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Model-Free Control for Constrained Mechanical Systems

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***Abstract*— This paper presents a model-free motion control technique for constrained mechanical systems. First, we exploit the inherent positive-definiteness of an unconstrained system's dynamics matrix to achieve local input-to-state stability (LISS) with respect to state estimation errors. Then, we impose holonomic constraints on our system and show that we can preserve the (LISS) property if the derivatives of the constraint equations are known. This suggests applications to robotics (particularly for robot manipulators) where the plant's dynamics may change abruptly or where interactions with the environment are difficult to model.**

I. INTRODUCTION

There is a large and longstanding body of literature pertaining to the control of mechanical systems. One of the most widespread control methods for such systems is feedback linearization (FBL), also referred to as computed-torque control within the robotics community. This method has seen broad application since the 1980s [1], which is easy to understand, given that it allows one to apply well-understood linear control design techniques to otherwise highly coupled, nonlinear systems.

An unfortunate drawback of feedback linearization is its high sensitivity to modeling errors. Many extensions to FBL exist to mitigate this issue [2]. Most either incorporate other classical nonlinear control design techniques [1] or they add some adaptive component [3]. Typically, these adaptive methods rely on partial knowledge of the underlying system or an assumption regarding its structure.

One can also take a more generalized approach to the control task via machine learning techniques. Chiefly, reinforcement learning (RL) has been used to design highly capable nonlinear controllers for systems which are otherwise difficult to even model [4]. Still, many such methods assume some knowledge of the underlying system [1] [5]. For example, researchers often train neural networks to capture unknown dynamics [6] [7]. However, the controllers these methods produce can have difficulty adapting to fast changes in their environment.

An alternative approach, referred to as model-free control [8], attempts to avoid learning any global model or policy [5]. Also known as "intelligent" PID control [9], this technique repeatedly learns and then discards a local model of the system [10], only using these models to determine an instantaneous control input [11]. We propose such an approach in this work. First we establish the stability of a model-free controller for a multidimensional unconstrained system. Then, we extend our results to constrained systems in which the constraints are described by holonomic functions.

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II. NOTATION

$\mathbb{R}$ denotes the set of real numbers while $\mathbb{R}_{>0}$ (resp. $\mathbb{R}_{\geq 0}$) denotes the set of positive (resp. nonnegative) real numbers. For $f : \mathbb{R}^n \to \mathbb{R}^m$ and for $x_0 \in \mathbb{R}^n$, $\frac{\partial f}{\partial x}|_{x_0}$ denotes the partial derivative of f, with respect to x evaluated at x_0. If the point of evaluation is clear we just write $\frac{\partial f}{\partial x}$.

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is uniformly bounded from above (resp. below) if there exists a constant $c \in \mathbb{R}$ such that $f_i(x) \leq c$ (resp. $f_i(x) \geq c$) for each $i \in \{1, 2, \ldots, m\}$ and all $x \in \mathbb{R}^n$. We say that a function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if $f(0) = 0$ and f is continuous and strictly increasing. We say that a function $g : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if for fixed s, $g(r, s)$ is a class $\mathcal{K}$ function with respect to r and for fixed r, $g(r, s)$ is a strictly decreasing function of s and $g(r, s) \to 0$ as $s \to 0$.

Given a set $\mathcal{D} \subset \mathbb{R}^n$, we denote the closure of $\mathcal{D}$ by $\overline{\mathcal{D}}$. We denote the orthogonal complement of $\mathcal{D}$, given by $\{y \in \mathbb{R}^n : x^T y = 0 \ \forall \ x \in \mathcal{D}\}$, with $\mathcal{D}^\perp$.

I^n denotes the $n \times n$ identity matrix. If its dimension is clear from context we write I. $\mathrm{Im}(A)$ denotes the image of $A \in \mathbb{R}^{m \times n}$, while $\mathrm{Ker}(A)$ denotes its kernel and $\mathrm{rank}(A)$ denotes its rank. For a symmetric matrix A, the expression $\lambda_{\max}(A)$ (resp. $\lambda_{\min}(A)$) denotes its maximum (resp. minimum) eigenvalue.

