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Control of Polytopic LPV Systems with Uncertain Initial Conditions

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***Abstract*— This paper focuses on the control design and analysis for nonstationary linear parameter-varying systems with affine parameter dependence and uncertain initial conditions. The uncertain initial state and the disturbance input are allowed to reside in two separate norm balls. Convex analysis and synthesis conditions are derived, and a reachability analysis result for systems with pointwise-bounded inputs is developed, enabling the construction of ellipsoids in which the state or some output of interest lies at specified time instants. The usefulness of the proposed approach is demonstrated through an illustrative example involving a two-mass rotational system.**

I. Introduction

This paper provides control analysis and synthesis results for nonstationary linear parameter-varying (NSLPV) systems with affine parameter dependence and uncertain initial conditions. These results are complementary to the ones provided in [1] for NSLPV systems formulated in a linear fractional transformation (LFT) framework and build on the works in [2], [3]. As in [1], the proposed approach focuses on eventually periodic systems [4], [5], i.e., systems composed of a finite horizon part and a subsequent periodic part, and involves solving a square ℓ_2 problem, where ℓ_2 denotes the space of square summable sequences. This problem has been addressed for linear time-invariant (LTI) systems in [6] and linear time-varying (LTV) systems in [7], [8]. The approach allows constraining the uncertain initial state to a Euclidean ball and the disturbance input to a separate ℓ_2-norm ball, and provides an upper bound on the ℓ_2-norm of the performance output. As demonstrated in our previous works [8], [9], [1] and the illustrative example herein, when dealing with eventually periodic systems with uncertain initial conditions, it may be possible to significantly improve the closed-loop performance by designing controllers that have a larger finite horizon length than the plant.

The analysis and synthesis conditions derived in this paper are in the form of linear matrix inequalities (LMIs) [10]. In addition, the paper gives a result on reachability analysis for NSLPV systems with inputs bounded pointwise in time. This result is achieved by expressing the system with a pointwise-bounded disturbance input as an unforced system with an uncertain initial state, where the input is viewed in this new formulation as a static linear time-varying perturbation (see, for instance, [11]). Then, invoking the analysis result for NSLPV systems with uncertain initial conditions leads to the reachability analysis result, which determines an ellipsoid in which the state or a subset of the state variables is guaranteed to reside at some specified time instant. The paper also provides an illustrative example involving a two-mass rotational system, based on similar examples given in [8], [1], to demonstrate the utility of the proposed approach. Lastly, the paper includes an appendix which gives necessary and sufficient analysis conditions for eventually periodic LTV systems with an uncertain initial state. This result can be easily deduced from the work in [8]; however, it is not explicitly stated there. The appendix also provides a version of the reachability analysis result for NSLPV systems formulated in an LFT framework. Further analysis results for uncertain time-varying systems can be found in [12].

The outline of the paper is as follows. The notation and problem formulation are presented in Section II. The analysis and synthesis results are provided in Section III, along with the result on reachability analysis for systems with pointwise-bounded inputs. An illustrative example demonstrating the usefulness of the approach is given in Section IV. Section V concludes the paper.

II. Preliminaries

A. Notation

The sets of nonnegative integers, real vectors of size n, real $n \times m$ matrices, and real symmetric $n \times n$ matrices are denoted by $\mathbb{N}_0$, $\mathbb{R}^n$, $\mathbb{R}^{n\times m}$, and $\mathbb{S}^n$, respectively. The block-diagonal augmentation of matrices $A_1, \ldots, A_N$ is denoted by $\mathrm{diag}(A_1, \ldots, A_N)$. The $n \times m$ zero matrix, $n \times n$ identity matrix, and vector of ones in $\mathbb{R}^n$ are denoted by $0_{n\times m}$, I_n, and $\mathbf{1}_n$, respectively. The transpose of a matrix X is denoted by X^T. Given a matrix $X \in \mathbb{S}^n$, we write $X \prec 0$ ($X \preceq 0$) to indicate that X is negative definite (negative semidefinite) and $X \succ 0$ ($X \succeq 0$) to indicate that X is positive definite (positive semidefinite). The Hilbert space ℓ_2^n is defined as the space of sequences $v = (v(0), v(1), \ldots)$, where $v(k) \in \mathbb{R}^{n(k)}$, such that $\sum_{k=0}^{\infty} v(k)^T v(k) < \infty$. The symbol ℓ_2^n is abbreviated to ℓ_2 when the dimensions are clear from the context. The ℓ_2-norm of a sequence $v \in \ell_2$ is defined as $\|v\|_{\ell_2}^2 := \sum_{k=0}^{\infty} v(k)^T v(k)$, and the Euclidean norm of a vector $d \in \mathbb{R}^m$ is denoted by $\|d\|_2$, i.e., $\|d\|_2^2 = d^T d$. The image and kernel of a linear map M are denoted by $\mathrm{Im}\, M$ and $\mathrm{Ker}\, M$, respectively.

A matrix sequence $X = (X(0), X(1), \ldots)$ is said to be (h, q)-eventually periodic for some integers $h \geq 0$ and $q \geq 1$ if $X(h + k) = X(h + q + k)$ for all $k \in \mathbb{N}_0$. A discrete-time LTV system is (h, q)-eventually periodic if all its state-space matrix sequences are (h, q)-eventually periodic [4], [5]. The

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class of eventually periodic systems contains finite horizon and periodic systems as special cases. An NSLPV system is defined by state-space equations of the form

$$x(k+1) = A(\delta(k), k)x(k) + B(\delta(k), k)w(k),$$
$$z(k) = C(\delta(k), k)x(k) + D(\delta(k), k)w(k),$$

where the scheduling parameter δ is not given a priori but is known to lie in some set $\boldsymbol{\delta}$, and the state-space matrix-valued functions have explicit dependence on both the scheduling parameter and time. An NSLPV system is (h, q)-eventually periodic if all its state-space matrix-valued functions are (h, q)-eventually periodic with respect to the explicit time dependence [13], i.e., $Q(\delta, h+k) = Q(\delta, h+q+k)$ for all $k \in \mathbb{N}_0$, $Q = A, B, C, D$.

B. Problem Formulation

Consider an (h, q)-eventually periodic NSLPV system G, defined by the following state-space equation:

$$\begin{bmatrix} x(k+1) \\ z(k) \\ y(k) \end{bmatrix} = \begin{bmatrix} A(\delta(k), k) & B_1(\delta(k), k) & B_2(k) \\ C_1(\delta(k), k) & D_{11}(\delta(k), k) & D_{12}(k) \\ C_2(k) & D_{21}(k) & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \\ u(k) \end{bmatrix}, \quad (1)$$

where $x(0) = x_0$. The symbols x, w, u, z, and y denote the state, disturbance input, control input, performance output, and measurement output, respectively, and their values at each discrete instant $k \in \mathbb{N}_0$ are vectors of dimensions $n(k)$, $n_w(k)$, $n_u(k)$, $n_z(k)$, and $n_y(k)$, respectively. The scheduling parameter δ takes values in a bounded, convex polytope $\mathcal{P} \subset \mathbb{R}^{n_p}$ with vertices $v_1, v_2, \ldots, v_r$, i.e., $\delta(k) \in \text{conv}\{v_1, v_2, \ldots, v_r\}$ (convex hull of the set of vertices) for all $k \in \mathbb{N}_0$. The set of parameter trajectories δ is defined as

$$\boldsymbol{\delta} = \{\delta : \mathbb{N}_0 \to \mathbb{R}^{n_p} \mid \delta(k) \in \text{conv}\{v_1, v_2, \ldots, v_r\} \, \forall \, k \in \mathbb{N}_0\}.$$

For ease of exposition, given a matrix-valued function Q with domain $\operatorname{dom} Q = \mathbb{R}^{n_p} \times \mathbb{N}_0$, we define $Q_i(k) := Q(v_i, k)$ for $i = 1, 2, \ldots, r$. The functions A, B_1, C_1, and D_{11} are assumed to have affine dependence on the parameter δ. It follows that $Q(\delta(k), k) \in \text{conv}\{Q_1(k), Q_2(k), \ldots, Q_r(k)\}$ for all $k \in \mathbb{N}_0$ and $Q = A, B_1, C_1, D_{11}$. For each $\delta \in \boldsymbol{\delta}$, the NSLPV system G reduces to an LTV system, denoted by G_δ, where the subscript indicates the specific parameter trajectory δ at which system G is evaluated. Thus, the NSLPV system G constitutes a set of LTV systems: $G = \{G_\delta \mid \delta \in \boldsymbol{\delta}\}$.

