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On Invariants for Open Hybrid Systems and their Interconnections

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***Abstract*— For a broad class of hybrid dynamical systems with inputs, termed *open hybrid inclusions*, a general interconnection model and solution concept are introduced. This model is employed to certify forward invariance of a set for the interconnection. The forward invariance notion allows for Zeno solutions and solutions that end prematurely – namely, maximal solutions that are not complete. Sufficient conditions for forward invariance of a set that are compositional and involve a properly defined scalar-valued barrier function are proposed. An example illustrates the ideas.**

I. Introduction

Compositions of dynamical systems are prevalent and emerge in a broad range of problems in science and engineering. Systems interconnected in series (or cascades) are of particular relevance to synchronization, cooperative control, and networked control, which have received significant attention in the literature, see, e.g., [1], [2], [3], to just list a few. A particularly important property to guarantee for networked dynamical systems is invariance of the resulting interconnection. By defining a set of points K where the evolution of the state of the interconnection should remain, the problem to solve is as follows:

Given a set K, determine if it is forward invariant for the interconnection, regardless of the value of the input.

Unfortunately, analyzing forward invariance for the full interconnection leads to conditions that depend on the entire state and input of the interconnection. The conditions involved from using such an approach are not local to each system. Very importantly, the approach does not scale with the number of systems in the interconnection.

Compositional approaches that certify forward invariance of an interconnection from properties of the individual systems are more effective than those that study the interconnection as a whole. A compositional approach to determine safety via barrier certificates for a class of continuous-time systems is presented in [4]; see also [5]. In [6], [7], a compositional approach based on passivity is presented to study and verify stability and safety of continuous-time systems. Also exploiting dissipativity properties, the work in [8] proposes abstractions of models of the systems involved in the interconnection to certify and verify, in a compositional manner, safety. In [9], and in the context of differential inclusions, the authors employ assume-guarantee contracts to certify specifications for interconnections by exploiting the properties of the subsystems. The work in [9] considers assume-guarantee contracts for forward invariance and, using tools from viability theory [10] – specifically, tangent cone-based conditions – shows that invariance of an interconnection can be established using assume-guarantee contracts for invariance. Also recently, in [11], the authors borrow ideas from the small-gain theorem for continuous-time systems and introduce the notion of compositional barrier certificates to assure safety of an interconnection.

In this paper, we propose conditions for forward invariance for interconnections that, as a difference to the work in the literature, allow for the subsystems to have hybrid dynamics. We consider hybrid dynamical systems within the framework in [12], [13], which can model hybrid automata, impulsive systems, differential inclusions, and difference inclusions with constraints. In this framework, a hybrid system has a state that can evolve continuously within a set called *the flow set* according to a differential inclusion and, at times, the state can jump instantaneously from a set called *the jump set* to a value determined by a difference inclusion. Specifically, we aim at the following:

i) Formulate conditions requiring local information of the state and input, and
ii) Exploit (over) approximations of the range of output values provided by the output of the systems connected to inputs of other systems.

The conditions provided in this paper do not require checking for solutions to the systems involved in the interconnection – in fact, the conditions are infinitesimal. By extending the notion of barrier certificate in [14] to the case of interconnections, we formulate conditions that individual barrier certificates need to satisfy at points where evolution of solutions is allowed, relative to the flow and jump set, and relative to the values that the outputs assigning inputs can take when interconnected. Due to the generality of the model considered, our results handle the more classical situation when continuous-time systems and discrete-time systems are interconnected, even under constraints.

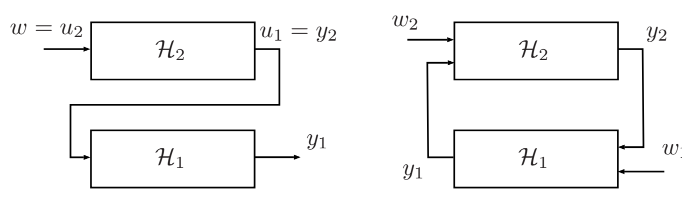

Fig. 1. Interconnections of hybrid systems: series (left) and parallel (right).

The remainder of the paper is organized as follows. An overview of interconnecting hybrid systems, along with

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a general interconnection model and the formal problem statement, are provided in Section II. The notion of barrier function and sufficient conditions for forward invariance are in Section III. An example is presented in Section IV.

