(\mathbf{A}, \mathbf{B})$ is controllable if and only if $(\mathbf{A}_p, \mathbf{B}_p)$ is controllable and $\det\left(\begin{bmatrix}\mathbf{A}_p & \mathbf{B}_p \\ \mathbf{H}_p & \mathbf{0}_{m\times m}\end{bmatrix}\right) \neq 0$.

2) Models of Uncertainty: The uncertainty $\mathbf{\Delta}(\mathbf{x}_p(t))$ can be modelled by function expansion using artificial basis functions in the absence of structural knowledge. The only thing available about the uncertainty is that it is known to be continuous and defined over a compact domain $D_p \subset \mathbb{R}^{n_p\times 1}$. A function approximator with the universal approximation capability can be used to model the uncertainty in this case. Among various function approximation schemes, the Radial

Basis Function Neural Network (RBF NN) will be used in this study, because better simplicity of further analysis is expected owing to its linear-in-parameter structure. The following assumption is satisfied for the *unstructured uncertainty*.

Assumption 1 (Unstructured Uncertainty)**.**
Let $\sigma_i(\mathbf{x}_p(t))$ denote a Gaussian RBF with its center at $\mathbf{c}_i$ and its width of μ_i, which is defined as follows,

$$\sigma_i(\mathbf{x}_p) = \exp\left(-\frac{\|\mathbf{x}_p - \mathbf{c}_i\|^2}{\mu_i}\right) \tag{8}$$

and let $\mathbf{\Sigma}(\mathbf{x}_p) = \begin{bmatrix} 1 & \sigma_2(\mathbf{x}_p) \cdots & \sigma_q(\mathbf{x}_p) \end{bmatrix}^T \in \mathbb{R}^{q\times 1}$ be the RBF vector. According to the universal approximation capability of RBF NN [13], there exists a unique constant ideal parameter $\mathbf{W}^* \in \mathbb{R}^{q\times m}$ that approximates the uncertainty $\mathbf{\Delta}(\mathbf{x}_p(t)) \in \mathbb{R}^{m\times 1}$ as closely as possible with a fixed number of given RBFs such that

$$\mathbf{\Delta}(\mathbf{x}_p(t)) = \mathbf{W}^{*T}\mathbf{\Sigma}(\mathbf{x}_p(t)) + \boldsymbol{\omega}(\mathbf{x}_p(t)) \tag{9}$$

holds for $\forall \mathbf{x}_p \in D_p \subset \mathbb{R}^{n_p\times 1}$. In Eq. (9), $\boldsymbol{\omega}(\mathbf{x}_p(t))$ is the minimal approximation error vector. Note that $\overline{\omega} \triangleq \sup_{\mathbf{x}_p\in D_p} \|\boldsymbol{\omega}(\mathbf{x}_p(t))\|$ can be arbitrarily small with sufficiently large number of RBFs at the cost of computation load.

3) Model Tracking Error Dynamics: The MRAC design philosophy is to synchronise the system state with the state of a reference model representing the desired closed-loop response. A reference model is the *ideal* closed-loop system obtainable with the baseline control law for the nominal system. Let us assume that there exists a full-state feedback baseline control law $\mathbf{u}_{\text{base}} = -\mathbf{K}\mathbf{x}$ such that the gain $\mathbf{K}$ satisfies $\mathbf{A}_r = \mathbf{A} - \mathbf{B}\mathbf{K}$ for a given Hurwitz matrix $\mathbf{A}_r$. Then, the reference model can be represented as

$$\begin{aligned} \dot{\mathbf{x}}_r(t) &= \mathbf{A}_r\mathbf{x}_r(t) + \mathbf{B}_r\mathbf{z}_{\text{cmd}}(t) \\ \mathbf{z}_r(t) &= \mathbf{H}\mathbf{x}_r(t) \end{aligned} \tag{10}$$

Given $\mathbf{A}_r$ is Hurwitz, there exists a symmetric positive definite matrix $\mathbf{P} \in \mathbb{R}^{n\times n}$ satisfying the following Lyapunov equation

$$\mathbf{A}_r{}^T\mathbf{P} + \mathbf{P}\mathbf{A}_r + \mathbf{Q} = \mathbf{0} \tag{11}$$

for any symmetric positive definite matrix $\mathbf{Q} \in \mathbb{R}^{n\times n}$

The control law for the uncertain system of Eq. (6) can be designed as

$$\mathbf{u} = \mathbf{u}_{\text{base}} - \mathbf{u}_{\text{ad}} = -\mathbf{K}\mathbf{x} - \mathbf{u}_{\text{ad}} \tag{12}$$

where $\mathbf{u}_{\text{base}}$ represents the *baseline control law*, and $\mathbf{u}_{\text{ad}}$ represents the *adaptive input*. Then, the *model tracking error* defined as $\mathbf{e}(t) \triangleq \mathbf{x}_r(t) - \mathbf{x}(t)$ evolves over time according to

$$\dot{\mathbf{e}}(t) = \mathbf{A}_r\mathbf{e}(t) + \mathbf{B}\boldsymbol{\epsilon}(t) \tag{13}$$

where $\boldsymbol{\epsilon}(t) = \mathbf{u}_{\text{ad}}(t) - \mathbf{\Delta}(\mathbf{x}_p(t)) \in \mathbb{R}^{m\times 1}$ denotes the *uncertainty approximation error*. The adaptive input can be designed to cancel the uncertainty from the tracking error dynamics as

$$\mathbf{u}_{\text{ad}}(t) = \hat{\mathbf{\Delta}}(\mathbf{x}_p(t)) = \hat{\mathbf{W}}^T(t)\,\mathbf{\Sigma}(\mathbf{x}_p(t)) \tag{14}$$

where $\hat{\mathbf{W}}(t)$ denotes the estimated parameter. The estimate $\hat{\mathbf{W}}$ should be as close as possible to the ideal value $\mathbf{W}^*$ to minimise the *parameter estimation error* denoted by $\tilde{\mathbf{W}}(t) \triangleq \hat{\mathbf{W}}(t) - \mathbf{W}^*$. Note that $\dot{\tilde{\mathbf{W}}} = \dot{\hat{\mathbf{W}}}$. The model tracking error dynamics given in Eq. (13) can be rewritten as

$$\dot{\mathbf{e}}(t) = \mathbf{A}_r\mathbf{e}(t) + \mathbf{B}\left[\tilde{\mathbf{W}}^T(t)\,\mathbf{\Sigma}(\mathbf{x}_p(t)) - \boldsymbol{\omega}(\mathbf{x}_p(t))\right] \tag{15}$$

Equation (15) shows that the parameter estimation error enters into the tracking error dynamics. The interaction of the two errors $\mathbf{e}$ and $\tilde{\mathbf{W}}$ within the feedback loop should be taken into account in the design of an adaptation law that generates $\dot{\hat{\mathbf{W}}}$.