It is assumed that s of the state variables have uncertain initial values for some positive integer $s \leq n(0)$; the initial values of the remaining $n(0) - s$ state variables are assumed to be zero. Thus, $x_0 = \Lambda x_0'$, where $\Lambda \in \mathbb{R}^{n(0) \times s}$ has full column rank and $x_0' \in \mathbb{R}^s$ comprises the uncertain initial state values. The vector x_0' is assumed to reside in some ellipsoid centered at the origin; namely, $x_0' \in \mathcal{E} = \{e \in \mathbb{R}^s \mid \|Pe\|_2 \leq 1\}$ for some $P \succ 0$. Defining $\xi_0 = P x_0'$, then ξ_0 takes values in a unit Euclidean ball centered at the origin, i.e., $\xi_0 \in \mathcal{B} = \{e \in \mathbb{R}^s \mid \|e\|_2 \leq 1\}$. The initial state x_0 can then be expressed as

$$x_0 = \Lambda P^{-1} \xi_0 = \Gamma \xi_0. \quad (2)$$

The plant G is assumed to be controlled by a feedback, (N, q)-eventually periodic NSLPV controller K, for some $N \geq h$, defined by the following state-space equation:

$$\begin{bmatrix} x_K(k+1) \\ u(k) \end{bmatrix} = \begin{bmatrix} A_K(\delta(k), k) & B_K(\delta(k), k) \\ C_K(\delta(k), k) & D_K(\delta(k), k) \end{bmatrix} \begin{bmatrix} x_K(k) \\ y(k) \end{bmatrix}, \quad (3)$$

where $x_K(k) \in \mathbb{R}^{m(k)}$, for some positive integer $m(k) \leq n(k)$, is the controller's state, $x_K(0) = 0$, and $\delta \in \boldsymbol{\delta}$ is the same parameter as the one that affects the plant's dynamics. The closed-loop system L is (N, q)-eventually periodic and defined by the state-space equation

$$\begin{bmatrix} x_L(k+1) \\ z(k) \end{bmatrix} = \begin{bmatrix} A_L(\delta(k), k) & B_L(\delta(k), k) \\ C_L(\delta(k), k) & D_L(\delta(k), k) \end{bmatrix} \begin{bmatrix} x_L(k) \\ w(k) \end{bmatrix}, \quad (4)$$

where $x_L(k) = [x(k)^T \; x_K(k)^T]^T \in \mathbb{R}^{n_L(k)}$ and $n_L(k) = n(k) + m(k)$. The closed-loop state-space matrix-valued functions can be expressed as follows:

$$A_L = \hat{A} + \underline{B} J \underline{C}, \qquad B_L = \hat{B} + \underline{B} J \underline{D}_{21}, \quad (5)$$
$$C_L = \hat{C} + \underline{D}_{12} J \underline{C}, \qquad D_L = D_{11} + \underline{D}_{12} J \underline{D}_{21},$$

where the dependence on $\delta(k)$ and k is suppressed, and

$$J = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}, \; \hat{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \; \hat{B} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \; \hat{C} = \begin{bmatrix} C_1 & 0 \end{bmatrix},$$
$$\underline{C} = \begin{bmatrix} 0 & I \\ C_2 & 0 \end{bmatrix}, \; \underline{B} = \begin{bmatrix} 0 & B_2 \\ I & 0 \end{bmatrix}, \; \underline{D}_{12} = \begin{bmatrix} 0 & D_{12} \end{bmatrix}, \; \underline{D}_{21} = \begin{bmatrix} 0 \\ D_{21} \end{bmatrix}.$$

The above block matrices have conformable partitions for the multiplication and addition operations described in (5), and so the dimensions of the 0's and I's can be easily deduced.

A feedback controller K is said to be a γ-admissible synthesis for plant G if this controller ensures that the closed-loop system L_δ is asymptotically stable for all $\delta \in \boldsymbol{\delta}$ and the following performance inequality holds:

$$\sup \{\|z\|_{\ell_2} \mid \|w\|_{\ell_2} \leq 1, \, \|\xi_0\|_2 \leq 1, \, \delta \in \boldsymbol{\delta}\} < \gamma. \quad (6)$$

Finding an admissible synthesis can be viewed as a square ℓ_2 problem, which is treated for LTI systems in [6], LTV systems in [7], [8], and NSLPV systems formulated in an LFT framework in [1].

III. Main Results

A. Analysis Result

For ease of exposition, the analysis result will be given for the open-loop system, which will also be denoted by G for simplicity, assuming that G_δ is asymptotically stable for all $\delta \in \boldsymbol{\delta}$. The state-space equation for this (h, q)-eventually periodic NSLPV system can be expressed as

$$\begin{bmatrix} x(k+1) \\ z(k) \end{bmatrix} = \begin{bmatrix} A(\delta(k), k) & B(\delta(k), k) \\ C(\delta(k), k) & D(\delta(k), k) \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}, \quad (7)$$

where $x(0) = \Gamma \xi_0$ as defined in Equation (2) and its preceding paragraph, $\delta \in \boldsymbol{\delta}$, and the state-space matrix-valued functions A, B, C, and D have affine dependence on the parameter.

Theorem 1: Consider a discrete-time (h, q)-eventually periodic NSLPV system G, defined by the state-space equation

(7), where $x(0) = \Gamma\xi_0$ as described in Equation (2) and its preceding paragraph. Then, G_δ is asymptotically stable for all $\delta \in \boldsymbol{\delta}$ and the performance inequality given in (6) holds if there exist positive scalars e, f_1, f_2, and t and positive definite matrices $X(k) \in \mathbb{S}^{n(k)}$ for $k = 0, 1, \ldots, N+q-1$ and some integer $N \geq h$ satisfying the following LMIs:

$$e + f_1 + f_2 < 2\gamma, \quad \Gamma^T X(0)\Gamma \prec f_1 I, \quad \begin{bmatrix} t & 1 \\ 1 & e \end{bmatrix} \succeq 0, \quad (8)$$

$$H_i(k)^T \begin{bmatrix} X(k+1) & 0 \\ 0 & tI \end{bmatrix} H_i(k) - \begin{bmatrix} X(k) & 0 \\ 0 & f_2 I \end{bmatrix} \prec 0, \quad (9)$$

for $k = 0, 1, \ldots, N+q-1$ and $i = 1, 2, \ldots, r$, where

$$X(N+q) = X(N) \quad \text{and} \quad H_i(k) = \begin{bmatrix} A_i(k) & B_i(k) \\ C_i(k) & D_i(k) \end{bmatrix}.$$

Proof: This result is proved in three steps. As in [8], [1], the first step involves constructing an $(h+1, q)$-eventually periodic NSLPV system, denoted by $\bar{G}$, that is isomorphic to G, where the system $\bar{G}$ has a zero initial condition and two input channels. The first input channel corresponds to the uncertain initial state and has a value of ξ_0 at time $k = 0$ and then becomes irrelevant afterwards; so, the dimensions of the values of this input for $k > 0$ are set to zero. The use of zero dimensions may be viewed as an abuse of notation, but it is allowed in our framework to simplify the presentation of the results. Alternatively, the state-space matrix blocks that are multiplied by this input for $k > 0$ can be set to zero to nullify its effect. The second input channel is the disturbance, which is only relevant at $k > 0$, and so its dimension is set to zero at $k = 0$. The advantage of defining this isomorphic system is that the performance inequality in (6) can now be expressed in terms of the square ℓ_2-induced norm of the system $\bar{G}_{\bar{\delta}}$ for $\bar{\delta} \in \boldsymbol{\delta}$. The second step then entails utilizing a theorem from [8], which is based on results from [6], [7], that relates the square ℓ_2-induced norm of an LTV system to the ℓ_2-induced norm of a scaled system. This theorem will be used in conjunction with an analysis result for polytopic NSLPV systems with zero initial conditions given in [3]. The last step involves carrying out some algebraic manipulations to obtain the linear matrix inequalities in the theorem statement.