Notation: The set of real numbers is denoted by $\mathbb{R}$, its subset of nonnegative real numbers by $\mathbb{R}_{\geq 0}$, the natural numbers including 0 by $\mathbb{N}$, the n-dimensional Euclidean space by $\mathbb{R}^n$, and the closed unit ball in Euclidean space centered at the origin by $\mathbb{B}$. Given a set S, $\overline{S}$ denotes its closure, ∂S denotes its boundary, and $\mathcal{U}(S)$ a neighborhood (open or closed). Given x and y, $\langle x, y\rangle$ denotes their inner product. Given a set $S \subset \mathbb{R}^n \times \mathbb{R}^m$, $\Pi(S)$ denotes the projection of S on its first component, i.e., $\{x \in \mathbb{R}^n \ : \ \exists u \text{ s.t. } (x, u) \in S\}$, $\Lambda(S)$ the projection on the second component, i.e., $\Lambda(S) := \{u \ : \ \exists x \in \mathbb{R}^n \text{ s.t. } (x, u) \in S\}$, and, given $x \in \mathbb{R}^n$, $\Psi(x, S)$ denotes the set of values u such that $(x, u) \in S$, i.e., $\Psi(x, S) := \{u \ : \ (x, u) \in S\}$. Given a set $S \subset \mathbb{R}^n \times \mathbb{R}^m$, a set $K \subset \mathbb{R}^n$, and a neighborhood of ∂K, denoted $\mathcal{U}(\partial K)$, define $\Phi(S, K, \mathcal{U}(\partial K)) := \{(x, u) \ : \ x \in (\mathcal{U}(\partial K) \setminus K) \cap \Pi(S), u \in \Psi(x, S)\}$. The operator Φ collects all points x nearby K that are in the projection of S to the space of x but not in K, along with all the associated values for u. Given a function f, $f^\circ(x, v)$ denotes the Clarke generalized derivative of f at x in the direction v and $\operatorname{rge} f$ denotes the range of f, i.e., $\operatorname{rge} f = f(\operatorname{dom} f)$, where $\operatorname{dom} f$ is the domain of definition of the function f.

II. Invariants for Compositions of Open Hybrid Systems with Local Information

A. Preliminaries

In this paper, we study interconnections of hybrid dynamical systems. We employ the framework in [13] to model hybrid systems with inputs. Following [13], the i-th hybrid system in the interconnection is denoted by $\mathcal{H}_i$, has state vector x_i, input u_i, and output y_i, where $i \in \{1, 2, \ldots, N\}$. A model for $\mathcal{H}_i$ is given by

$$\mathcal{H}_i : \begin{cases} (x_i, u_i) \in C_i & \dot{x}_i \in F_i(x_i, u_i) \\ (x_i, u_i) \in D_i & x_i^+ \in G_i(x_i, u_i) \\ & y_i = h_i(x_i, u_i) \end{cases} \tag{1}$$

The state x_i can evolve as follows:

- Flow: x_i is allowed to evolve continuously – namely, to flow – when, for given input u_i, the condition

  $$(x_i, u_i) \in C_i$$

  is satisfied. The set C_i is a subset of the state and input space. This set can include constraints that the state and the input have to satisfy during flows. During flows, x_i changes continuously according to the differential inclusion

  $$\dot{x}_i \in F_i(x_i, u_i)$$

  In simple words, given an input u_i, the state x_i evolves continuously, with a velocity defined by F_i when (x_i, u_i) is in the set C_i.
- Jump: The state x_i might also experience instantaneous changes – namely, jumps. Jumps are allowed when the condition

  $$(x_i, u_i) \in D_i$$

  is satisfied. The set D_i is a subset of the state and input space. Similar to C_i, the set D_i can include constraints that the state and the input have to satisfy for jumps to occur. When a jump occurs, the new value of the state, which is denoted x_i^+, is assigned via the difference inclusion

  $$x_i^+ \in G_i(x_i, u_i)$$

  In other words, the state x_i is instantaneously reset to a value given by G_i when (x_i, u_i) belongs to D_i.

The output of $\mathcal{H}_i$ is defined by the function h_i as

$$y_i = h_i(x_i, u_i)$$

The hybrid system model $\mathcal{H}_i$ is defined by the data $(C_i, F_i, D_i, G_i, h_i)$, where

- $C_i \subset \mathbb{R}^{n_i} \times \mathbb{R}^{m_i}$ is the flow set,
- $F_i : \mathbb{R}^{n_i} \times \mathbb{R}^{m_i} \rightrightarrows \mathbb{R}^{n_i}$ is the flow map,
- $D_i \subset \mathbb{R}^{n_i} \times \mathbb{R}^{m_i}$ is the jump set,
- $G_i :\subset \mathbb{R}^{n_i} \times \mathbb{R}^{m_i} \rightrightarrows \mathbb{R}^{n}$ is the jump map, and
- $h_i : \mathbb{R}^{n_i} \times \mathbb{R}^{m_i} \rightrightarrows \mathbb{R}^{r_i}$ is the output map.

Since F_i and G_i may give rise to a differential inclusion and a difference inclusion with inputs, respectively, we refer to $\mathcal{H}_i$ as an *open hybrid inclusion*. The data of $\mathcal{H}_i$ is $(C_i, F_i, D_i, G_i, h_i)$. When it is important to denote the data of $\mathcal{H}_i$ explicitly, we write $\mathcal{H}_i = (C_i, F_i, D_i, G_i, h_i)$.

In this paper, solutions to $\mathcal{H}_i$ are given in terms of pairs of hybrid arcs and hybrid inputs on hybrid time domains; see [12], [13]. In Section III, a notion of solution for the interconnection of hybrid dynamical systems is presented.