C. Filtered System Dynamics

This section describes the regressor filtering scheme which is similar to the one based on first-order filter as described in [8]. From Eq. (13), we have

$$\begin{aligned} &\mathbf{u}_{\text{ad}}(t) - \mathbf{B}^\dagger\left[\dot{\mathbf{e}}(t) - \mathbf{A}_r\mathbf{e}(t)\right] \\ &\quad = \mathbf{\Delta}(\mathbf{x}_p(t)) = \mathbf{W}^{*T}\mathbf{\Sigma}(\mathbf{x}_p(t)) + \boldsymbol{\omega}(\mathbf{x}_p(t)) \end{aligned} \tag{16}$$

where $(\cdot)^\dagger$ denotes the Moore-Penrose pseudoinverse. The column linear independence of $\mathbf{B}_p$ assures full column rank of $\mathbf{B}$. Therefore, $\mathbf{B}^\dagger = \left(\mathbf{B}^T\mathbf{B}\right)^{-1}\mathbf{B}^T$. Assuming $\mathbf{e}(t_0) = \mathbf{0}$ without loss of generality, the Laplace transform of Eq. (16) can be written as

$$\begin{aligned} &\mathbf{u}_{\text{ad}}(s) - \mathbf{B}^\dagger\left(s\mathbf{I}_{n\times n} - \mathbf{A}_r\right)\mathbf{e}(s) \\ &\quad = \mathbf{\Delta}(\mathbf{x}_p) = \mathbf{W}^{*T}\mathbf{\Sigma}(s) + \boldsymbol{\omega}(s) \end{aligned} \tag{17}$$

Consider a stable linear first-order low-pass filter represented as $F(s) = \frac{1}{\tau_f s + 1}$ with $\tau_f > 0$. Multiplying both sides of Eq. (17) by $F(s)$ yields the filtered uncertainty as

$$\begin{aligned} &\mathbf{u}_{\text{ad}_f}(s) - \mathbf{B}^\dagger\left[\frac{1}{\tau_f}\mathbf{e}(s) - \left(\frac{1}{\tau_f}\mathbf{I}_{n\times n} + \mathbf{A}_r\right)\mathbf{e}_f(s)\right] \\ &\quad = \mathbf{\Delta}_f(\mathbf{x}_p) = \mathbf{W}^{*T}\mathbf{\Sigma}_f(s) + \boldsymbol{\omega}_f(s) \end{aligned} \tag{18}$$

where the subscript f is used to denote a signal filtered by $F(s)$, i.e., $\boldsymbol{\alpha}_f(s) = F(s)\,\boldsymbol{\alpha}(s)$. The inverse Laplace transform of Eq. (18) yields the filtered system dynamics as follows:

$$\boldsymbol{\chi}(t) \triangleq \boldsymbol{\xi}(t) - \frac{1}{\tau_f}\mathbf{B}^\dagger\mathbf{e}(t) = \mathbf{W}^{*T}\boldsymbol{\eta}(t) + \boldsymbol{\delta}(t) \tag{19}$$

$$\dot{\boldsymbol{\xi}}(t) = \frac{1}{\tau_f}\left[\mathbf{u}_{\text{ad}}(t) + \mathbf{B}^\dagger\left(\frac{1}{\tau_f}\mathbf{I}_{n\times n} + \mathbf{A}_r\right)\mathbf{e}(t) - \boldsymbol{\xi}(t)\right] \tag{20}$$

$$\dot{\boldsymbol{\eta}}(t) = \frac{1}{\tau_f}\left(\mathbf{\Sigma}(\mathbf{x}_p(t)) - \boldsymbol{\eta}(t)\right) \tag{21}$$

$$\dot{\boldsymbol{\delta}}(t) = \frac{1}{\tau_f}\left(\boldsymbol{\omega}(\mathbf{x}_p(t)) - \boldsymbol{\delta}(t)\right) \tag{22}$$

with $\boldsymbol{\xi}(t_0) = \mathbf{0}_{m\times 1}$, $\boldsymbol{\eta}(t_0) = \mathbf{0}_{q\times 1}$, $\boldsymbol{\delta}(t_0) = \mathbf{0}_{m\times 1}$ The output $\boldsymbol{\chi}(t)$ is a function of known signals $\boldsymbol{\xi}(t)$ and

$\mathbf{e}(t)$, and the filtered regressor $\boldsymbol{\eta}(t)$ is also a known signal. Therefore, the system described by Eqs. (19)-(22) provides the equations to perform linear regression.

III. Uncertainty Learning Under Finite Excitation

This section develops a new adaptation law for the case of unstructured uncertainty by extending the algorithm presented in [12] and discussed in [4] which was developed assuming full knowledge of the basis function for the uncertainty. Also, this section presents the analysis of stability and performance of the overall closed-loop system under the proposed scheme.

A. Design of Adaptation Law

Let us assume FE in the filtered regressor.

Assumption 2 (Finite Excitation of Filtered Regressor)**.** *The filtered regressor $\boldsymbol{\eta}(t)$ verifies FE over $[t_s, t_e]$.*

The learning residual $\boldsymbol{\delta}$ is generally nonzero for the unstructured uncertainties. It can be inferred from Eq. (19) that the unknown parameter $\mathbf{W}^*$ can be estimated with some nonremovable amount of bounded error from the measurable signals $\boldsymbol{\chi}(t)$ and $\boldsymbol{\eta}(t)$. This intuition will be shown later as the exponentially convergent ultimate uniform boundedness around the neighborhood of $\left(\mathbf{e}, \tilde{\mathbf{W}}\right) = (\mathbf{0}, \mathbf{0})$.