The $(h+1, q)$-eventually periodic NSLPV system $\bar{G}$ is defined by the following state-space equation:

$$\begin{bmatrix} \bar{x}(k+1) \\ \bar{z}(k) \end{bmatrix} = \begin{bmatrix} \bar{A}(\bar{\delta}(k), k) & \bar{B}(\bar{\delta}(k), k) \\ \bar{C}(\bar{\delta}(k), k) & \bar{D}(\bar{\delta}(k), k) \end{bmatrix} \begin{bmatrix} \bar{x}(k) \\ \bar{w}(k) \end{bmatrix},$$

where $\bar{x}(0) = 0$, $\bar{w} = (\xi_0, w(0), w(1), \ldots)$, $\bar{x}(k) \in \mathbb{R}^{\bar{n}(k)}$, $\bar{z}(k) \in \mathbb{R}^{\bar{n}_z(k)}$, $\bar{\delta}(k{+}1) = \delta(k)$, and $\bar{Q}(\bar{\delta}(k{+}1), k{+}1) = Q(\delta(k), k)$ for $Q = A, B, C, D$ and $k \in \mathbb{N}_0$. The value of $\bar{\delta}(0)$ is irrelevant but constrained to lie in the set $\operatorname{conv}\{v_1, v_2, \ldots, v_r\}$ so that $\bar{\delta} \in \boldsymbol{\delta}$. All the state-space matrices are set equal to zero at $k = 0$ except for the input matrix which is set equal to Γ, i.e., $\bar{B}(\bar{\delta}(0), 0) = \Gamma$. It is not difficult to see that $\bar{x} = (0, x)$ and $\bar{z} = (0, z)$, and so, $\bar{n} = (\bar{n}(0), n)$ and $\bar{n}_z = (\bar{n}_z(0), n_z)$, where the values of $\bar{n}(0)$ and $\bar{n}_z(0)$ are inconsequential.

The inequality in (6) can be expressed in terms of the square ℓ_2-induced norm of $\bar{G}_{\bar{\delta}}$ for $\bar{\delta} \in \boldsymbol{\delta}$, which is defined as

$$\|\bar{G}_{\bar{\delta}}\|_{\mathrm{sq}} = \sup_{\|\xi_0\|_2 \leq 1,\, \|w\|_{\ell_2} \leq 1} \|\bar{G}_{\bar{\delta}}(\xi_0, w)\|_{\ell_2},$$

where the symbol $\bar{G}_{\bar{\delta}}$ is not merely used here to refer to the LTV system in question but also to denote the corresponding linear operator mapping $\mathbb{R}^s \oplus \ell_2^{n_w}$ to $\ell_2^{\bar{n}_z}$. Then, the inequality in (6) is equivalent to $\|\bar{G}_{\bar{\delta}}\|_{\mathrm{sq}} < \gamma$ for all $\bar{\delta} \in \boldsymbol{\delta}$.

Invoking [8, Theorem 3], for each $\bar{\delta} \in \boldsymbol{\delta}$, $\|\bar{G}_{\bar{\delta}}\|_{\mathrm{sq}} < \gamma$ if and only if there exist positive scalars e, f_1, and f_2 such that $e + f_1 + f_2 < 2\gamma$ and $\|E^{-\frac{1}{2}} \bar{G}_{\bar{\delta}} F^{-\frac{1}{2}}\|_{\ell_2 \to \ell_2} < 1$, where

$$E = \operatorname{diag}(eI_{\bar{n}_z(0)}, eI_{n_z(0)}, eI_{n_z(1)}, \ldots),$$
$$F = \operatorname{diag}(f_1 I_s, f_2 I_{n_w(0)}, f_2 I_{n_w(1)}, \ldots).$$

In addition, applying Theorem 3(i) and Proposition 5(i) from [3] to the $(h+1, q)$-eventually periodic NSLPV system $E^{-\frac{1}{2}} \bar{G} F^{-\frac{1}{2}}$, along with an appropriate permutation, the Schur complement formula, and some algebraic manipulations, we get $\|E^{-\frac{1}{2}} \bar{G}_{\bar{\delta}} F^{-\frac{1}{2}}\|_{\ell_2 \to \ell_2} < 1$ for all $\bar{\delta} \in \boldsymbol{\delta}$ if there exist positive definite matrices $\bar{X}(k) \in \mathbb{S}^{\bar{n}(k)}$ for $k = 0, 1, \ldots, N+q$ and some integer $N \geq h$ such that

$$\bar{H}_i(k)^T \begin{bmatrix} \bar{X}(k{+}1) & 0 \\ 0 & e^{-1} I \end{bmatrix} \bar{H}_i(k) - \begin{bmatrix} \bar{X}(k) & 0 \\ 0 & f(k) I \end{bmatrix} \prec 0, \quad (10)$$

for $k = 0, 1, \ldots, N{+}q$ and $i = 1, 2, \ldots, r$, where $f(0) = f_1$, $f(k) = f_2$ for $k > 0$, $\bar{X}(N+q+1) = \bar{X}(N+1)$, and

$$\bar{H}_i(k) = \begin{bmatrix} \bar{A}_i(k) & \bar{B}_i(k) \\ \bar{C}_i(k) & \bar{D}_i(k) \end{bmatrix}.$$

At $k = 0$, inequality (10) simplifies to $\Gamma^T \bar{X}(1)\Gamma \prec f_1 I$. Notice that the value of the positive definite matrix $\bar{X}(0)$ is inconsequential. Defining the matrix sequence $X(k)$ for $k = 0, 1, \ldots, N+q-1$ such that $X(k) = \bar{X}(k+1)$, inequality (10) can be equivalently expressed as $\Gamma^T X(0)\Gamma \prec f_1 I$ and

$$H_i(k)^T \begin{bmatrix} X(k{+}1) & 0 \\ 0 & e^{-1} I \end{bmatrix} H_i(k) - \begin{bmatrix} X(k) & 0 \\ 0 & f_2 I \end{bmatrix} \prec 0. \quad (11)$$

Introducing a variable $t \geq e^{-1} > 0$, or equivalently,

$$\begin{bmatrix} t & 1 \\ 1 & e \end{bmatrix} \succeq 0,$$

it is not difficult to see that (9), along with the aforementioned condition on t, is equivalent to (11). ■