B. General Interconnections and Problem Formulation

We consider an interconnection of N hybrid dynamical systems, in which each system is modeled as an *open hybrid inclusion*. Specifically, for each $i \in \mathcal{V} := \{1, 2, \ldots, N\}$, the i-th hybrid dynamical system in the interconnection is given by $\mathcal{H}_i$ as in (1), with data $(C_i, F_i, D_i, G_i, h_i)$. The interconnection between the N hybrid dynamical systems in the family

$$\{\mathcal{H}_i = (C_i, F_i, D_i, G_i)\}_{i \in \mathcal{V}}$$

is determined by the *interconnectivity graph*

$$\Gamma = (\mathcal{V}, \mathcal{E}, \mathcal{G}, \{\varphi_i, s_i\}_{i \in \mathcal{V}})$$

The set $\mathcal{E}$ collects the edges given by pairs (ℓ, k) indicating that the output of $\mathcal{H}_k$ is connected to the input of $\mathcal{H}_\ell$. The matrix $\mathcal{G}$ is the adjacency matrix. Its (ℓ, k)-th entry $g_{\ell k}$ is equal to one if $(\ell, k) \in \mathcal{E}$. (Note that $(\mathcal{V}, \mathcal{E}, \mathcal{G})$ defines a directed graph.) Without loss of generality, the function φ_i assigns the first s_i components of the input of $\mathcal{H}_i$ using the output of the systems that connect to it, namely,

$$u_i = \left(\varphi_i\left(\{y_k\}_{k \in \mathcal{N}(i)}\right), w_i\right) \tag{2}$$

where $\mathcal{N}(i)$ is the set of indices corresponding to the neighbors that are connected to $\mathcal{H}_i$ and w_i are the components of the input u_i that are not assigned via φ_i. The pair $(\{\mathcal{H}_i\}_{i\in\mathcal{V}}, \Gamma)$ defines the interconnection, which is denoted $\mathcal{H}^{\text{int}}$.

A hybrid model of the interconnection $\mathcal{H}^{\text{int}}$ is given by

$$\mathcal{H}^{\text{int}}: \begin{cases} (x,w) \in C & \dot{x} \in F(x,w) \\ (x,w) \in D & x^+ \in G(x,w) \\ & y = h(x,w) \end{cases} \tag{3}$$

where $x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^n$ is the state with $n = \sum_{i=1}^N n_i$, $w = \{w_i\}_{i\in\mathcal{V}:\ s_i<m_i} \in \mathbb{R}^{m_w}$ collects the components of the inputs of each system that are not connected to outputs of other systems – namely, $w_i \in \mathbb{R}^{m_{w_i}}$ is the vector of inputs of $\mathcal{H}_i$ that are unassigned, in which case $s_i < m_i$ – and $y \in \mathbb{R}^p$ is defined by the function h which collects the desired outputs of the individual systems.

To define the data C, F, D, and G of $\mathcal{H}^{\text{int}}$, a rule (or semantics) for flows and jumps of the individual systems needs to be formulated. In this paper, we employ the following *interconnection rules*:

1) For a solution to the interconnection to flow, all systems in the family $\{\mathcal{H}_i\}_{i\in\mathcal{V}}$ should be able to flow, i.e., for each $i \in \mathcal{V}$, the flow conditions imposed by $\mathcal{H}_i$ need to be satisfied;
2) A solution to the interconnection jumps when at least one system in the family $\{\mathcal{H}_i\}_{i\in\mathcal{V}}$ is able to jump, i.e., these exists $i \in \mathcal{V}$ such that the jump conditions imposed by $\mathcal{H}_i$ are satisfied.

Following these interconnection rules, the flow map of $\mathcal{H}^{\text{int}}$ is given by

$$F(x,w) := (\widetilde{F}_1(x,w), \widetilde{F}_2(x,w), \ldots, \widetilde{F}_N(x,w)) \tag{4}$$

where $\widetilde{F}_1(x,w) := F_i(x_i, \varphi_i(\{y_k\}_{k\in\mathcal{N}(i)}), w_i)$ with y_k being the output of $\mathcal{H}_k$, which depends on the state x_k and the input u_k (which, in addition, depends on the assignment (2)). The flow set C is given by

$$\{(x,w)\ :\ (x_i,u_i) \in C_i\ \forall i \in \mathcal{V},\ u_i \text{ as in (2)}, \quad w = \{w_i\}_{i\in\mathcal{V}:\ s_i<m_i}\} \tag{5}$$

The jump set D is defined as

$$\{(x,w)\ : \exists i \in \mathcal{V}\ :\ (x_i,u_i) \in D_i,\ u_i \text{ as in (2)}, \quad w = \{w_i\}_{i\in\mathcal{V}:\ s_i<m_i}\} \tag{6}$$

and, at each $(x,w) \in D$, the jump map is given by

$$G(x,w) := \bigcup_{i\in\mathcal{V}} \widetilde{G}_i(x_i,w_i) \tag{7}$$

where $\widetilde{G}_i : \mathbb{R}^n \times \mathbb{R}^{m_{w_i}} \rightrightarrows \mathbb{R}^n$ is nonempty on

$$\{(x_i,w_i)\ :\ (x_i,u_i) \in D_i,\ (u_i,w_i) \text{ satisfying (2) }\}$$

and empty elsewhere. To properly reset the component of x associated to the state of system that jumps, the i-th entry of $\widetilde{G}_i$ is equal to $G_i(x_i, \varphi_i(\{y_k\}_{k\in\mathcal{N}(i)}), w_i)$, and, for each $k \in \mathcal{V} \setminus \{i\}$, its k-th entry is equal to x_k.