For $A \in \mathbb{R}^{m \times n}$ we use $\mathrm{P}_A : \mathbb{R}^n \to \mathbb{R}^n$ to denote the orthogonal projection operator mapping $\mathbb{R}^n$ onto $\mathrm{Im}(A)$. If A has full column rank, the identity $\mathrm{P}_A = A(A^T A)^{-1} A^T$ holds. We use $\mathrm{P}_A^\perp$ to denote the orthogonal projection onto the complement of $\mathrm{Im}(A)$ such that $\mathrm{P}_A + \mathrm{P}_A^\perp = I$.

III. SYSTEM MODEL

We restrict our focus to Euler-Lagrange (EL) systems which are defined by:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + V(q) = u, \tag{1}$$

where $q \in \mathbb{R}^n$ and $\dot{q} \in \mathbb{R}^n$ denote generalized positions and velocities respectively, $M : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ denotes the dynamics matrix, $C : \mathbb{R}^{2n} \to \mathbb{R}^{n \times n}$ denotes the Coriolis and centrifugal force matrix, $V : \mathbb{R}^n \to \mathbb{R}^n$ denotes the force from gravity, and $u \in \mathbb{R}^n$ is a vector of control inputs.

Further still, we only consider EL systems for which M is uniformly bounded from above and below, i.e:

$$\underline{\sigma} I \preceq M(q) \preceq \overline{\sigma} I \quad \forall \ q \in \mathbb{R}^n, \tag{2}$$

and $\underline{\sigma}, \overline{\sigma} > 0$. This is not an especially restrictive assumption, as it covers a large subclass of serial robot manipulators [12].

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We assume that we have access to instantaneous position measurements, but that other components of the state are unknown. If we place (1) in an affine form and define our output to be the position, q, our system satisfies:

$$\ddot{q} = \alpha(q, \dot{q}) + \beta(q)u, \quad h(q, \dot{q}) = q, \tag{3}$$

where $\alpha = -M^{-1}(C\dot{q} + V)$ and $\beta = M^{-1}$.

IV. MODEL-FREE CONTROLLER DESIGN

A. Controller Derivation

Our goal is to design a dynamic controller that stabilizes (3). For now, assume that we have full knowledge of the state such that we can select a linear control law:

$$k(q, \dot{q}) = -k_p q - k_d \dot{q}, \quad k_p, k_d > 0, \tag{4}$$

which stabilizes the double integrator $\ddot{q} = u$. For brevity, we'll refer to this control law as k, dropping its arguments.

Define the error term $e_{\ddot{q}} = \ddot{q} - k$ and dynamic controller $\dot{u} = -\gamma e_{\ddot{q}}$ for $\gamma > 0$. Substituting these into (3) gives us:

$$\begin{aligned} \ddot{q} &= k + e_{\ddot{q}} \\ \dot{u} &= -\gamma e_{\ddot{q}}. \end{aligned} \tag{5}$$

Note that $\beta(q)$ is nonsingular for all $q \in \mathbb{R}^n$, so we can define the continuous function $u^* = \beta^{-1}(k - \alpha)$ satisfying:

$$k = \alpha + \beta u^*. \tag{6}$$

With u^*, define an input error term $e_u = u - u^*$ for which $e_{\ddot{q}} = \beta e_u$ holds. Then (5) is equivalent to:

$$\begin{aligned} \ddot{q} &= k + \beta e_u \\ \dot{e}_u &= -\gamma \beta e_u - \dot{u}^*. \end{aligned} \tag{7}$$

We will use (7) to show stability for our proposed controller.