B. Synthesis Result

Theorem 2: Consider (h, q)-eventually periodic NSLPV plant G defined in (1), where $x(0) = \Gamma\xi_0$ as described in Equation (2) and its preceding paragraph. Then, there exists a γ-admissible (N, q)-eventually periodic NSLPV synthesis K for G for some scalar $\gamma > 0$ and integer $N \geq h$ if there exist positive scalars e, f_1, f_2, p, t, and positive definite matrices $R(k), S(k) \in \mathbb{S}^{n(k)}$ for $k = 0, 1, \ldots, N+q-1$ such that the following LMIs hold:

$$e + f_1 + f_2 < 2\gamma, \qquad \Gamma^T S(0)\Gamma \prec f_1 I, \quad (12)$$

$$N_R(k)^T \left\{ M_i(k) \begin{bmatrix} R(k) & 0 \\ 0 & pI \end{bmatrix} M_i(k)^T - \begin{bmatrix} R(k+1) & 0 \\ 0 & eI \end{bmatrix} \right\} N_R(k) \prec 0, \quad (13)$$

$$N_S(k)^T \left\{ M_i(k)^T \begin{bmatrix} S(k+1) & 0 \\ 0 & tI \end{bmatrix} M_i(k) - \begin{bmatrix} S(k) & 0 \\ 0 & f_2 I \end{bmatrix} \right\} N_S(k) \prec 0, \quad (14)$$

$$\begin{bmatrix} R(k) & I \\ I & S(k) \end{bmatrix} \succeq 0, \quad \begin{bmatrix} p & 1 \\ 1 & f_2 \end{bmatrix} \succeq 0, \quad \begin{bmatrix} t & 1 \\ 1 & e \end{bmatrix} \succeq 0, \quad (15)$$

for $k = 0, 1, \ldots, N+q-1$ and $i = 1, 2, \ldots, r$, where

$$R(N+q) = R(N), \quad S(N+q) = S(N),$$
$$\operatorname{Im} N_R(k) = \operatorname{Ker} \begin{bmatrix} B_2(k)^T & D_{12}(k)^T \end{bmatrix}, \; N_R(k)^T N_R(k) = I,$$
$$\operatorname{Im} N_S(k) = \operatorname{Ker} \begin{bmatrix} C_2(k) & D_{21}(k) \end{bmatrix}, \; N_S(k)^T N_S(k) = I,$$

and $M_i(k) = \begin{bmatrix} A_i(k) & B_{1i}(k) \\ C_{1i}(k) & D_{11i}(k) \end{bmatrix}$.

Proof: For an (N,q)-eventually periodic NSLPV system K with a zero initial state to be a γ-admissible synthesis for plant G, the resulting closed-loop system for each $\delta \in \boldsymbol{\delta}$, i.e., L_δ, defined in (4), must be asymptotically stable and the performance inequality in (6) must hold. Since B_2, C_2, D_{12}, and D_{21} are not parameter dependent, then A_L, B_L, C_L, and D_L, defined in (5), have affine dependence on the parameter, and so Theorem 1 can be applied to the closed-loop system L. Thus, L satisfies the aforementioned stability and performance requirements if there exist positive scalars e, f_1, f_2, and t and positive definite matrices $X_L(k) \in \mathbb{S}^{n_L(k)}$ for $k = 0, 1, \ldots, N+q-1$ satisfying the LMIs in (8) and (9) equivalently expressed for the closed-loop system as

$$e + f_1 + f_2 < 2\gamma, \; \begin{bmatrix} \Gamma \\ 0_{m(0)\times s} \end{bmatrix}^T X_L(0) \begin{bmatrix} \Gamma \\ 0_{m(0)\times s} \end{bmatrix} \prec f_1 I, \quad (16)$$

$$\begin{bmatrix} -X_L(k+1)^{-1} & A_{Li}(k) & B_{Li}(k) & 0 \\ A_{Li}(k)^T & -X_L(k) & 0 & C_{Li}(k)^T \\ B_{Li}(k)^T & 0 & -f_2 I & D_{Li}(k)^T \\ 0 & C_{Li}(k) & D_{Li}(k) & -eI \end{bmatrix} \prec 0, \quad (17)$$

for all $k = 0, 1, \ldots, N+q-1$ and $i = 1, 2, \ldots, r$, where $X_L(N+q) = X_L(N)$.

The rest of the proof follows the same approach used in [14] to obtain the synthesis conditions for LTV systems, which is based on the methods developed in [15], [16]. Namely, inequality (17) can be equivalently expressed as

$$T_i(k) + V(k)^T J_i(k)^T U(k) + U(k)^T J_i(k) V(k) \prec 0, \quad (18)$$

where, using the closed-loop parametrization from (5),

$$T_i(k) = \begin{bmatrix} -X_L(k+1)^{-1} & \hat{A}_i(k) & \hat{B}_i(k) & 0 \\ \hat{A}_i(k)^T & -X_L(k) & 0 & \hat{C}_i(k)^T \\ \hat{B}_i(k)^T & 0 & -f_2 I & D_{11i}(k)^T \\ 0 & \hat{C}_i(k) & D_{11i}(k) & -eI \end{bmatrix},$$

$$U(k) = \begin{bmatrix} \underline{B}(k)^T & 0_{v_1(k)\times n_L(k)} & 0_{v_1(k)\times n_w(k)} & \underline{D}_{12}(k)^T \end{bmatrix},$$
$$V(k) = \begin{bmatrix} 0_{v_2(k)\times n_L(k+1)} & \underline{C}(k) & \underline{D}_{21}(k) & 0_{v_2(k)\times n_z(k)} \end{bmatrix},$$

$$J_i(k) = \begin{bmatrix} A_{Ki}(k) & B_{Ki}(k) \\ C_{Ki}(k) & D_{Ki}(k) \end{bmatrix},$$

with $v_1(k) = m(k+1) + n_u(k)$ and $v_2(k) = m(k) + n_y(k)$. From [15], [16], there exists a matrix $J_i(k)$ satisfying inequality (18) if and only if

$$W_U(k)^T T_i(k) W_U(k) \prec 0, \; W_V(k)^T T_i(k) W_V(k) \prec 0, \quad (19)$$

for $k = 0, 1, \ldots, N+q-1$ and $i = 1, 2, \ldots, r$, where $\operatorname{Im} W_U(k) = \operatorname{Ker} U(k)$, $W_U(k)^T W_U(k) = I$, $\operatorname{Im} W_V(k) = \operatorname{Ker} V(k)$, and $W_V(k)^T W_V(k) = I$.

The matrix $X_L(k)$ and its inverse are partitioned as

$$X_L(k) = \begin{bmatrix} S(k) & X_{12}(k) \\ X_{12}(k)^T & X_{22}(k) \end{bmatrix}, \quad X_L(k)^{-1} = \begin{bmatrix} R(k) & Y_{12}(k) \\ Y_{12}(k)^T & Y_{22}(k) \end{bmatrix}, \quad (20)$$

where $S(k), R(k) \in \mathbb{S}^{n(k)}$ and $X_{22}(k), Y_{22}(k) \in \mathbb{S}^{m(k)}$. From [15], given positive definite matrices $S(k)$ and $R(k)$ in $\mathbb{S}^{n(k)}$ and a positive integer $m(k)$, there exists a positive definite matrix $X_L(k)$ satisfying (20) if and only if

$$\begin{bmatrix} R(k) & I \\ I & S(k) \end{bmatrix} \succeq 0 \text{ and } \operatorname{rank} \begin{bmatrix} R(k) & I \\ I & S(k) \end{bmatrix} \leq n(k) + m(k).$$

Furthermore, it is not difficult to show that the inequalities in (19) are, respectively, equivalent to

$$N_R(k)^T \left\{ M_i(k) \begin{bmatrix} R(k) & 0 \\ 0 & f_2^{-1} I \end{bmatrix} M_i(k)^T - \begin{bmatrix} R(k+1) & 0 \\ 0 & eI \end{bmatrix} \right\} N_R(k) \prec 0,$$

$$N_S(k)^T \left\{ M_i(k)^T \begin{bmatrix} S(k+1) & 0 \\ 0 & e^{-1} I \end{bmatrix} M_i(k) - \begin{bmatrix} S(k) & 0 \\ 0 & f_2 I \end{bmatrix} \right\} N_S(k) \prec 0,$$

which themselves are equivalent to (13) and (14), along with the last two LMIs in (15). Last, notice that the second LMI in (16) is the same as the second LMI in (12). ■