To define the notion of solution for $\mathcal{H}^{\text{int}}$ in (3), we introduce the following objects.

Definition 2.1 (hybrid time and domain): A *compact hybrid time domain* is a set of the form

$$E := \bigcup_{i=0}^{J-1} \Big([t_i, t_{i+1}] \times \{i\}\Big) \tag{8}$$

where $J \in \mathbb{N}$, and $0 = t_0 \leq t_1 \leq \cdots \leq t_J$. A *hybrid time domain* is the union of a nondecreasing sequence of compact hybrid time domains $E_1 \subset E_2 \subset E_3 \subset \ldots$. Each element $(t,j) \in E$ denotes the elapsed hybrid time, which indicates that t seconds of flow time and j jumps have occurred.

Definition 2.2 (hybrid input): A function $w : \operatorname{dom} w \to \mathbb{R}^{m_w}$ is a hybrid input if $\operatorname{dom} w$ is a hybrid time domain and if, for each $j \in \mathbb{N}$, the function $t \mapsto w(t,j)$ is Lebesgue measurable and locally essentially bounded on the interval $I_w^j := \{t\ :\ (t,j) \in \operatorname{dom} w\ \}$.

Definition 2.3 (hybrid arc): A function $x : \operatorname{dom} x \to \mathbb{R}^n$ is a hybrid arc if $\operatorname{dom} x$ is a hybrid time domain and if, for each $j \in \mathbb{N}$, the function $t \mapsto x(t,j)$ is locally absolutely continuous on the interval I_x^j.

We are ready to introduce a notion of solution for $\mathcal{H}^{\text{int}}$.

Definition 2.4 (solution to $\mathcal{H}^{\text{int}}$): A hybrid input w and a hybrid arc x define a solution (x,w) to the hybrid system $\mathcal{H}^{\text{int}}$ in (3) if

(S0) $(x(0,0), w(0,0)) \in \overline{C}$ or $(x(0,0), w(0,0)) \in D$, and $\operatorname{dom} x = \operatorname{dom} w$ $(= \operatorname{dom}(x,w))$;

(S1) For each $j \in \mathbb{N}$ such that $I^j_{(x,w)}$ has a nonempty interior $\text{int}(I^j_{(x,u)})$, $t \mapsto (x(t,j), w(t,j))$ satisfies

$$(x(t,j), w(t,j)) \in C \qquad \text{for all } t \in \text{int}(I^j_{(x,w)})$$

and

$$\frac{d}{dt}x(t,j) \in F(x(t,j), w(t,j)) \quad \text{ for almost all } t \in I^j_{(x,u)}$$

(S2) For each $(t,j) \in \operatorname{dom}(x,w)$ such that $(t, j+1) \in \operatorname{dom}(x,w)$, $(t,j) \mapsto (x(t,j), w(t,j))$ satisfies

$$(x(t,j), w(t,j)) \in D$$

and

$$x(t,j+1) \in G(x(t,j), w(t,j))$$

A solution pair (x,w) to $\mathcal{H}^{\text{int}}$ is said to be *complete* if $\operatorname{dom}(x,w)$ is unbounded. It is said to be *maximal* if there does not exist another pair $(x,w)'$ such that (x,w) is a truncation of $(x,w)'$ to some proper subset of $\operatorname{dom}(x,w)'$. A solution (x,w) is Zeno if it is complete and the projection of $\operatorname{dom}(x,w)$ to $\mathbb{R}_{\geq 0}$ is bounded. For more details about solutions to hybrid systems with inputs, see [15], [13].

To formally state the problem to solve, we introduce the following forward invariance notion of a set for the interconnection $\mathcal{H}^{\text{int}}$ in (3).

Definition 2.5 (uniform forward pre-invariance): Given a hybrid system $\mathcal{H}^{\text{int}}$ as in (3), a set $K \subset \mathbb{R}^n$ is said to be *forward pre-invariant for $\mathcal{H}^{\text{int}}$ uniformly in w* if for every solution pair (x,w) with $x(0,0) \in K$, the state component x satisfies $x(t,j) \in K$ for all $(t,j) \in \operatorname{dom}(x,w)$.

Remark 2.6: The term "pre" in uniform forward pre-invariance is included to capture the situation when maximal solutions from K are not complete. When maximal solutions to $\mathcal{H}^{\text{int}}$ from K are complete, then forward pre-invariance becomes forward invariance.

Following the problem outlined in Section I, we are ready to state the problem to solve.