B. Stability Analysis

Definition 4.1: [13] Given a dynamical system $\dot{x} = f(x)$, the origin is said be *stable* if for each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that every solution $x(t)$ satisfies:

$$\|x(0)\| < \delta(\varepsilon) \implies \|x(t)\| < \varepsilon, \ \forall t \geq 0. \tag{8}$$

The origin is *locally asymptotically stable* if it is stable and δ can be chosen such that:

$$\|x(0)\| < \delta \implies \lim_{t \to \infty} x(t) = 0. \tag{9}$$

Given a controlled dynamical system $\dot{x} = f(x, u(x))$, where u is a parameterized control policy, we say that the origin is *semi-globally asymptotically stable* if for any compact set $\mathcal{D}$ that contains the origin, it is always possible to choose controller parameters that render the origin asymptotically stable for any initial condition $x(0) \in \mathcal{D}$.

In order to make our analysis more concise, define:

$$z \stackrel{\text{def}}{=} \begin{bmatrix} q \\ \dot{q} \end{bmatrix}, \quad z_1 \stackrel{\text{def}}{=} q, \quad z_2 \stackrel{\text{def}}{=} \dot{q}, \tag{10}$$

such that $\dot{z}_2 = -k_p z_1 - k_d z_2 + e_{\ddot{q}}$ holds. In what follows, we will switch between z and $(q, \dot{q})$ where convenient.

Given that $\ddot{q} = k$ is a stable linear system, there exists a quadratic Lyapunov function $V_z(z) = z^T P z$ such that $\dot{V}_z(z) = -z^T Q z$ where P and Q are constant positive-definite matrices [14]. We can use P to define a candidate Lyapunov function for (7):

$$V(z, e_u) = z^T P z + \frac{1}{2} e_u^T e_u, \tag{11}$$

with derivative:

$$\begin{aligned} \dot{V}(z, e_u) &= \begin{bmatrix} z_2 \\ k + \beta e_u \end{bmatrix}^T P z + z^T P \begin{bmatrix} z_2 \\ k + \beta e_u \end{bmatrix} + e_u^T \dot{e}_u \\ &= -z^T Q z + 2 z^T P \begin{bmatrix} 0 \\ \beta e_u \end{bmatrix} - e_u^T \beta^{-1} \beta \dot{u}^* - \gamma e_u^T \beta e_u. \end{aligned} \tag{12}$$

Differentiating (6) yields:

$$-k_p \dot{q} - k_d \ddot{q} = \frac{\partial \alpha}{\partial q} \dot{q} + \frac{\partial \alpha}{\partial \dot{q}} \ddot{q} + \dot{\beta} u^* + \beta \dot{u}^*, \tag{13}$$

which we can rearrange and combine with (3) to get:

$$\begin{aligned} \beta \dot{u}^* &= -k_p \dot{q} - k_d \ddot{q} - \frac{\partial \alpha}{\partial q} \dot{q} - \frac{\partial \alpha}{\partial \dot{q}} \ddot{q} - \dot{\beta} u^* \\ &= -k_p \dot{q} - k_d \left(-k_p q - k_d \dot{q} + \beta e_u \right) \\ &\quad - \frac{\partial \alpha}{\partial q} \dot{q} - \frac{\partial \alpha}{\partial \dot{q}} \left(-k_p q - k_d \dot{q} + \beta e_u \right) - \tilde{\beta} \dot{q}. \end{aligned} \tag{14}$$

Observe that in the preceding line, we replace $\dot{\beta} u^*$ with $\tilde{\beta} \dot{q}$, highlighting the linear dependence of $\dot{\beta}$ on $\dot{q}$.

Every term in (14) is the product of a continuous function and a term q, $\dot{q}$, or e_u. As such, the magnitude of each term can be bounded by a linear function of the state in (7) if we restrict that state to a compact set.

Theorem 4.1: The origin of the system in (7) is semi-globally asymptotically stable.