The γ-admissible (N,q)-eventually periodic controller K is constructed from the solutions e, f_2, $R(k)$, and $S(k)$ for $k = 0, 1, \ldots, N+q-1$ as follows. First, form the (h,q)-eventually periodic NSLPV plant G_c, which is also (N,q)-eventually periodic for any $N \geq h$, where the state-space matrix sequences for the LTV system $G_{\mathrm{c}i}$ corresponding to each vertex v_i of the polytope $\mathcal{P}$, for $i = 1, \ldots, r$, are given compactly below for $k = 0, 1, \ldots, N+q-1$:

$$\begin{bmatrix} A_i(k) & f_2^{-\frac{1}{2}} B_{1i}(k) & B_{2i}(k) \\ e^{-\frac{1}{2}} C_{1i}(k) & (ef_2)^{-\frac{1}{2}} D_{11i}(k) & e^{-\frac{1}{2}} D_{12i}(k) \\ C_{2i}(k) & f_2^{-\frac{1}{2}} D_{21i}(k) & 0 \end{bmatrix}.$$

Following the procedure given in [14], use $R(k)$ and $S(k)$ to construct a controller ($A_{Ki}(k)$, $B_{Ki}(k)$, $C_{Ki}(k)$, $D_{Ki}(k)$) for each (N,q)-eventually periodic LTV system $G_{\mathrm{c}i}$. Then, at each instant k, express $\delta(k)$ as $\delta(k) = \sum_{i=1}^r \eta_i v_i$, where $\eta_i \in [0,1]$ and $\sum_{i=1}^r \eta_i = 1$. The scheduled controller at time k would then be ($\sum_{i=1}^r \eta_i A_{Ki}(k)$, $\sum_{i=1}^r \eta_i B_{Ki}(k)$, $\sum_{i=1}^r \eta_i C_{Ki}(k)$, $\sum_{i=1}^r \eta_i D_{Ki}(k)$).

C. Reachability Analysis with Pointwise-Bounded Inputs

Consider the state equation

$$x(k+1) = A(\delta(k), k)x(k) + B(k)w(k), \qquad (21)$$

which defines an asymptotically stable system for all $\delta \in \boldsymbol{\delta}$. Suppose that $x(0) = 0$, n_w is constant for simplicity, and $w(k)$ lies in the set $\Omega_k \subset \mathbb{R}^{n_w}$ for all $k \in \mathbb{N}_0$, where

$$\Omega_k = \{a = (a_1, \dots, a_{n_w}) \,|\, |a_i| \le \bar{w}_i(k),\ i = 1, \dots, n_w\},$$

for some $\bar{w}_i(k)$. The disturbance input w resides in the set

$$\mathcal{W} = \{w = (w(0), w(1), \dots) \,|\, w(k) \in \Omega_k \text{ for all } k \in \mathbb{N}_0\}.$$

Introducing an additional state α, where $\alpha(k+1) = \alpha(k)$, $\alpha(0) = 1$, equation (21) can be equivalently expressed as

$$\begin{aligned}\hat{x}(k+1) &= \hat{A}(\delta(k), \delta_w(k), k)\hat{x}(k) \\ &= \begin{bmatrix} A(\delta(k), k) & B(k)E(k)\delta_w(k) \\ 0 & 1 \end{bmatrix} \hat{x}(k), \qquad (22)\end{aligned}$$

where $\hat{x}(k) = (x(k), \alpha(k))$, $E(k)$ is a scaling matrix defined as $E(k) = \operatorname{diag}(\bar{w}_1(k), \bar{w}_2(k), \dots, \bar{w}_{n_w}(k))$, and δ_w is a static LTV perturbation that takes values in the set

$$\boldsymbol{\delta}_w = \{\delta_w : \mathbb{N}_0 \to \mathbb{R}^{n_w} \,|\, \delta_w(k) \in \mathcal{B}_\infty(0,1) \text{ for all } k \in \mathbb{N}_0\},$$

with $\mathcal{B}_\infty(0,1)$ denoting a norm ball (in the ∞-norm) of radius one centered about the origin, i.e.,

$$\mathcal{B}_\infty(0,1) = \{a \in \mathbb{R}^{n_w} \,|\, \|a\|_\infty \le 1\}.$$

The state equations (21) and (22) define equivalent systems in the sense that the set of achievable state trajectories for system (21) for $x(0) = 0$, $\delta \in \boldsymbol{\delta}$, and $w \in \mathcal{W}$ is the same as that for system (22) for $\hat{x}(0) = (0_{n(0)\times 1}, 1)$ and $(\delta, \delta_w) \in \boldsymbol{\delta} \times \boldsymbol{\delta}_w$. The parameter values $(\delta(k), \delta_w(k))$ lie in $\mathcal{P} \times \mathcal{B}_\infty(0,1)$, where $\mathcal{P}$ is a closed, convex polytope with vertices $v_1, \dots, v_r$. The norm ball $\mathcal{B}_\infty(0,1)$ is a hypercube with $\kappa = 2^{n_w}$ vertices, denoted $\mu_1, \dots, \mu_\kappa$. We define $\mathcal{I} = \{1, \dots, r\} \times \{1, \dots, \kappa\}$ and $\hat{A}_{(i,j)}(k) = \hat{A}(v_i, \mu_j, k)$ for $(i,j) \in \mathcal{I}$ and $k \in \mathbb{N}_0$, where $\hat{A}$ is defined in (22).

System (22) is an unforced NSLPV system with a nonzero initial state. This NSLPV system is not asymptotically stable in general because the state matrix in the new formulation has an additional eigenvalue of one. However, reachability analysis is conducted over finite time intervals, and finite horizon LTV systems of finite horizon length h are viewed as $(h, 1)$-eventually periodic systems where the state-space matrices in the periodic part are all set to zero. Thus, finite horizon LTV systems are asymptotically stable. The state variable α has a constant value for all time instants, i.e., $\alpha(k) = \alpha(0)$ for all $k \in \mathbb{N}_0$, and so the value of $\alpha(0)$ can be regarded as a scaling factor of the parameter δ_w. In addition, the parameter values $\delta_w(k)$ lie in $\mathcal{B}_\infty(0,1)$, which is a hypercube symmetric about the origin. Thus, the initial condition $\alpha(0) = 1$ can be replaced with the inequality $|\alpha(0)| \le 1$ in reachability analysis without adding conservatism since worst-case analysis would still correspond to the case where $\alpha(0) = 1$. We can now state the following result.

Theorem 3: Consider the state equation in (21), which defines an asymptotically stable system for all $\delta \in \boldsymbol{\delta}$, where $x(0) = 0$ and $w \in \mathcal{W}$. Given some $k' \in \mathbb{N}_0 \backslash \{0\}$, the state at time k' lies in the interior of the ellipsoid $\mathcal{E} = \{a \in \mathbb{R}^{n(k')} \,|\, a^T \tilde{P} a \le 1\}$, i.e., $x(k') \in \operatorname{int} \mathcal{E}$, where $\tilde{P} = (1/\gamma^2)P$ is obtained by solving the following convex optimization problem for some $c > 0$:

$$\begin{aligned} \text{minimize} \quad & -\log\det P + c\, g \qquad (23) \\ \text{subject to} \quad & \begin{bmatrix} 0_{n(0)\times 1} \\ 1 \end{bmatrix}^T X(0) \begin{bmatrix} 0_{n(0)\times 1} \\ 1 \end{bmatrix} < g, \\ & \hat{A}_{(i,j)}(k)^T X(k{+}1) \hat{A}_{(i,j)}(k) - X(k) \prec 0 \\ & \qquad \text{for } k = 0, 1, \dots, k'-1, \\ & \hat{A}_{(i,j)}(k')^T X(k'{+}1) \hat{A}_{(i,j)}(k') \\ & \qquad + \operatorname{diag}(P, 0) - X(k') \prec 0 \text{ for } (i,j) \in \mathcal{I}, \\ & P \succ 0,\ X(k) \succ 0 \text{ for } k = 0, 1, \dots, k'+1. \end{aligned}$$

The variables of this optimization problem are $g = \gamma^2 \in \mathbb{R}$, $P \in \mathbb{S}^{n(k')}$, and $X(k) \in \mathbb{S}^{n(k)+1}$ for $k = 0, 1, \dots, k'+1$.