Problem ($\star$): Given an interconnectivity graph Γ, a family of dynamical systems

$$\{\mathcal{H}_i = (C_i, F_i, D_i, G_i)\}_{i\in\mathcal{V}}$$

and a collection of sets $\{K_i\}_{i\in\mathcal{V}}$ with

$$K_i \subset \mathbb{R}^{n_i} \qquad \forall i \in \mathcal{V}$$

defining sets to render invariant for $\mathcal{H}_i$, determine local conditions at each agent guaranteeing that the set

$$K := K_1 \times K_2 \times \ldots \times K_N \tag{9}$$

is forward pre-invariant for the interconnection $\mathcal{H}^{\text{int}}$ in (3) uniformly in w.

III. Sufficient Conditions for Invariance of Interconnections using Local Information

To formulate sufficient conditions that solve Problem ($\star$), given information about the possible input values for each system, we define an over approximation of the output sets Y_i for each system in Section III-A. Using these sets, and in that same section, we characterize the range of solution pairs to $\mathcal{H}_i$ and $\mathcal{H}^{\text{int}}$. The proposed sufficient conditions are given in Section III-B.

A. Definitions and Properties of Solutions

For each $i \in \mathcal{V}$, and with φ_i defining (via (2)) the assignment of the input u_i of $\mathcal{H}_i$ using the output of its neighbors $\{\mathcal{H}_k\}_{k\in\mathcal{N}(i)}$, we denote by $\widehat{Y}_k$ an over approximation[1] of the set of output values that the solutions to $\mathcal{H}_k$ can attain, which is denoted Y^k. With this information, we define $\widetilde{C}_i$ and $\widetilde{D}_i$ as the effective flow and jump sets for $\mathcal{H}_i$ as follows:

- We define the set $\mathcal{I}_i^c$ collecting the values of u_i that, through the assignment in (2), are possible during flows in light of the effect of the outputs of the systems that are connected to $\mathcal{H}_i$, namely, $\{\mathcal{H}_k\}_{k\in\mathcal{N}(i)}$. This set is defined as

$$\mathcal{I}_i^c = \left\{u_i = (\widetilde{u}_i, w_i) \in \Lambda(C_i) : \widetilde{u}_i \in \varphi_i\left(\{\widehat{Y}_k\}_{k\in\mathcal{N}(i)}\right)\right\}$$

  where $\widetilde{u}_i$ represents the input components of u_i that are assigned through φ_i. Then, the effective flow set for $\mathcal{H}_i$ resulting from the interconnection is

$$\widetilde{C}_i = C_i \cap (\mathbb{R}^{n_i} \times \mathcal{I}_i^c) \tag{10}$$

- The set $\mathcal{I}_i^d$ collects the values of u_i that are possible at jumps of $\mathcal{H}_i$ under the effect of the outputs of $\{\mathcal{H}_k\}_{k\in\mathcal{N}(i)}$. We define this set as

$$\mathcal{I}_i^d = \left\{u_i = (\widetilde{u}_i, w_i) \in \Lambda(D_i) : \widetilde{u}_i \in \varphi_i\left(\{\widehat{Y}_k\}_{k\in\mathcal{N}(i)}\right)\right\}$$

  Then, the effective jump set for $\mathcal{H}_i$ resulting from the interconnection is

$$\widetilde{D}_i = D_i \cap (\mathbb{R}^{n_i} \times \mathcal{I}_i^c) \tag{11}$$

For series interconnections, the definition of the effective flow set $\widetilde{C}_i$ and jump set $\widetilde{D}_i$ is explicit and these sets can be constructed sequentially, starting from the first system that is in series, and continuing down the interconnection. For interconnections with feedback, the definition is unavoidably implicit and typically requires to solve for these sets simultaneously.

Remark 3.1: When the possible values of the output of the k-th neighbor to $\mathcal{H}_i$ are known, then $\widehat{Y}_k$ can be chosen to be equal to Y_k. Without any such information, the set $\widehat{Y}_k$ can simply be chosen as $\mathbb{R}^{r_k}$. A smaller choice for $\widehat{Y}_k$ might be possible if one is able to identify a set including all possible values attained by the output of $\mathcal{H}_k$. Such a potentially smaller choice than $\mathbb{R}^{r_k}$ may lead to smaller effective flow and jump sets, $\widetilde{C}_i$ and $\widetilde{D}_i$, respectively. Reducing the size of these sets is beneficial for the forthcoming sufficient conditions for forward invariance. Such information can be obtained from knowing the data of each system and the interconnection assignment. In applications, logging the evolution of the output of the systems and employing reachability tools can aid in obtaining such information.

The following result characterizes the range of the solutions pairs to $\mathcal{H}^{\text{int}}$ in (3) in terms of the sets $\widetilde{C}_i$, $\widetilde{D}_i$, and $G_i(\widetilde{D}_i)$. It follows directly from the construction of the sets $\widetilde{C}_i$ and $\widetilde{D}_i$, the data of $\mathcal{H}^{\text{int}}$, and the definition of solution in Definition 2.4.