Let the *information matrix* $\boldsymbol{\Omega}(t)$ and the *auxiliary matrix* $\mathbf{M}(t)$ be defined by

$$\begin{aligned} \dot{\boldsymbol{\Omega}}(t) &= -k(t)\,\boldsymbol{\Omega}(t) + \boldsymbol{\eta}(t)\,\boldsymbol{\eta}^T(t) \\ \dot{\mathbf{M}}(t) &= -k(t)\,\mathbf{M}(t) + \boldsymbol{\eta}(t)\,\boldsymbol{\chi}^T(t) \end{aligned} \tag{23}$$

with $\boldsymbol{\Omega}(t_0) = \mathbf{0}_{q\times q}$ and $\mathbf{M}(t_0) = \mathbf{0}_{q\times m}$ where $k(t)$ is a scalar forgetting factor satisfying $0 < k_L \le k(t) \le k_U$. One possible example introduced in [12] for the design of forgetting factor is to set

$$k(t) = k_L + (k_U - k_L)\tanh(\vartheta \|\dot{\boldsymbol{\eta}}\|) \tag{24}$$

where $\vartheta > 0$ is a constant design parameter, and $\dot{\boldsymbol{\eta}}$ is calculated according to Eq. (21).

Central to the adaptation algorithm of [12] is the idea of intermittent signal holding which is to perform selective update of the *adequate information matrix* $\boldsymbol{\Omega}_a(t)$ and the *adequate auxiliary matrix* $\mathbf{M}_a(t)$ as follows:

$$\begin{aligned} t_a &\triangleq \max\left\{ \underset{\tau\in[t_0,t]}{\arg\max}\, \mathcal{F}(\boldsymbol{\Omega}(\tau)) \right\} \\ \boldsymbol{\Omega}_a(t) &\triangleq \boldsymbol{\Omega}(t_a) \\ \mathbf{M}_a(t) &\triangleq \mathbf{M}(t_a) \end{aligned} \tag{25}$$

where $\mathcal{F}(\cdot)$ represents a chosen metric for quantifying the quality of information matrix. The intention behind performing selective update as described in Eq. (25) is to ensure monotonic increase in $\mathcal{F}(\boldsymbol{\Omega}_a(t))$, i.e., $\frac{d\mathcal{F}(\boldsymbol{\Omega}_a(t))}{dt} \ge 0$ for $\forall t \ge t_0$. One representative example for the choice of the information measure is to set $\mathcal{F}(\cdot) = \lambda_{\min}(\cdot)$.

The new adaptation law for the unstructured uncertainty case is proposed as follows:

$$\dot{\hat{\mathbf{W}}}(t) = \begin{cases} -\boldsymbol{\Gamma}_w \left[\boldsymbol{\Sigma}(\mathbf{x}_p(t))\,\mathbf{e}^T(t)\,\mathbf{P}\mathbf{B} \right. \\ \qquad \left. + R\left(\boldsymbol{\Omega}_a(t)\,\hat{\mathbf{W}}(t) - \mathbf{M}_a(t)\right) + \kappa\hat{\mathbf{W}}(t)\right] \\ \quad \text{if } \operatorname{rank}(\boldsymbol{\Omega}_a(t)) < q \\ -\boldsymbol{\Gamma}_w \left[\boldsymbol{\Sigma}(\mathbf{x}_p(t))\,\mathbf{e}^T(t)\,\mathbf{P}\mathbf{B} \right. \\ \qquad \left. + R\left(\boldsymbol{\Omega}_a(t)\,\hat{\mathbf{W}}(t) - \mathbf{M}_a(t)\right)\right] \\ \quad \text{if } \operatorname{rank}(\boldsymbol{\Omega}_a(t)) = q \end{cases} \tag{26}$$

where $\boldsymbol{\Gamma}_w > 0$ is a constant adaptation gain matrix, $\kappa > 0$ is a constant scalar gain for the robustification term, $R > 0$ is a scalar relative weight on the parameter-estimation-based modification term, and $\mathbf{P} = \mathbf{P}^T > 0$ is the solution of Eq. (11) for a given $\mathbf{Q} = \mathbf{Q}^T > 0$.

B. Stability and Performance Analysis

The solution of Eq. (23) can be written as

$$\begin{aligned} \boldsymbol{\Omega}(t) &= \int_{t_0}^{t} \exp\left(-\int_{\tau}^{t} k(\nu)\,d\nu\right) \boldsymbol{\eta}(\tau)\,\boldsymbol{\eta}^T(\tau)\,d\tau \\ \mathbf{M}(t) &= \int_{t_0}^{t} \exp\left(-\int_{\tau}^{t} k(\nu)\,d\nu\right) \boldsymbol{\eta}(\tau)\,\boldsymbol{\chi}^T(\tau)\,d\tau \end{aligned} \tag{27}$$

From Eqs. (19) and (27), it is clear that

$$\mathbf{M}(t) = \boldsymbol{\Omega}(t)\,\mathbf{W}^* + \mathbf{N}(t) \tag{28}$$

where

$$\mathbf{N}(t) = \int_{t_0}^{t} \exp\left(-\int_{\tau}^{t} k(\nu)\,d\nu\right) \boldsymbol{\eta}(\tau)\,\boldsymbol{\delta}^T(\tau)\,d\tau \tag{29}$$

Let us consider $\lambda_{\min}(\cdot)$ for $\mathcal{F}(\cdot)$ in Eq. (25). It is obvious from Eqs. (25) and (28) that

$$\mathbf{M}_a(t) = \boldsymbol{\Omega}_a(t)\,\mathbf{W}^* + \mathbf{N}_a(t) \tag{30}$$

where $\mathbf{N}_a(t) \triangleq \mathbf{N}(t_a)$.

In Lemma 2, the adequate information matrix is shown to be positive definite after FE. Using this result, the stability of the equilibrium point is shown in Theorem 1, and the transient performance guarantee is given in Corollary 1.

Lemma 2 (Positive Definiteness and Minimum Eigenvalue of Adequate Information Matrix)**.** *With the FE condition as stated in Assumption 2 and the choice of $\mathcal{F}(\cdot)$ by $\lambda_{\min}(\cdot)$,*

- $\boldsymbol{\Omega}_a(t) \ge 0$ *for* $\forall t \ge t_0$.
- $\boldsymbol{\Omega}_a(t) > 0$ *for* $\forall t \ge t_e$.
- $\lambda_{\min}(\boldsymbol{\Omega}_a(t)) \ge \lambda_{\min}(\boldsymbol{\Omega}_a(t_e)) > 0$ *for* $\forall t \ge t_e$.

Proof. See [12].

□

In the unstructured uncertainty case, $\boldsymbol{\delta}(t)$ is unknown, but it is bounded as explained in Lemma 3. Using this result, the boundedness of the mismatch term $\mathbf{N}_a(t)$ is shown in Lemma 4. The boundedness of unknown signals shown in these Lemmas is essential in the following analysis.