Proof: To start, we construct a $(k'+1, 1)$-eventually periodic system, as first discussed in [17]. The state equation is expressed as in (22) for $k = 0, 1, \dots, k'$, and the output equation is chosen as $z(k) = 0$ for $k \ne k'$ and $z(k') = [P^{1/2}\ 0]\hat{x}(k') = P^{1/2}x(k')$. For $k > k'$, all the state-space matrices are set equal to zero. The initial state is $\hat{x}(0) = (0, \alpha(0))$, with $|\alpha(0)| \le 1$. Then, applying Theorem 1, we conclude that $\|z\|_{\ell_2} < \gamma$ for all $(\delta, \delta_w) \in \boldsymbol{\delta} \times \boldsymbol{\delta}_w$ and $|\alpha(0)| \le 1$ if the analysis conditions in the statement of Theorem 1 hold. Notice that the eventually periodic system in this case is not subjected to a forcing input, i.e., the input and feedthrough matrices are empty for all $k \in \mathbb{N}_0$. Also, based on how the performance output is defined, we have $\|z\|_{\ell_2} = \|z(k')\|_2 = (x(k')^T P x(k'))^{1/2}$; thus, the inequality $\|z\|_{\ell_2} < \gamma$ can be equivalently expressed as $x(k')^T P x(k') < \gamma^2$, i.e., $x(k') \in \operatorname{int} \mathcal{E}$. The rest of this proof centers around proving that the LMIs $(8) - (9)$, along with the constraints on their matrix variables, are equivalent to the constraints of the optimization problem (23).

As the input and feedthrough matrices are empty for all $k \in \mathbb{N}_0$, LMI (9), expressed for this system, simplifies to

$$\hat{A}_{(i,j)}(k)^T \hat{X}(k{+}1) \hat{A}_{(i,j)}(k) + e^{-1} \hat{C}(k)^T \hat{C}(k) - \hat{X}(k) \prec 0,$$

for $(i,j) \in \mathcal{I}$ and $k = 0, 1, \dots, k'+1$, where $\hat{C}(k) = 0$ for $k \ne k'$ and $\hat{C}(k') = [P^{1/2}\ 0]$, and t is set equal to e^{-1}, eliminating the need for the last inequality in (8) (see the end of the proof of Theorem 1). Multiplying both sides of the above inequality by e, defining $X(k) = e\hat{X}(k)$ for all k, and setting all the state-space matrices equal to zero for $k > k'$, we retrieve the corresponding inequalities in the constraints of the optimization problem (23). As for the first two inequalities in (8), they become in this case $e + f_1 < 2\gamma$ and $\Gamma^T \hat{X}(0) \Gamma < f_1$, where $\Gamma = [0_{1\times n(0)}\ 1]^T$; the latter inequality can be equivalently expressed as $\Gamma^T X(0) \Gamma < e f_1$ by multiplying both sides of this inequality by e. From the preceding, it is not difficult to see that for a given value of the sum $e + f_1$, we would like to choose the values of e and f_1

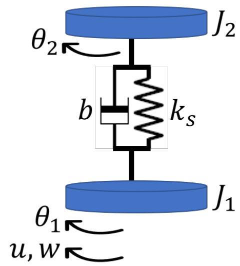

Fig. 1. Two-mass rotational system.

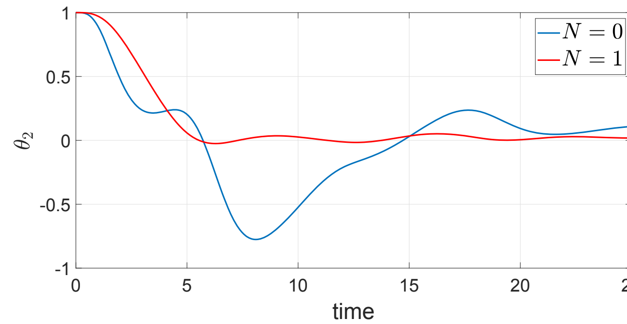

Fig. 2. Time history of the state variable θ_2.

such that the product ef_1 is maximized, which is achieved by choosing $e = f_1$. It follows that the inequalities, $e + f_1 < 2\gamma$ and $\Gamma^T X(0)\Gamma < ef_1$, can be replaced with $\Gamma^T X(0)\Gamma < \gamma^2$.

Concerning the objective function in (23), originally we wanted to minimize the volume of the ellipsoid $\mathcal{E}$, that is, we wanted the cost function to be $\log\det \tilde{P}^{-1}$ or, equivalently, $-\log\det P + n(k')\log\gamma^2$. But this function is not convex in γ^2. As a viable alternative, we opted to minimize the convex function, $-\log\det P + c\gamma^2$, for some positive scalar c which controls the trade-off between the two terms in the cost. ■

Remark 1: In Theorem 3, it is not difficult to modify the convex optimization problem (23) to find an ellipsoid that contains all the possible values of some output of interest, z, at time k'. This ellipsoid constitutes an overapproximated reachable set for the output z at time k'. Suppose that $z(k') = C(k')x(k')$, where $C(k') \in \mathbb{R}^{n_z(k') \times n(k')}$. The required modification in this case can be deduced from the proof and entails replacing the term $\mathrm{diag}(P, 0)$ that appears in the constraints of the optimization problem (23) with the term $\mathrm{diag}(C(k')^T P C(k'), 0)$, where the matrix variable P is in $\mathbb{S}^{n_z(k')}$.

Remark 2: In the optimization problem (23), the first term in the objective function, i.e., $\log\det P^{-1}$, could be replaced, for instance, with $\mathrm{trace}(P^{-1})$ or the spectral norm $\|P^{-1}\|_2$. This change would generally lead to different ellipsoids.

IV. Illustrative Example

The example is based on the ones given in [8], [1] and deals with designing an LPV controller for a two-mass rotational system, shown in Fig. 1, and conducting reachability analysis. The semidefinite programs are solved using Yalmip [18] with Mosek [19]. All computations are carried out in Matlab R2022a on a Lenovo laptop with Intel Core i7-8650U CPU and 16 GB of RAM running Windows 11 Pro.

A. Control Design

The symbols J_i and θ_i in Fig. 1 denote the moment of inertia and the angular displacement of body i, respectively, for $i = 1, 2$. The symbol k_s denotes the spring constant, and b denotes the damping coefficient. The control input u is the torque applied to body 1. The values used for the system parameters are as follows: $J_1 = 1$, $J_2 = 0.1$, and $b = 0.004$. The spring constant k_s is assumed to be not known a priori but available for measurement at each time instant, namely, $k_s(\delta(k)) = 0.025\delta(k) + 0.075$, where the scheduling parameter δ satisfies $|\delta(k)| \leq 1$ for all $k \in \mathbb{N}_0$.