Lemma 3.2: Every solution (x, w) to $\mathcal{H}^{\text{int}}$ satisfies

$$\operatorname{rge}(x, w) \subset \left(\bigcap_{i\in\mathcal{V}} \overline{\widetilde{C}}_i\right) \cup \left(\bigcup_{i\in\mathcal{V}} \widetilde{D}_i\right) \cup \left(\bigcup_{i\in\mathcal{V}} \left(G_i(\widetilde{D}_i) \times \Lambda\left(\overline{\widetilde{C}}_i \cup \widetilde{D}_i\right)\right)\right) \tag{12}$$

where the sets $\{\widetilde{C}_i\}_{i\in\mathcal{V}}$ and $\{\widetilde{D}_i\}_{i\in\mathcal{V}}$ are defined via (10) *and* (11)*, respectively.*

B. Sufficient Conditions using Local Barrier Certificates

This section presents sufficient conditions guaranteeing forward pre-invariance of the set K in (9). The conditions provided are in terms of barrier certificates, namely, state-dependent scalar functions that are nonincreasing at points in an outer neighborhood of the set K guarantee that solutions cannot leave K. Inspired by [14], [13], we define the following notion of barrier function candidate for interconnections.

Definition 3.3: (barrier function candidate for interconnections) Given the family of hybrid systems

[1]Note that the inputs of $\mathcal{H}_k$ may depend on the values of the output of other systems, which may include $\mathcal{H}_i$ if the interconnection assignment includes feedback.

$\{\mathcal{H}_i = (C_i, F_i, D_i, G_i)\}_{i\in\mathcal{V}}$, the interconnection graph $\Gamma = (\mathcal{V}, \mathcal{E}, \mathcal{G}, \{\varphi_i, s_i\}_{i\in\mathcal{V}})$, and the collections of closed sets $\{K_i\}_{i\in\mathcal{V}}$, the collection of functions $\{B_i\}_{i\in\mathcal{V}}$ define a *barrier function candidate for the interconnection* $\mathcal{H}^{int} = (\{\mathcal{H}_i\}_{i\in\mathcal{V}}, \Gamma)$ with respect to K in (9) if the following properties hold: for each $i \in \mathcal{V}$,

1) $K_i = \{x_i \in \Pi(\overline{C_i}) \cup \Pi(D_i) \ : \ B_i(x_i) \leq 0 \}$;
2) For some open neighborhood $\mathcal{U}_i$ of ∂K_i, B_i is locally Lipschitz on $(\mathcal{U}_i(\partial K_i) \setminus K_i) \cap \overline{\Pi(C_i)}$.

Remark 3.4: The condition in item 1 requires the existence of B_i such that every point in the given set K_i is in the zero sub-level set of B_i, restricted to $\Pi(\overline{C_i} \cup D_i)$. In turn, for points in $\Pi(\overline{C_i} \cup D_i)$ but not in K_i, the value of B_i is positive. Item 2 assumes basic regularity properties to be able to take derivatives of B_i in terms of the Clarke generalized derivative, which is required to impose conditions that render B_i nonincreasing along solutions. Note that the conditions in Definition 3.3 only depend on information available at each agent; in particular, they do not depend on the effective flow and jump sets $\widetilde{C}_i$ and $\widetilde{D}_i$. However, to assure that a barrier certificate guarantees forward pre-invariance of K, information about the input to each system provided by the neighbors can be exploited, as the following result states.

Theorem 3.5: (invariance of interconnections using barrier functions) Given the family of hybrid systems $\{\mathcal{H}_i = (C_i, F_i, D_i, G_i)\}_{i\in\mathcal{V}}$*, the interconnection graph* $\Gamma = (\mathcal{V}, \mathcal{E}, \mathcal{G}, \{\varphi_i, s_i\}_{i\in\mathcal{V}})$*, the collection of closed sets* $\{K_i\}_{i\in\mathcal{V}}$*, and the collection of functions* $\{B_i\}_{i\in\mathcal{V}}$ *defining a barrier function candidate for the interconnection* $\mathcal{H}^{int} = (\{\mathcal{H}_i\}_{i\in\mathcal{V}}, \Gamma)$ *with respect to* K *in* (9)*, the set* K *is forward pre-invariant for* $\mathcal{H}^{int}$ *in* (3) *uniformly in* w *if the following properties hold:*[2] *for each* $i \in \mathcal{V}$,

$$B_i^\circ(x_i, f_i) \leq 0 \qquad \forall f_i \in F_i(x_i, u_i), \quad \forall (x_i, u_i) \in \widetilde{C}_i \cap \Phi(C_i, K_i, \mathcal{U}_i), \tag{13}$$

$$B_i(g_i) \leq 0 \qquad \forall g_i \in G_i(x_i, u_i), \quad \forall (x_i, u_i) \in \widetilde{D}_i \cap (K_i \times \mathbb{R}^{m_i}), \tag{14}$$

$$G_i(x_i, u_i) \subset \Pi(C_i) \cup \Pi(D_i) \quad \forall (x_i, u_i) \in \widetilde{D}_i \cap (K_i \times \mathbb{R}^{m_i}) \tag{15}$$

where $\mathcal{U}_i := \mathcal{U}(\partial K_i)$ *and* $\Phi(C_i, K_i, \mathcal{U}_i)$ *collects all points* x_i *nearby* K_i *that are in the projection of* C_i *to* $\mathbb{R}^{n_i}$ *but not in* K*, along with all the associated values for* u_i*; see Notation in Section I.*