Lemma 3 (Uniform Boundedness of $\boldsymbol{\eta}(t)$ and $\boldsymbol{\delta}(t)$). *The signals $\boldsymbol{\eta}(t)$ and $\boldsymbol{\delta}(t)$ are uniformly bounded as follows*

$$\begin{aligned} \|\boldsymbol{\eta}(t)\| &\leq \sqrt{q} \\ \|\boldsymbol{\delta}(t)\| &\leq \bar{\omega} \end{aligned} \tag{31}$$

Proof. It is obvious that $\|\boldsymbol{\Sigma}(\mathbf{x}_p(t))\| \leq \sqrt{q}$, because the Gaussian RBF given by Eq. (8) satisfies $0 < \sigma_i(\mathbf{x}_p) \leq 1$. Also, we have $\|\boldsymbol{\omega}(\mathbf{x}_p(t))\| \leq \bar{\omega}$ from Assumption 1. Then, the bounds on $\boldsymbol{\eta}(t)$ and $\boldsymbol{\delta}(t)$ can be obtained as Eq. (31) by applying Lemma 1 to Eqs. (21)-(22). □

Lemma 4 (Uniform Boundedness of Mismatch Term). *Let D_p be the compact set in which the RBF NN approximation holds as explained in Assumption 1. If $\mathbf{x}_p \in D_p$ for $\forall t \geq t_0$, then the mismatch term $\mathbf{N}_a(t)$ in Eq. (30) is uniformly bounded as follows*

$$\|\mathbf{N}_a(t)\|_F \leq \frac{\bar{\omega}\sqrt{q}}{k_L} \tag{32}$$

Proof. Since $0 < k_L \leq k(t) \leq k_U$, we have

$$\begin{aligned} 0 < \exp(-k_U(t-\tau)) &\leq \exp\left(-\int_\tau^t k(\nu)\,d\nu\right) \\ &\leq \exp(-k_L(t-\tau)) \end{aligned} \tag{33}$$

From Eq. (33) and Lemma 3, the upper bound on $\mathbf{N}(t)$ can be obtained from the triangle inequality as follows:

$$\begin{aligned} \|\mathbf{N}(t)\|_F &\leq \int_{t_0}^t \exp\left(-\int_\tau^t k(\nu)\,d\nu\right) \left\|\boldsymbol{\eta}(\tau)\boldsymbol{\delta}^T(\tau)\right\|_F d\tau \\ &= \int_{t_0}^t \exp\left(-\int_\tau^t k(\nu)\,d\nu\right) \|\boldsymbol{\delta}(\tau)\|\,\|\boldsymbol{\eta}(\tau)\|\,d\tau \\ &\leq \int_{t_0}^t \exp(-k_L(t-\tau))\,\bar{\omega}\sqrt{q}\,d\tau \leq \frac{\bar{\omega}\sqrt{q}}{k_L} \end{aligned} \tag{34}$$

Equation (32) is an obvious consequence of Eq. (34). □

From Eqs. (15), (30), and (26), the closed-loop system dynamics of the tracking error $\mathbf{e}$ and the parameter estimation error $\tilde{\mathbf{W}}$ can be written as follows:

$$\begin{aligned} \dot{\mathbf{e}} &= \mathbf{A}_r\mathbf{e} + \mathbf{B}\left[\tilde{\mathbf{W}}^T\boldsymbol{\Sigma}(\mathbf{x}_p) - \boldsymbol{\omega}(\mathbf{x}_p)\right] \qquad \mathbf{e}(t_0) = \mathbf{0} \\ \dot{\tilde{\mathbf{W}}} &= \begin{cases} -\boldsymbol{\Gamma}_w\left[\boldsymbol{\Sigma}(\mathbf{x}_p)\mathbf{e}^T\mathbf{P}\mathbf{B} + (R\boldsymbol{\Omega}_a + \kappa\mathbf{I}_{q\times q})\tilde{\mathbf{W}} \right. \\ \qquad \left. - \mathbf{N}_a + \kappa\mathbf{W}^*\right], \quad \text{if } \operatorname{rank}(\boldsymbol{\Omega}_a(t)) < q \\ -\boldsymbol{\Gamma}_w\left[\boldsymbol{\Sigma}(\mathbf{x}_p)\mathbf{e}^T\mathbf{P}\mathbf{B} + R\boldsymbol{\Omega}_a\tilde{\mathbf{W}} - \mathbf{N}_a\right] \\ \quad \text{if } \operatorname{rank}(\boldsymbol{\Omega}_a(t)) = q \end{cases} \end{aligned} \tag{35}$$

The uniform ultimate boundedness of the closed-loop trajectories around $\left(\mathbf{e}, \vec{\tilde{\mathbf{W}}}\right) = (\mathbf{0}, \mathbf{0})$ is shown in Theorem 1, and the ultimate bound is given as a performance guarantee in Corollary 1.

Theorem 1 (Uniform Ultimate Boundedness of Errors). *Let $D \triangleq \left\{\mathbf{x} \mid \mathbf{x}_p \in D_p \subset \mathbb{R}^{n_p\times 1},\ \mathbf{e}_{z_I} \in D_I \subset \mathbb{R}^{m\times 1}\right\}$ be a compact set where D_p is the compact set in which the RBF NN approximation holds, i.e., $\|\boldsymbol{\omega}(\mathbf{x}_p)\| \leq \bar{\omega}$ in D_p, and D_I is an arbitrary closed bounded subset of $\mathbb{R}^{m\times 1}$ containing $\mathbf{0}_{m\times 1}$. Let $\alpha \triangleq \max_{\mathbf{x}\in D}\|\mathbf{x}\|$, and let $B_\alpha \triangleq \{\|\mathbf{x}\| \leq \alpha\}$ so that $B_\alpha \subset D$. Consider the Lyapunov function defined by Eq. (36), and let β be the minimum possible value of V such that $\dot{V} < 0$ is guaranteed for all $\left(\mathbf{e}, \tilde{\mathbf{W}}\right)$ outside of and at the boundary of the set $\Omega_\beta \triangleq \left\{\left(\mathbf{e}, \tilde{\mathbf{W}}\right) \Big|\ V\left(\mathbf{e}, \tilde{\mathbf{W}}\right) \leq \beta\right\}$. Let $\gamma \geq \beta$ and $\Omega_\gamma \triangleq \left\{\left(\mathbf{e}, \tilde{\mathbf{W}}\right) \Big|\ V\left(\mathbf{e}, \tilde{\mathbf{W}}\right) \leq \gamma\right\}$.*

Suppose that the following assumptions hold:

(i) $\mathbf{x}(t_0) \in B_\alpha$;
(ii) $V\left(\mathbf{e}(t_0), \tilde{\mathbf{W}}(t_0)\right) \in \Omega_\gamma$;
(iii) The reference model is BIBO stable such that $\|\mathbf{x}_r(t)\| \leq \alpha - \sqrt{\frac{2\gamma}{\lambda_{\min}(\mathbf{P})}}$ for $\forall t \geq t_0$.