The equations of motion of this system are given in [8]. These equations are discretized using the Euler method with a sampling time $T = 0.1$ and then expressed in state-space form, where the state x is chosen as $x = (\theta_2, \dot{\theta}_2, \theta_1, \dot{\theta}_1)$. As in [8], the disturbance input w is in the form of a torque applied to body 1, designating the control input inaccuracies. The measurement and performance outputs are chosen as $y = \theta_2$ and $z = (\theta_2, u)$, respectively. The system is assumed to be initially at rest with $\theta_1(0) = \theta_2(0)$ having uncertain values, and so the initial state x_0 can be expressed as $x_0 = \Gamma\xi_0$, where $\Gamma = [1\ 0\ 1\ 0]^T$ and $\xi_0 = \theta_2(0)$.

Applying Theorem 2, the semidefinite program, minimize γ subject to the synthesis conditions (12)–(15), is solved for $N = 0, 1, 5, 10, 100$. The corresponding optimal values obtained are $\gamma^* = 32.9080, 6.3845, 6.2155, 6.0868, 6.0295$, with the largest problem taking less than 3 seconds to solve. Thus, just increasing the length of the finite horizon of the controller by one, i.e., designing a $(1, 1)$-eventually periodic NSLPV controller, results in a significant improvement in performance over a stationary LPV controller. Increasing the γ-values for the cases $N = 0$ and $N = 1$ by 5%, we re-solve the corresponding synthesis feasibility problems and use the obtained synthesis solutions to construct a stationary LPV controller and a $(1, 1)$-eventually periodic NSLPV controller.

Simulations are then performed to compare the performances of these two controllers. In these simulations, the continuous-time LPV plant model is used, with the scheduling parameter chosen as $\cos(t)$ for continuous time t. The disturbance input is applied as the output of the zero-order hold with a sampling time of $T = 0.1$ and uniformly distributed pseudorandom values generated using the Matlab function *rand* such that $|w(k)| \leq 0.1$ for all $k \in \mathbb{N}_0$. The significant improvement in performance achieved by using the $(1, 1)$-eventually periodic NSLPV controller is evident in Figures 2 and 3. Comparable outcomes are also observed when applying the results of [1].

B. Reachability Analysis

Consider the $(1, 1)$-eventually periodic closed-loop system obtained using the designed $(1, 1)$-eventually periodic NSLPV controller, and assume that the initial state of this closed-loop system is equal to zero. The formulation used in Section III-C for analysis of finite horizon NSLPV systems

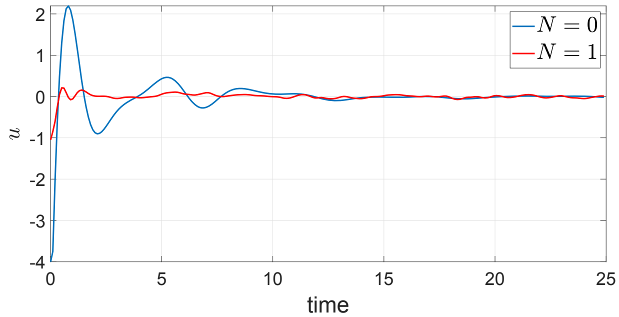

Fig. 3. Time history of the control input u.

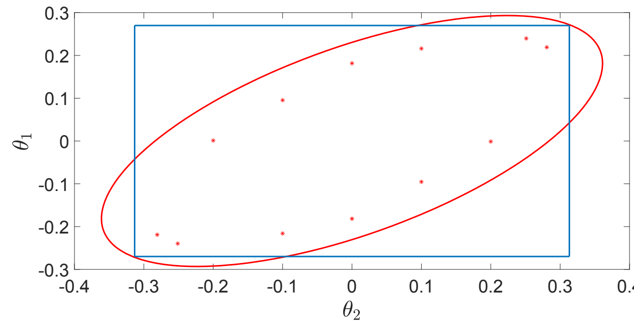

Fig. 4. Bounding sets in which the points $(\theta_2(k'), \theta_1(k'))$ reside.

with pointwise-bounded inputs is adopted here. Expressing the closed-loop state equation as in (22), where in this case $E = \bar{w} = 0.1$ and $|\delta_w(k)| \leq 1$ for all $k \in \mathbb{N}_0$, and choosing the output of interest $z = (\theta_2, \theta_1)$, we apply Theorem 3 and solve a modified version of the optimization problem (23) for $c = 10$ and $k' = 100$ (see Remark 1). The ellipse in which the values of the state variables θ_2 and θ_1 lie at time instant k' is plotted in Fig. 4. The wall-clock time for solving the optimization problem is about 4.5 seconds. By choosing the performance output $z = \theta_i$ for $i = 1, 2$, bounds on the values of θ_i at time k' can be computed by applying Theorem 1. These bounds are depicted in the rectangle shown in Fig. 4. The region bounded by the ellipse and that bounded by the rectangle, as well as the intersection of these two regions, represent overapproximated reachable sets at discrete instant k' for the system under consideration.

Some values of the state variables θ_2 and θ_1 at discrete instant k' are also plotted. These points are evidently symmetric about the origin. Half of these points are obtained by solving nonlinear optimization problems, based on the discrete-time NSLPV closed-loop system, for the disturbance and scheduling parameter sequences that would minimize the objective functions $\theta_2(k')$, $\theta_1(k')$, and $\theta_1(k')$ for specific values of $\theta_2(k')$, namely, $\theta_2(k') = 0, -0.1, 0.1, 0.2$. The rest of the points are also obtained from the solutions of the aforementioned optimization problems by applying each disturbance sequence found, multiplied by -1, along with the associated parameter sequence. These points demonstrate that the bounds computed using the proposed approach are satisfactory. The nonlinear optimization problems are solved using Yalmip, along with the solver fmincon [20].

V. Conclusions

This paper deals with designing controllers and analyzing the performance of eventually periodic NSLPV systems with affine parameter dependence and uncertain initial conditions. Analysis and synthesis results involving convex conditions are provided. A reachability analysis result for systems with pointwise-bounded inputs is also given, and an example demonstrating the usefulness of the approach is presented.

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Appendix

A. Analysis Result for LTV Systems

Computing the square ℓ_2-induced norm of the (isomorphic) closed-loop system for appropriate choices of $\delta \in \boldsymbol{\delta}$

can provide useful lower bounds on the performance index in (6). Given a specific $\delta \in \boldsymbol{\delta}$, an NSLPV system would reduce to an LTV system. Suppose that the nominal model of the NSLPV system is an (h_0, q_0)-eventually periodic LTV system. Then, as long as the parameter trajectory δ is chosen to be eventually periodic, say, (h_δ, q_δ)-eventually periodic, i.e., $\delta(h_\delta + k) = \delta(h_\delta + q_\delta + k)$ for all $k \in \mathbb{N}_0$, the resulting LTV system will be (h, q)-eventually periodic, where $h = \max(h_0, h_\delta)$ and q is the least common multiple of q_0 and q_δ. An LTV counterpart to Theorem 1 is provided below, which gives necessary and sufficient conditions for asymptotic stability and performance (as defined in (6) sans the condition on δ).

Theorem 4 (LTV case): Consider an (h, q)-eventually periodic discrete-time LTV system defined by the state-space equations $x(k+1) = A(k)x(k) + B(k)w(k)$ and $z(k) = C(k)x(k) + D(k)w(k)$, where $x(0) = \Gamma\xi_0$ as described in Equation (2) and its preceding paragraph, and $x(k) \in \mathbb{R}^{n(k)}$. Then, this system is asymptotically stable and the performance inequality given in (6) (sans the condition on δ) holds if and only if there exist positive scalars e, f_1, f_2, and t and positive definite matrices $X(k) \in \mathbb{S}^{n(k)}$ for $k = 0, 1, \ldots, h + q - 1$ satisfying the following LMIs:

$$e + f_1 + f_2 < 2\gamma, \quad \Gamma^T X(0)\Gamma \prec f_1 I, \quad \begin{bmatrix} t & 1 \\ 1 & e \end{bmatrix} \succeq 0, \quad (24)$$

$$H(k)^T \begin{bmatrix} X(k+1) & 0 \\ 0 & tI \end{bmatrix} H(k) - \begin{bmatrix} X(k) & 0 \\ 0 & f_2 I \end{bmatrix} \prec 0, \quad (25)$$

for $k = 0, 1, \ldots, h + q - 1$, where $X(h + q) = X(h)$ and

$$H(k) = \begin{bmatrix} A(k) & B(k) \\ C(k) & D(k) \end{bmatrix}.$$

The proof of this result is deducible from that of Theorem 1 and so is omitted. Given any (h_δ, q_δ)-eventually periodic $\delta \in \boldsymbol{\delta}$ for some integers $h_\delta \geq 0$ and $q_\delta \geq 1$, the preceding result can be applied to obtain the aforementioned lower bound, which is the optimal value of the semidefinite programming problem, minimize γ subject to (24) and (25).