Remark 3.6: The effective flow set $\widetilde{C}_i$ and jump set $\widetilde{D}_i$ enter (13)-(15) in a modular manner, as intersections to the set of points at which the infinitesimal conditions therein are to be checked. Due to the construction of the map Φ, condition (13) has to be checked on an outer neighborhood (of any size) around the set K_i; see the interconnection in Section IV. When B_i is continuously differentiable, we can replace the generalize Clarke derivative $B_i^\circ(x_i, f_i)$ by the inner product $\langle \nabla B_i(x_i), f_i \rangle$. On the other hand, to prevent solutions from jumping outside of K_i, condition (14) is checked at all points in D_i with state component that is in K_i, as long as is allowed by the interconnection assignment – which is encoded by $\widetilde{D}_i$. Note that the intersection by $\widetilde{C}_i$ and $\widetilde{D}_i$ reduce the set of points at which these conditions are to be checked. As pointed out in Remark 3.1, these sets can be reduced when information about the output of its neighbors is available.

[2]Conditions (13)-(15) are written in terms of u_i, but note that the interconnection assignment in (2) is encoded in those conditions through $\widetilde{C}_i$ and $\widetilde{D}_i$, which, in turn, lead to w_i's collecting the unassigned inputs to $\mathcal{H}_i$.

C. Establishing Uniform Forward Invariance

With forward pre-invariance of a set K (uniformly in w) established for $\mathcal{H}^{int}$ using Theorem 3.5, one may want to show that maximal solutions form K are complete, so that K is forward invariance (uniformly in w); see Remark 2.6. Showing that maximal solutions to $\mathcal{H}^{int}$ are complete can established using results in [16], [17], which, in particular, exploit the ideas in [12, Proposition 2.10 and Proposition 6.10] along with linear growth or boundedness of the flow map. Note that the class of inputs w would need to be restricted, due the difficulty of assuring viability of flows in the presence of state constraints.

IV. Example: Interconnections of Two Thermostats

We consider interconnections defined by coupling between models of temperature of two rooms. The temperature of each room is controlled by an independent thermostat system. The temperature of the first room is denoted z_1 and the temperature of the second room is denoted z_2. The thermostat systems control the heater present in each room with capacity $z_{1,\Delta} > 0$ and $z_{2,\Delta} > 0$, respectively. The external temperature to each room is denoted by $z_{1,\text{out}}$ and $z_{2,\text{out}}$, respectively.

The hybrid systems $\mathcal{H}_1$ and $\mathcal{H}_2$ are identical and defined as follows. For each $i \in \{1, 2\}$, the state of $\mathcal{H}_i$ is $x_i = (z_i, q_i)$, where $z_i \in \mathbb{R}$ is temperature and $q_i \in Q := \{0, 1\}$ is a logic state that when equal to zero indicates that the i-th heater is off and when equal to one indicates that it is on. The data $(C_i, D_i, F_i, G_i, h_i)$ of $\mathcal{H}_i$ is given by[3]

$$\begin{aligned}
C_i &= (\{x_i \in \mathbb{R} \times Q \ : \ z_i \geq z_{i,\min}, q_i = 0 \} \\
&\quad \cup \{x_i \in \mathbb{R} \times Q \ : \ z_i \leq z_{i,\max}, q_i = 1 \}) \times U_i \\
F_i(x_i, u_i) &= \begin{bmatrix} -z_i + \begin{bmatrix} z_{i,\Delta} & 1 \end{bmatrix} \begin{bmatrix} q_i \\ u_i \end{bmatrix} \\ 0 \end{bmatrix} \quad \forall (x_i, u_i) \in C_i, \\
D_i &= \{x_i \in \mathbb{R} \times Q \ : \ z_i \leq z_{i,\min}, q_i = 0 \} \\
&\quad \cup \{x_i \in \mathbb{R} \times Q \ : \ z_i \geq z_{i,\max}, q_i = 1 \} \\
G_i(x_i) &= \begin{bmatrix} z_i \\ \delta(q_i) \end{bmatrix} \qquad \forall x_i \in D_i \\
h_i(x_i) &= z_i
\end{aligned}$$

where $U_i \subset \mathbb{R}$ defines the possible values for u_i, $\delta(q_i) = 1 - q_i$ toggles q_i from zero to one when z_i is smaller than or equal the threshold $z_{i,\min}$, and from one to zero when z_i is larger than or equal to the threshold $z_{i,\max}$.

[3]Since the input of $\mathcal{H}_i$ affects the flows only, for simplicity, it is omitted from D_i and G_i.