Then, with the control law given by Eqs. (12) and (14), the adaptation law given by Eq. (26), and the FE condition as stated in Assumption 2, the trajectory $\mathbf{e}(t)$ and $\vec{\tilde{\mathbf{W}}}(t)$ are uniformly ultimately bounded for all $t \geq t_0$.

Proof. Consider the following positive definite and radially unbounded Lyapunov candidate function.

$$V\left(\mathbf{e}, \tilde{\mathbf{W}}\right) = \frac{1}{2}\mathbf{e}^T\mathbf{P}\mathbf{e} + \frac{1}{2}\operatorname{tr}\left(\tilde{\mathbf{W}}^T{\boldsymbol{\Gamma}_w}^{-1}\tilde{\mathbf{W}}\right) \tag{36}$$

Note that $V(\mathbf{0}, \mathbf{0}) = 0$, and $V\left(\mathbf{e}, \tilde{\mathbf{W}}\right) > 0$ for $\forall\left(\mathbf{e}, \tilde{\mathbf{W}}\right) \neq (\mathbf{0}, \mathbf{0})$. Let $\boldsymbol{\xi} \triangleq \begin{bmatrix}\mathbf{e}^T & \vec{\tilde{\mathbf{W}}}^T\end{bmatrix}^T$, then the Lyapunov candidate function given by Eq. (36) is bounded from below and above as follows:

$$\begin{aligned} \frac{1}{2}\min\left\{\lambda_{\min}(\mathbf{P}), \lambda_{\min}\left({\boldsymbol{\Gamma}_w}^{-1}\right)\right\}\|\boldsymbol{\xi}\|^2 &\leq V\left(\mathbf{e}, \tilde{\mathbf{W}}\right) \\ &\leq \frac{1}{2}\max\left\{\lambda_{\max}(\mathbf{P}), \lambda_{\max}\left({\boldsymbol{\Gamma}_w}^{-1}\right)\right\}\|\boldsymbol{\xi}\|^2 \end{aligned} \tag{37}$$

Consider the positive definite and radially unbounded Lyapunov candidate function given by Eq. (36). Let $\boldsymbol{\xi} \triangleq \begin{bmatrix}\mathbf{e}^T & \vec{\tilde{\mathbf{W}}}^T\end{bmatrix}^T$, then the Lyapunov candidate function of Eq. (36) is bounded from below and above as Eq. (37). As explained in Lemma 2, under the FE condition of Assumption 2, there exists $t_e > t_0$ such that $\boldsymbol{\Omega}_a(t) > 0$, $\forall t \geq t_e$. Therefore, the adaptation law given by Eq. (26) switches from the first one to the second one at some t_e, as $\operatorname{rank}(\boldsymbol{\Omega}_a(t))$ becomes populated to the full rank.

Suppose that there exists the the boundary value of the Lyapunov function, $\beta > 0$, as described in the statement of this Theorem. The β will be clearly defined below. From the assumption (ii), $V\left(\mathbf{e}(t_0), \tilde{\mathbf{W}}(t_0)\right) \in \Omega_\gamma$, it can be shown that $\|\mathbf{e}\| \leq \sqrt{\frac{2V}{\lambda_{\min}(\mathbf{P})}} \leq \sqrt{\frac{2\gamma}{\lambda_{\min}(\mathbf{P})}}, \forall t \geq t_0$, because $V\left(\mathbf{e}, \tilde{\mathbf{W}}\right) \geq \frac{1}{2}\lambda_{\min}(\mathbf{P})\|\mathbf{e}\|^2$ for $\forall t \geq t_0$, $\Omega_\beta \subseteq \Omega_\gamma$, and $\dot{V} < 0$ outside Ω_β. Next, from the assumption (iii), it can be shown that $\|\mathbf{x}\| = \|\mathbf{x}_r - \mathbf{e}\| \leq \|\mathbf{x}_r\| + \|\mathbf{e}\| \leq \alpha$ for all $t \geq t_0$, using the result shown above. Then, for all $t \geq t_0$, $\mathbf{x}(t) \in D$ because $B_\alpha \subset D$, and therefore the RBF NN approximation holds.

Consider first the case of $\operatorname{rank}(\boldsymbol{\Omega}_a(t)) < q$, which corresponds to the time interval $t_0 \leq t < t_e$. From Eqs. (11), and (35), the time derivative of Eq. (36) along the trajectory of the closed-loop system can be obtained as:

$$\begin{aligned}\dot{V}\left(\mathbf{e},\tilde{\mathbf{W}}\right) &= \mathbf{e}^T\mathbf{P}\left(\mathbf{A}_r\mathbf{e}+\mathbf{B}\left[\tilde{\mathbf{W}}^T\boldsymbol{\Sigma}\left(\mathbf{x}_p\right)-\boldsymbol{\omega}\left(\mathbf{x}_p\right)\right]\right)\\ &\quad -\operatorname{tr}\left(\tilde{\mathbf{W}}^T\left[\boldsymbol{\Sigma}\left(\mathbf{x}_p\right)\mathbf{e}^T\mathbf{PB}+\left(R\boldsymbol{\Omega}_a+\kappa\mathbf{I}_{q\times q}\right)\tilde{\mathbf{W}}\right.\right.\\ &\quad \left.\left.-\ \mathbf{N}_a+\kappa\mathbf{W}^*\right]\right)\\ &= -\frac{1}{2}\mathbf{e}^T\mathbf{Q}\mathbf{e}-\operatorname{tr}\left(\tilde{\mathbf{W}}^T\left(R\boldsymbol{\Omega}_a+\kappa\mathbf{I}_{q\times q}\right)\tilde{\mathbf{W}}\right)\\ &\quad -\mathbf{e}^T\mathbf{PB}\boldsymbol{\omega}\left(\mathbf{x}_p\right)+\operatorname{tr}\left(\tilde{\mathbf{W}}^T\left[\mathbf{N}_a-\kappa\mathbf{W}^*\right]\right)\end{aligned}\tag{38}$$