Proof: Define $V(z, e_u)$ as in (11). For any compact set $\mathcal{D} \subset \mathbb{R}^{3n}$ containing the origin, it is possible to define a bounded and open sublevel set of V:

$$\mathcal{D}_c = \left\{ (z, e_u) \in \mathbb{R}^{3n} : V(z, e_u) < c \right\}, \tag{15}$$

where $c > 0$, such that $\mathcal{D} \subset \mathcal{D}_c$.

We can see from (14) that z and e_u enter $\beta \dot{u}^*$ linearly. Thus, there must exist a constant $\tilde{d} > 0$ for which:

$$\|\beta \dot{u}^*\| \leq \tilde{d} \left(\|z\| + \|e_u\| \right), \tag{16}$$

holds over $\overline{\mathcal{D}}$. Defining $d = \max \left\{ \tilde{d}, \|\beta\|, \|\beta^{-1}\|, \|2P\| \right\}$, (2), (12), and (16) imply the following bound for $\dot{V}(z, e_u)$:

$$\begin{aligned} \dot{V}(z, e_u) \leq &-\lambda_{\min}(Q) \|z\|^2 + 2d^2 \|z\| \|e_u\| \\ &+ \left(d^2 - \frac{\gamma}{\sigma} \right) \|e_u\|^2. \end{aligned} \tag{17}$$

It's clear from the quadratic form in (17) that $\dot{V}$ becomes negative-definite for large enough $\gamma > 0$. For such γ, $\mathcal{D}_c$ is rendered invariant and trajectories with initial conditions $x(0) \in \mathcal{D}$ approach the origin asymptotically. ■

C. Practical Considerations

In the previous section, we assumed full knowledge of our system's state. However, our output equations only provide position information. This means that in order to implement the controller in (5), we must estimate the first two derivatives of q. Because derivative estimates are highly sensitive to noise, we need to show that the stability of our controller is robust to state estimation errors.

Definition 4.2: [13] A dynamical system $\dot{x} = f(x, u)$ is *input-to-state stable* (ISS) if for every initial condition $x(0) \in \mathbb{R}^n$ and every bounded, continuous input $u(t) \in \mathbb{R}^m$ for $t \geq 0$, the solution $x(t)$ for $t \geq 0$ exists and satisfies:

$$\|x(t)\| \leq \eta(\|x(0)\|, t) + \xi\left(\sup_{0\leq\tau\leq t} \|u(\tau)\|\right), \quad t \geq 0, \tag{18}$$

where $\eta(s, t)$ is a class $\mathcal{KL}$ function and $\xi(s)$ is of class $\mathcal{K}$.

A dynamical system $\dot{x} = f(x, u)$ is *locally input-to-state stable* (LISS) if there exists a constant $r > 0$ such that for all initial conditions $x(0) \in \mathbb{R}^n$ satisfying $\|x(0)\| \leq r$ and all admissible inputs $u(t)$ satisfying $\sup_{t\geq 0} \|u(t)\| \leq r$, (18) holds for some $\eta(s, t) \in \mathcal{KL}$ and $\xi(s) \in \mathcal{K}$.

Our model-free controller relies on the assumption that our system can be made to evolve on a compact set. Noise or large estimation errors have the potential to drive our system outside any bounded set. Thus, the notion of LISS is particularly important.

Theorem 4.2: [13] A system $\dot{x} = f(x, u)$ is ISS if and only if there exists a continuously differentiable, radially unbounded, and positive-definite function $V : \mathbb{R}^n \to \mathbb{R}$, along with continuous functions $\xi_1, \xi_2 \in \mathcal{K}$, such that for every $u \in \mathbb{R}^m$:

$$\|x\| \geq \xi_2(\|u\|) \implies \dot{V}(x, u) \leq -\xi_1(\|x\|). \tag{19}$$

Assume that we have an estimator for $\dot{q}$ and $\ddot{q}$ which takes q as its input and produces derivative estimates $\hat{\dot{q}}$ and $\hat{\ddot{q}}$ with bounded error. We can use these estimates to construct an estimate of the acceleration error:

$$\hat{e}_{\ddot{q}} = \hat{\ddot{q}} + k = e_{\ddot{q}} + \epsilon, \tag{20}$$

where $\epsilon = \hat{\ddot{q}} - \ddot{q} + k_d(\hat{\dot{q}} - \dot{q})$. The expression for our dynamic controller then becomes $\dot{u} = -\gamma\hat{e}_{\ddot{q}}$, yielding:

$$\begin{aligned} \ddot{q} &= k + \beta e_u \\ \dot{e}_u &= -\gamma\beta e_u - \gamma\epsilon - \dot{u}^*. \end{aligned} \tag{21}$$

Theorem 4.3: There exists $\gamma > 0$ rendering the system in (21) LISS with respect to the estimation error ϵ.

Proof: Define a Lyapunov function $V(z, e_u)$ as in (11) and with it, define an open sublevel set $\mathcal{D}_c$ containing the origin. Using an argument analogous to that in the poof of Theorem 4.1, we can derive an upper bound:

$$\begin{aligned} \dot{V}(z, e_u, \epsilon) \leq &-\lambda_{\min}(Q)\|z\|^2 + d^2\|z\|\|e_u\| \\ &+ \left(d^2 - \frac{\gamma}{\bar{\sigma}}\right)\|e_u\|^2 + \gamma\|e_u\|\|\epsilon\|, \end{aligned} \tag{22}$$

which holds for all $(z, e_u) \in \overline{\mathcal{D}}_c$.

We can always choose γ large enough that the expression:

$$-\lambda_{\min}(Q)\|z\|^2 + d^2\|z\|\|e_u\| + \left(d^2 - \frac{\gamma}{\bar{\sigma}}\right)\|e_u\|^2, \tag{23}$$

becomes negative-definite. Since (23) is quadratic with respect to (z, e_u), there exists $a > 0$ such that we can upper bound (23) by $-a\left(\|z\|^2 + \|e_u\|^2\right)$, letting us relax (22) to:

$$\begin{aligned} \dot{V}(z, e_u, \epsilon) &\leq -a\left(\|z\|^2 + \|e_u\|^2\right) + \gamma\bar{\sigma}^{-1}\|e_u\|\|\epsilon\| \\ &\leq -a\|(z, e_u)\|^2 + \gamma\bar{\sigma}^{-1}\|(z, e_u)\|\|\epsilon\|. \end{aligned} \tag{24}$$

Next, pick $0 < \theta < 1$ and substitute the identity:

$$\|(z, e_u)\|^2 = (1 - \theta)\|(z, e_u)\|^2 + \theta\|(z, e_u)\|^2, \tag{25}$$

into (24) to yield:

$$\begin{aligned} \dot{V}(z, e_u, \epsilon) \leq &-a(1 - \theta)\|(z, e_u)\|^2 - a\theta\|(z, e_u)\|^2 \\ &+ \gamma\bar{\sigma}^{-1}\|\epsilon\|\left(\|(z, e_u)\|\right). \end{aligned} \tag{26}$$

It follows that $\dot{V}$ is negative-definite for:

$$\|(z, e_u)\| \geq \frac{\gamma}{a\theta\bar{\sigma}}\|\epsilon\|. \tag{27}$$

Choose any $r > 0$ for which $\mathcal{B}_r \subset \mathcal{D}_c$ and let $r_\epsilon = \frac{a\theta\bar{\sigma}}{\gamma}r$. If $\sup_{t\geq 0} \|\epsilon(t)\| \leq r_\epsilon$ then $\dot{V}$ is negative-definite outside $\mathcal{B}_r$, including the boundary of $\mathcal{D}_c$. Thus, $\mathcal{D}_c$ is invariant and (27) indicates that our system is ISS within $\mathcal{D}_c$. ■

It's worth noting that in the previous proof, there was no restriction placed on the size of $\mathcal{D}_c$. This implies that for any bounded set $\mathcal{D}$, we can always find a sublevel set $\mathcal{D}_c$ containing $\mathcal{D}$ which can be rendered invariant for large enough γ and small enough ϵ. However, we cannot recover the ISS property, because for any fixed γ there is always an error threshold beyond which $\mathcal{D}_c$ may lose its invariance.