Consider the series interconnection depicted in Figure 1 (left) defined by the interconnection assignment

$$u_1 = z_2, \qquad u_2 = z_{2,\text{out}} \tag{16}$$

This assignment represents the situation when the outside temperature for the first room is equal to the temperature of the second room (e.g., the first room is physically contained in the second room). Following the model in Section II-B, $w = u_2$ for the resulting interconnection $\mathcal{H}^{\text{int}}$, whose model is given in (3) with $N = 2$. For given parameters $z_{1,\min} < z_{1,\max}$ and $z_{2,\min} < z_{2,\max}$ determining the desired range of temperatures for each room, the objective is to determine conditions on U_1, U_2, $z_{1,\Delta}$, and $z_{2,\Delta}$ to keep z_1 in $[z_{1,\min}, z_{1,\max}]$ and z_2 in $[z_{2,\min}, z_{2,\max}]$ for all hybrid time if the temperatures start within those ranges. This objective consists of rendering the set K in (9) with $N = 2$ forward invariant uniformly in w for the resulting interconnection, where, for each $i \in \{1, 2\}$,

$$K_i = [z_{i,\min}, z_{i,\max}] \times Q \subset \mathbb{R} \times Q \tag{17}$$

and, to assure that every maximal solution to $\mathcal{H}^{\text{int}}$ from K is complete.

To certify this property, consider the barrier function $B_i : \mathbb{R} \times Q \to \mathbb{R}$ defined for each $x_i = (z_i, q_i) \in \mathbb{R} \times Q$ as the locally Lipschitz function

$$B_i(x_i) := \max\{z_i - z_{i,\max}, z_{i,\min} - z_i\}$$

To have $B_i(x_i) \leq 0$ on K_i and $B_i(x_i) > 0$ outside K_i, note that

$$\max\{z_i - z_{i,\max}, z_{i,\min} - z_i\} \leq 0$$

if and only if $z_i - z_{i,\max} \leq 0$ and $z_{i,\min} - z_i \leq 0$. This condition implies that z_i should satisfy $z_{i,\min} \leq z_i$ and $z_i \leq z_{i,\max}$ for z_i to belong to K_i, which, since q_i is unrestricted, leads to the set K_i defined in (17). Hence, since $C_i \cup D_i = \mathbb{R} \times Q$, $\{B_i\}_{i\in\{1,2\}}$ is a barrier function candidate for the interconnection.

Using the definition of C_i and K_i, pick $\mathcal{U}_i = \mathcal{U}(\partial K_i)$ with $\mathcal{U}(\partial K_i)$ an open neighborhood of ∂K_i. When $q_i = 1$, this open neighborhood leads to points in the z_i component that are in the set $(z_{i,\max}, z_{i,\max} + \epsilon_i)$ for some $\epsilon_i > 0$, and, at such points, $B_i(x_i) = z_i - z_{i,\max}$. Similarly, when $q_i = 0$, and for the same value of ϵ_i, this open neighborhood collects z_i points in the set $(z_{i,\min} - \epsilon_i, z_{i,\min})$, and, at such points, $B_i(x_i) = z_{i,\min} - z_i$. It can be shown by analyzing the variation of B_i along flows that B_i is nonincreasing along flows if

$$\left.\begin{array}{c} w < z_{2,\max} < z_{1,\max} \\ w + z_{2,\Delta} > z_{2,\min} > z_{1,\min} - z_{1,\Delta} \end{array}\right\} \quad \forall w \in U_2 \tag{18}$$

which require U_2 to be a compact subset of $\mathbb{R}$. Hence, since $\widetilde{C}_i \subset C_i$, (13) holds. Since the jump map G_i is such that z_i remains constant at jumps, then $B_i(G_i(x_i)) = B_i(x_i)$ for each $x_i \in D_i \cap K_i$. Then, for each point $x_i \in D_i \cap K_i$, $B_i(G_i(x_i)) \leq 0$; hence, since $\widetilde{D}_i \subset D_i$, (14) holds. Using the same argument, it is straightforward to show that (15) holds. By Theorem 3.5, K as in (9), with K_i in (17), is pre-forward invariant for $\mathcal{H}^{\text{int}}$, uniformly in $w = z_{2,\text{out}}$ taking values on a compact set U_2 on which the conditions in (18) hold – note that U_1 can be taken to be equal to $\mathbb{R}$. To assure that every maximal solution is complete, we assume that w belongs to the class of piecewise-continuous functions taking values from U_2. The conditions mentioned in Section III-C hold due to fact that, for each $i \in \{1, 2\}$, $C_i \backslash D_i$ is open, F_i is smooth, and the jump map takes points in the jump set back to the flow set. Moreover, $\mathcal{H}^{\text{int}}$ does not have Zeno solutions since the distance between $G(D)$ and D is uniformly lower bounded by a positive constant and F_i has linear growth (hence, no finite escape times are possible).

V. Conclusion

The proposed compositional sufficient conditions for forward invariance of K in (9) employ a scalar-valued barrier function for each system. In light of the constructions in [14], they can be replaced by vector-valued barrier functions to allow for K_i to be given by the intersection of finitely many sublevel sets of barrier functions. Future work includes developing algorithms to approximate the output sets $\widehat{Y}_i$ using over approximations of reachable sets given by multiple Lyapunov functions. The work in [9] also provides motivation to extend assume-guarantee contracts to encode and reason about invariance properties of $\mathcal{H}^{\text{int}}$.

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