$\boldsymbol{\Omega}_a(t)$ is only positive semidefinite while $\operatorname{rank}(\boldsymbol{\Omega}_a(t))<q$. Using Eq. (34), the upper bound for Eq. (38) can be obtained as follows

$$\begin{aligned}\dot{V}\left(\mathbf{e},\tilde{\mathbf{W}}\right) &\le -\left[\frac{1}{2}\lambda_{\min}\left(\mathbf{Q}\right)\|\mathbf{e}\|-c_1\right]\|\mathbf{e}\|\\ &\quad -\left[\kappa\left\|\tilde{\mathbf{W}}\right\|_F-\left(c_2+c_3\right)\right]\left\|\tilde{\mathbf{W}}\right\|_F\\ &= -\frac{1}{2}\lambda_{\min}\left(\mathbf{Q}\right)\left(\|\mathbf{e}\|-\frac{c_1}{\lambda_{\min}\left(\mathbf{Q}\right)}\right)^2\\ &\quad -\kappa\left(\left\|\tilde{\mathbf{W}}\right\|_F-\frac{c_2+c_3}{2\kappa}\right)^2+\frac{{c_1}^2}{2\lambda_{\min}\left(\mathbf{Q}\right)}+\frac{\left(c_2+c_3\right)^2}{4\kappa}\\ &\triangleq f_{\text{UB}}\left(\|\mathbf{e}\|,\left\|\tilde{\mathbf{W}}\right\|_F\right)\end{aligned}\tag{39}$$

where $c_1=\|\mathbf{PB}\|\bar{\omega}$, $c_2=\kappa\|\mathbf{W}^*\|_F$, and $c_3=\frac{\bar{\omega}\sqrt{q}}{k_L}$. Therefore, $\dot{V}<0$, $\forall t<t_e$, if $\left(\mathbf{e},\tilde{\mathbf{W}}\right)$ is outside of the compact set $\Theta\triangleq\left\{\left(\mathbf{e},\tilde{\mathbf{W}}\right)\middle|\ f_{\text{UB}}\left(\|\mathbf{e}\|,\left\|\tilde{\mathbf{W}}\right\|_F\right)\ge 0\right\}$. The boundary $\partial\Theta$ is a level set on which $f_{\text{UB}}\left(\|\mathbf{e}\|,\left\|\tilde{\mathbf{W}}\right\|_F\right)=0$, and it is an ellipse centered at $\left(\|\mathbf{e}\|,\left\|\tilde{\mathbf{W}}\right\|_F\right)=\left(\frac{c_1}{\lambda_{\min}(\mathbf{Q})},\frac{(c_2+c_3)}{2\kappa}\right)$. The compact set Θ is inside of that boundary, including the boundary itself. Let us define β as

$$\beta\triangleq(1+\zeta)\cdot\max_{\left(\mathbf{e},\tilde{\mathbf{W}}\right)\in\Theta}V\left(\mathbf{e},\tilde{\mathbf{W}}\right)\tag{40}$$

where $0\le\zeta\ll 1$ is an arbitrarily small number. The boundary $\partial\Omega_\beta=\left\{\left(\mathbf{e},\tilde{\mathbf{W}}\right)\middle|\ V\left(\mathbf{e},\tilde{\mathbf{W}}\right)=\beta\right\}$ is an ellipse centered at $\left(\|\mathbf{e}\|,\left\|\tilde{\mathbf{W}}\right\|_F\right)=(0,0)$. Note that $\Theta\subseteq\Omega_\beta$. Also note that $\dot{V}<0$ on $\partial\Omega_\beta$ for $\zeta>0$, or except the point of contact between $\partial\Omega_\beta$ and $\partial\Theta$ for $\zeta=0$. Therefore, the compact set Ω_β is positive invariant. Moreover, the solution $\left(\mathbf{e}(t),\tilde{\mathbf{W}}(t)\right)$ that starts outside of Ω_β will ultimately enter the set Ω_β within some finite time. Thus, the solution $\left(\mathbf{e}(t),\tilde{\mathbf{W}}(t)\right)$ is uniformly ultimately bounded.

The analysis for the case of $\operatorname{rank}(\boldsymbol{\Omega}_a(t))=q$, which corresponds to $t\ge t_e$, can be performed similarly. Using Eqs. (11) and (35), the time derivative of Eq. (36) along the trajectory of the closed-loop system can be obtained as follows:

$$\begin{aligned}\dot{V}\left(\mathbf{e},\tilde{\mathbf{W}}\right) &= \mathbf{e}^T\mathbf{P}\left(\mathbf{A}_r\mathbf{e}+\mathbf{B}\left[\tilde{\mathbf{W}}^T\boldsymbol{\Sigma}\left(\mathbf{x}_p\right)-\boldsymbol{\omega}\left(\mathbf{x}_p\right)\right]\right)\\ &\quad -\operatorname{tr}\left(\tilde{\mathbf{W}}^T\left[\boldsymbol{\Sigma}\left(\mathbf{x}_p\right)\mathbf{e}^T\mathbf{PB}+R\boldsymbol{\Omega}_a\tilde{\mathbf{W}}-\mathbf{N}_a\right]\right)\\ &= -\frac{1}{2}\mathbf{e}^T\mathbf{Q}\mathbf{e}-R\operatorname{tr}\left(\tilde{\mathbf{W}}^T\boldsymbol{\Omega}_a\tilde{\mathbf{W}}\right)\\ &\quad -\mathbf{e}^T\mathbf{PB}\boldsymbol{\omega}\left(\mathbf{x}_p\right)+\operatorname{tr}\left(\tilde{\mathbf{W}}^T\mathbf{N}_a\right)\end{aligned}\tag{41}$$

The upper bound for Eq. (41) can be obtained as follows

$$\begin{aligned}\dot{V}\left(\mathbf{e},\tilde{\mathbf{W}}\right) &\le -\left[\frac{1}{2}\lambda_{\min}\left(\mathbf{Q}\right)\|\mathbf{e}\|-c_1\right]\|\mathbf{e}\|\\ &\quad -\left[R\lambda_{\min}\left(\boldsymbol{\Omega}_a\left(t_e\right)\right)\left\|\tilde{\mathbf{W}}\right\|_F-c_3\right]\left\|\tilde{\mathbf{W}}\right\|_F\\ &= -\frac{1}{2}\lambda_{\min}\left(\mathbf{Q}\right)\left(\|\mathbf{e}\|-\frac{c_1}{\lambda_{\min}\left(\mathbf{Q}\right)}\right)^2\\ &\quad -R\lambda_{\min}\left(\boldsymbol{\Omega}_a\left(t_e\right)\right)\left(\left\|\tilde{\mathbf{W}}\right\|_F-\frac{c_3}{2R\lambda_{\min}\left(\boldsymbol{\Omega}_a\left(t_e\right)\right)}\right)^2\\ &\quad +\frac{{c_1}^2}{2\lambda_{\min}\left(\mathbf{Q}\right)}+\frac{{c_3}^2}{4R\lambda_{\min}\left(\boldsymbol{\Omega}_a\left(t_e\right)\right)}\end{aligned}\tag{42}$$

The rest of the proof to show the uniform ultimate boundedness is identical to the case where $t_0\le t<t_e$.