B. Reachability Analysis for LFT Systems

A version of Theorem 3 for NSLPV systems with rational dependence on the parameters, formulated in an LFT framework [1], is given next. The system equations in this case are

$$\begin{aligned} \begin{bmatrix} x(k+1) \\ \varphi(k) \end{bmatrix} &= \begin{bmatrix} A_{ss}(k) & A_{sp}(k) & B_s(k) \\ A_{ps}(k) & A_{pp}(k) & B_p(k) \end{bmatrix} \begin{bmatrix} x(k) \\ \vartheta(k) \\ w(k) \end{bmatrix}, \\ \vartheta(k) &= \operatorname{diag}(\delta_1(k) I_{r_1(k)}, \ldots, \delta_{n_p}(k) I_{r_{n_p}(k)})\varphi(k), \\ &= \Delta(k)\varphi(k) \end{aligned} \quad (26)$$

where $x(k) \in \mathbb{R}^{n(k)}$, $\vartheta(k), \varphi(k) \in \mathbb{R}^{n_\vartheta(k)}$, $\sum_{i=1}^{n_p} r_i(k) = n_\vartheta(k)$, $x(0) = 0$, $\delta = (\delta_1, \ldots, \delta_{n_p}) \in \boldsymbol{\delta}$, with

$$\boldsymbol{\delta} = \{\delta : \mathbb{N}_0 \to \mathbb{R}^{n_p} \,|\, |\delta_i(k)| \leq 1 \; \forall\, i = 1, \ldots, n_p,\, k \in \mathbb{N}_0\}.$$

The LFT system defined in (26) is assumed to be asymptotically stable for all $\delta \in \boldsymbol{\delta}$. The input matrix sequence of the NSLPV system in this case may depend on the parameter δ, unlike that in the state equation (21). Assuming a pointwise-bounded input as described in Section III-C, the equations in (26) can be equivalently expressed as

$$\begin{aligned} \begin{bmatrix} \hat{x}(k+1) \\ \hat{\varphi}(k) \end{bmatrix} &= \begin{bmatrix} \hat{A}_{ss}(k) & \hat{A}_{sp}(k) \\ \hat{A}_{ps}(k) & \hat{A}_{pp}(k) \end{bmatrix} \begin{bmatrix} \hat{x}(k) \\ \hat{\vartheta}(k) \end{bmatrix}, \\ \hat{\vartheta}(k) &= \operatorname{diag}(\Delta(k), \delta_{w,1}(k), \ldots, \delta_{w,n_w}(k))\hat{\varphi}(k), \end{aligned} \quad (27)$$

where n_w is assumed to be constant for simplicity, $\hat{x}$, $\delta_w = (\delta_{w,1}, \ldots, \delta_{w,n_w})$, and E (which appears below) are defined in Section III-C, and

$$\hat{A}_{ss}(k) = \begin{bmatrix} A_{ss}(k) & 0_{n(k)\times 1} \\ 0_{1\times n(k)} & 1 \end{bmatrix},$$

$$\hat{A}_{sp}(k) = \begin{bmatrix} A_{sp}(k) & B_s(k)E(k) \\ 0_{1\times n_\vartheta(k)} & 0_{1\times n_w} \end{bmatrix},$$

$$\hat{A}_{ps}(k) = \begin{bmatrix} A_{ps}(k) & 0_{n_\vartheta(k)\times 1} \\ 0_{n_w\times n(k)} & \mathbf{1}_{n_w} \end{bmatrix},$$

$$\hat{A}_{pp}(k) = \begin{bmatrix} A_{pp}(k) & B_p(k)E(k) \\ 0_{n_w\times n_\vartheta(k)} & 0_{n_w\times n_w} \end{bmatrix}.$$

A counterpart of Theorem 3 for NSLPV systems formulated in an LFT framework can now be given based on the analysis result of [1].

Theorem 5 (LFT case): Consider the system equations in (26), which define an asymptotically stable system for all $\delta \in \boldsymbol{\delta}$, where $x(0) = 0$ and $w \in \mathcal{W}$ (as defined in Section III-C). Given some $k' \in \mathbb{N}_0 \backslash \{0\}$, the state at time k' lies in the interior of the ellipsoid $\mathcal{E} = \{a \in \mathbb{R}^{n(k')} \,|\, a^T \tilde{P} a \leq 1\}$, i.e., $x(k') \in \operatorname{int} \mathcal{E}$, where $\tilde{P} = (1/\gamma^2)P$ is obtained by solving the following convex optimization problem for some $c > 0$:

minimize $\quad -\log\det P + c\, g$

subject to

$$\begin{bmatrix} 0_{n(0)\times 1} \\ 1 \end{bmatrix}^T X_s(0) \begin{bmatrix} 0_{n(0)\times 1} \\ 1 \end{bmatrix} < g,$$

$$\hat{M}(k)^T \begin{bmatrix} X_s(k+1) & 0 \\ 0 & X_p(k) \end{bmatrix} \hat{M}(k) - \begin{bmatrix} X_s(k) & 0 \\ 0 & X_p(k) \end{bmatrix} \prec 0,$$

for $k = 0, 1, \ldots, k' - 1$, where $\hat{M}(k) = \begin{bmatrix} \hat{A}_{ss}(k) & \hat{A}_{sp}(k) \\ \hat{A}_{ps}(k) & \hat{A}_{pp}(k) \end{bmatrix}$,

$$\hat{M}(k')^T \begin{bmatrix} X_s(k'+1) & 0 \\ 0 & X_p(k') \end{bmatrix} \hat{M}(k') + \begin{bmatrix} P & 0 \\ 0 & 0_{v\times v} \end{bmatrix} - \begin{bmatrix} X_s(k') & 0 \\ 0 & X_p(k') \end{bmatrix} \prec 0, \; v = n_\vartheta(k') + n_w + 1,$$

$$P \succ 0, \; X_s(k) \succ 0, \text{ for } k = 0, 1, \ldots, k' + 1,$$

$$\begin{aligned} X_p(k) = \operatorname{diag}(&X_p^{(1)}(k), \ldots, X_p^{(n_p)}(k), \\ &X_{p,w}^{(1)}(k), \ldots, X_{p,w}^{(n_w)}(k)) \succ 0, \; X_p^{(i)}(k) \in \mathbb{S}^{r_i(k)}, \end{aligned}$$

$$X_{p,w}^{(j)}(k) \in \mathbb{R}, \; i = 1, \ldots, n_p, \; j = 1, \ldots, n_w, \; k = 1, \ldots, k'.$$

The variables of this optimization problem are $g = \gamma^2 \in \mathbb{R}$, $P \in \mathbb{S}^{n(k')}$, $X_s(k) \in \mathbb{S}^{n(k)+1}$ for $k = 0, 1, \ldots, k' + 1$, and $X_p(k)$ for $k = 0, 1, \ldots, k'$.

The proof uses [1, Theorem 1] and follows a similar argument to that in the proof of Theorem 3.