V. ADDING CONSTRAINTS

A. Model Derivation

Now, let's impose a set of m holonomic constraints on (1) by requiring that $\phi(q) = 0$ hold for some smooth function $\phi : \mathbb{R}^n \to \mathbb{R}^m$ that evaluates to zero at the origin. Assume also that $\text{rank}(\frac{\partial\phi}{\partial q}) = m$ holds across $\mathbb{R}^n$. Then the constrained system satisfies:

$$\begin{aligned} M(q)\ddot{q} + C(q, \dot{q})\dot{q} + V(q) &= \frac{\partial\phi}{\partial q}^T\lambda + u \\ \phi(q) &= 0, \end{aligned} \tag{28}$$

where $\lambda \in \mathbb{R}^m$ represents a vector of constraint forces.

Clearly, λ depends on u. We can exploit this to make the constraints implicit in our model. First left-multiply $\ddot{q}$ by $\frac{\partial\phi}{\partial q}$:

$$\frac{\partial\phi}{\partial q}\ddot{q} = \frac{\partial\phi}{\partial q}M^{-1}\left(\frac{\partial\phi}{\partial q}^T\lambda - C\dot{q} - V + u\right), \tag{29}$$

then rearrange terms to yield an expression for λ:

$$\lambda = \left(\frac{\partial\phi}{\partial q}M^{-1}\frac{\partial\phi}{\partial q}^T\right)^{-1}\frac{\partial\phi}{\partial q}\left(\ddot{q} + M^{-1}\left(C\dot{q} + V - u\right)\right), \tag{30}$$

where $\text{rank}\left(\frac{\partial\phi}{\partial q}\right) = m$ implies that $\left(\frac{\partial\phi}{\partial q}M^{-1}\frac{\partial\phi}{\partial q}^T\right)^{-1}$ exists.

Differentiating $\phi(q) = 0$ gives us:

$$\frac{\partial \phi}{\partial q}\dot{q} = 0 \quad \text{and} \quad \psi(q, \dot{q}) + \frac{\partial \phi}{\partial q}\ddot{q} = 0, \tag{31}$$

where $\psi_i(q, \dot{q}) = \dot{q}^T \frac{\partial^2 \phi_i}{\partial q^2}\dot{q}$. The second equation in (31) lets us express λ as a function of q, $\dot{q}$, and e_u. Substitute (30) and (31) back into (28) to conclude:

$$\ddot{q} = \tilde{M}\left(-C\dot{q} - V + u\right) + \tilde{\psi}, \tag{32}$$

where:

$$\tilde{M} = \left(M^{-1} - M^{-1}\frac{\partial \phi}{\partial q}^T \left(\frac{\partial \phi}{\partial q} M^{-1} \frac{\partial \phi}{\partial q}^T\right)^{-1} \frac{\partial \phi}{\partial q} M^{-1}\right), \tag{33}$$

and:

$$\tilde{\psi} = -M^{-1}\frac{\partial \phi}{\partial q}^T \left(\frac{\partial \phi}{\partial q} M^{-1} \frac{\partial \phi}{\partial q}^T\right)^{-1} \psi, \tag{34}$$

Because M is a symmetric positive-definite matrix, it has a unique positive-definite square root such that:

$$\tilde{M} = M^{-1/2}\, \mathrm{P}_v^{\perp}\, M^{-1/2}, \tag{35}$$

for $v = M^{-1/2}\frac{\partial \phi}{\partial q}^T$. The rank of $\mathrm{P}_v^{\perp}$ is $(n-m)$ and its eigenvectors x satisfy:

$$\lambda x = \mathrm{P}_v^{\perp}\, x = \left(\mathrm{P}_v^{\perp}\right)^2 x = \lambda^2 x, \tag{36}$$

so every eigenvalue of $\tilde{M}$ equals either zero or one. Thus, $\tilde{M}$ itself must be positive semi-definite.