The geometrical representation depicted in Fig. 1 summarizes the proof. □

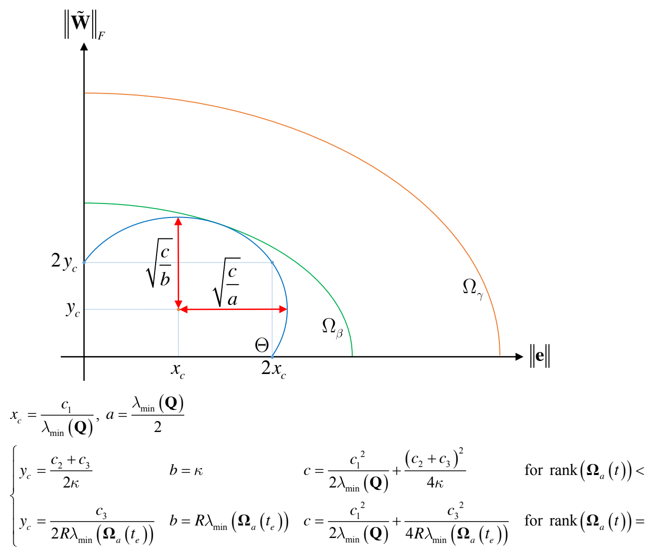

Fig. 1. Geometrical Representation of the Stability Analysis

Corollary 1 (Performance Guarantee for the Case of Unstructured Uncertainty)**.** *Let*

$$\begin{aligned}\alpha_1 &\triangleq \frac{\min\left\{\lambda_{\min}\left(\mathbf{Q}\right),2\kappa\right\}}{\max\left\{\lambda_{\max}\left(\mathbf{P}\right),\lambda_{\max}\left({\boldsymbol{\Gamma}_w}^{-1}\right)\right\}},\\ \alpha_2 &\triangleq \frac{\min\left\{\lambda_{\min}\left(\mathbf{Q}\right),2R\lambda_{\min}\left(\boldsymbol{\Omega}_a\left(t_e\right)\right)\right\}}{\max\left\{\lambda_{\max}\left(\mathbf{P}\right),\lambda_{\max}\left({\boldsymbol{\Gamma}_w}^{-1}\right)\right\}},\\ \psi_1 &\triangleq \frac{{c_1}^2}{2\lambda_{\min}\left(\mathbf{Q}\right)}+\frac{\left(c_2+c_3\right)^2}{4\kappa},\\ \psi_2 &\triangleq \frac{{c_1}^2}{2\lambda_{\min}\left(\mathbf{Q}\right)}+\frac{{c_3}^2}{4R\lambda_{\min}\left(\boldsymbol{\Omega}_a\left(t_e\right)\right)}.\end{aligned}$$

For some constant $\theta\in(0,1)$, the bounds for the Lyapunov function given by Eq. (36) can be derived as follows:

$$V(t)\le\begin{cases}\left(V\left(t_0\right)-\dfrac{\psi_1}{\theta(1-\theta)\alpha_1}\right)\exp\left(-\theta\alpha_1\left(t-t_0\right)\right)+\dfrac{\psi_1}{\theta(1-\theta)\alpha_1}\\ \text{for } t_0\le t\le t_e\\ \left(V\left(t_e\right)-\dfrac{\psi_2}{\theta(1-\theta)\alpha_2}\right)\exp\left(-\theta\alpha_2\left(t-t_e\right)\right)+\dfrac{\psi_2}{\theta(1-\theta)\alpha_2}\\ \text{for } t\ge t_e\end{cases}\tag{43}$$

where

$$V(t_0) \le \frac{1}{2}\lambda_{\max}\left(\boldsymbol{\Gamma}_w{}^{-1}\right)\left\|\tilde{\mathbf{W}}(t_0)\right\|_F^2$$
$$V(t_e) \le \left(\frac{1}{2}\lambda_{\max}\left(\boldsymbol{\Gamma}_w{}^{-1}\right)\left\|\tilde{\mathbf{W}}(t_0)\right\|_F^2 - \frac{\psi_1}{\theta(1-\theta)\alpha_1}\right) \cdot \exp\left(-\theta\alpha_1(t_e - t_0)\right) + \frac{\psi_1}{\theta(1-\theta)\alpha_1} \quad (44)$$

The bounds for $\|\mathbf{e}(t)\|$ *and* $\left\|\tilde{\mathbf{W}}(t)\right\|_F$ *can be obtained by substituting Eqs. (43)-(44) into*

$$\|\mathbf{e}(t)\| \le \sqrt{\frac{2V(t)}{\lambda_{\min}(\mathbf{P})}}$$
$$\left\|\tilde{\mathbf{W}}(t)\right\|_F \le \sqrt{\frac{2V(t)}{\lambda_{\min}\left(\boldsymbol{\Gamma}_w{}^{-1}\right)}} \quad (45)$$

Proof. Let $V(t) := V\left(\mathbf{e}(t), \tilde{\mathbf{W}}(t)\right)$. Consider first the time interval in which $\boldsymbol{\Omega}_a$ is rank deficient, namely $t \in [t_0, t_e]$. For some constant $\theta \in (0,1)$, Eq. (39) can be rewritten using Eq. (37) as:

$$\begin{aligned}
\dot{V}(t) \le& -\theta\left[\frac{1}{2}\lambda_{\min}(\mathbf{Q})\|\mathbf{e}\|^2 + \kappa\left\|\tilde{\mathbf{W}}\right\|_F^2\right] \\
&- \frac{1}{2}(1-\theta)\lambda_{\min}(\mathbf{Q})\left(\|\mathbf{e}\| - \frac{c_1}{\lambda_{\min}(\mathbf{Q})}\right)^2 \\
&- (1-\theta)\kappa\left(\left\|\tilde{\mathbf{W}}\right\|_F - \frac{c_2 + c_3}{2\kappa}\right)^2 \\
&+ \frac{1}{1-\theta}\left[\frac{{c_1}^2}{2\lambda_{\min}(\mathbf{Q})} + \frac{(c_2+c_3)^2}{4\kappa}\right] \\
\le& -\theta\frac{1}{2}\min\left\{\lambda_{\min}(\mathbf{Q}), 2\kappa\right\}\|\boldsymbol{\xi}\|^2 \\
&+ \frac{1}{1-\theta}\left[\frac{{c_1}^2}{2\lambda_{\min}(\mathbf{Q})} + \frac{(c_2+c_3)^2}{4\kappa}\right] \\
\le& -\theta\alpha_1 V(t) + \frac{\psi_1}{1-\theta}
\end{aligned} \quad (46)$$

Eq. (43) for $t_0 \le t \le t_e$ can be derived by applying the comparison lemma to Eq. (46).

Next, consider the right-infinite time interval in which $\boldsymbol{\Omega}_a$ is full rank, namely $t \ge t_e$. For some constant $\theta \in (0,1)$, Eq. (42) can be rewritten using Eq. (37) as follows:

$$\begin{aligned}
\dot{V}(t) \le& -\theta\left[\frac{1}{2}\lambda_{\min}(\mathbf{Q})\|\mathbf{e}\|^2 + R\lambda_{\min}(\boldsymbol{\Omega}_a(t_e))\left\|\tilde{\mathbf{W}}\right\|_F^2\right] \\
&- \frac{1}{2}(1-\theta)\lambda_{\min}(\mathbf{Q})\left(\|\mathbf{e}\| - \frac{c_1}{\lambda_{\min}(\mathbf{Q})}\right)^2 \\
&- (1-\theta)R\lambda_{\min}(\boldsymbol{\Omega}_a(t_e))\left(\left\|\tilde{\mathbf{W}}\right\|_F - \frac{c_3}{2R\lambda_{\min}(\boldsymbol{\Omega}_a(t_e))}\right)^2 \\
&+ \frac{1}{1-\theta}\left[\frac{{c_1}^2}{2\lambda_{\min}(\mathbf{Q})} + \frac{{c_3}^2}{4R\lambda_{\min}(\boldsymbol{\Omega}_a(t_e))}\right] \\
\le& -\theta\frac{1}{2}\min\left\{\lambda_{\min}(\mathbf{Q}), 2R\lambda_{\min}(\boldsymbol{\Omega}_a(t_e))\right\}\|\boldsymbol{\xi}\|^2 \\
&+ \frac{1}{1-\theta}\left[\frac{{c_1}^2}{2\lambda_{\min}(\mathbf{Q})} + \frac{{c_3}^2}{4R\lambda_{\min}(\boldsymbol{\Omega}_a(t_e))}\right] \\
\le& -\theta\alpha_2 V(t) + \frac{\psi_2}{1-\theta}
\end{aligned} \quad (47)$$

Applying the comparison lemma to Eq. (47) yields Eq. (43) for $t \ge t_e$. □

IV. Conclusions

A composite model reference adaptive control algorithm is extended to be capable of learning unstructured but matched uncertainties without requiring excessive degree of excitation. A selective update algorithm that performs intermittent holding is combined with the memory-based regressor extension scheme to avoid rank deficiency of the information matrix after being encountered with a finite amount of excitation. The proposed scheme updates the adequate information matrix only when its quality in terms a chosen metric can be improved. The closed-loop stability analysis showed that the extended algorithm guarantees exponentially convergent uniform ultimate boundedness of the errors in the unstructured uncertainty case.

References

[1] A. Gahlawat, A. Lakshmanan, L. Song, A. Patterson, Z. Wu, N. Hovakimyan, and E. A. Theodorou, "Contraction $\mathcal{L}_1$-Adaptive Control using Gaussian Processes," in *3rd Conference on Learning for Dynamics and Control*, June 2021, pp. 1027–1040. [Online]. Available: https://proceedings.mlr.press/v144/gahlawat21a.html

[2] S. M. Richards, N. Azizan, J.-J. S. Slotine, and M. Pavone, "Adaptive-Control-Oriented Meta-Learning for Nonlinear Systems," in *Robotics: Science and Systems*, Virtual, July 2021.

[3] Y. Pan, A. Bobtsov, M. Darouach, and Y.-H. Joo, "Learning from Adaptive Control Under Relaxed Excitation Conditions," *International Journal of Adaptive Control and Signal Processing*, vol. 33, no. 12, pp. 1723–1725, 2019.

[4] R. Ortega, V. Nikiforov, and D. Gerasimov, "On Modified Parameter Estimators for Identification and Adaptive Control. A Unified Framework and Some New Schemes," *Annual Reviews in Control*, vol. 50, pp. 278–293, 2020.

[5] K. Guo and Y. Pan, "Composite Adaptation and Learning for Robot Control: A Survey," *Annual Reviews in Control*, vol. Online Published, 2022.

[6] J.-J. E. Slotine and W. Li, "On the Adaptive Control of Robot Manipulators," *International Journal of Robotics Research*, vol. 6, no. 3, pp. 49–59, 1987.

[7] ——, "Composite Adaptive Control of Robot Manipulators," *Automatica*, vol. 25, no. 4, pp. 509–519, 1989.

[8] E. Lavretsky, "Combined/Composite Model Reference Adaptive Control," *IEEE Transactions on Automatic Control*, vol. 54, no. 11, pp. 2692–2697, 2009.

[9] G. Chowdhary, "Concurrent Learning for Convergence in Adaptive Control without Persistency of Excitation," Ph.D. dissertation, Georgia Institute of Technology, 2010.

[10] G. Chowdhary and E. N. Johnson, "Theory and Flight-Test Validation of a Concurrent-Learning Adaptive Controller," *Journal of Guidance, Control, and Dynamics*, vol. 34, no. 2, pp. 592–607, 2011.

[11] G. Chowdhary, M. Mühlegg, and E. N. Johnson, "Exponential Parameter and Tracking Error Convergence Guarantees for Adaptive Controllers without Persistency of Excitation," *International Journal of Control*, vol. 87, no. 8, pp. 1583–1603, 2014.

[12] N. Cho, H.-S. Shin, Y. Kim, and A. Tsourdos, "Composite Model Reference Adaptive Control with Parameter Convergence Under Finite Excitation," *IEEE Transactions on Automatic Control*, vol. 63, no. 3, pp. 811–818, 2018.

[13] J. Park and I. W. Sandberg, "Approximation and Radial-Basis-Function Networks," *Neural Computation*, vol. 5, no. 2, pp. 305–316, 1993.