Every term on the right-hand side of (32) is a smooth function of the state. We can combine the terms that don't include u into a single term and rewrite (32) as:

$$\ddot{q} = \alpha + \beta u, \tag{37}$$

where:

$$\alpha = \tilde{M}\left(-C\dot{q} - V\right) + \tilde{\psi} \text{ and } \beta = \tilde{M}. \tag{38}$$

B. Intermediate Results

Theorem 5.1: (Taylor's Theorem in Several Variables)[15] Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is smooth on an open convex set $\mathcal{D}$. If $x \in \mathcal{D}$ and $x + h \in \mathcal{D}$, then:

$$f(x+h) = f(x) + \frac{\partial f}{\partial x}\Big|_x h + R(x, h), \tag{39}$$

where the remainder is given in Lagrange's form by:

$$R(x,h) = \sum_{i,j \in \{1,\ldots,n\}} \frac{\partial^2 f_i}{\partial x_i, x_j}\Big|_{x+ch} \frac{h_i h_j}{2}, \tag{40}$$

for some $c \in (0,1)$. If, in addition, $\frac{\partial f}{\partial x}$ is bounded on $\mathcal{D}$, it also holds that:

$$|R(x,h)| \le r(x)\|h\|^2, \tag{41}$$

for a continuous function $r : \mathbb{R}^n \to \mathbb{R}_{>0}$.

Corollary 5.1: Given a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ and an open, bounded, and convex set $\mathcal{D} \subset \mathbb{R}^n$, there exists $c > 0$ such that for all points $x, y \in \mathcal{D}$:

$$\left|\frac{\partial f}{\partial x}\Big|_x (y-x)\right| \le |f(y) - f(x)| + c\|y - x\|^2. \tag{42}$$

Proof: An application of Taylor's theorem gives us the following identity:

$$\frac{\partial f}{\partial x}\Big|_x (y-x) = f(x) - f(y) + R(x,y). \tag{43}$$

Taking the absolute value of both sides of this equation and applying the triangle inequality yields:

$$\left|\frac{\partial f}{\partial x}\Big|_x (y-x)\right| \le |f(y) - f(x)| + |R(x,y)|. \tag{44}$$

Since $\mathcal{D}$ is bounded, its closure $\overline{\mathcal{D}}$ is compact and $\frac{\partial f}{\partial x}$ must be bounded on $\mathcal{D}$. Thus, $|R(x,y)| \le r(x)\|y - x\|^2$ for a continuous function $r(x)$, according to Theorem 5.1. Define $c = \max_{x \in \overline{\mathcal{D}}} r(x)$ which satisfies the claim. ■

Lemma 5.1: Let $\mathcal{D} \subset \mathbb{R}^n$ denote an open, bounded, and convex set containing the origin and let $f : \mathcal{D} \to \mathbb{R}$ be a smooth function satisfying $f(q) = 0$ on a nontrivial subset $\mathcal{U}$ of $\mathcal{D}$ that includes the origin. Then, there exists $c > 0$ such that:

$$\left|\frac{\partial f}{\partial q} q\right| \le c\|q\|^2 \quad \text{for} \quad q \in \mathcal{U}. \tag{45}$$

Proof: Let $y = 0$ and $x = q$ and apply Corollary 5.1. ■

Lemma 5.2: For $\tilde{M}$ as defined in (33):

$$\mathrm{Ker}(\tilde{M}) = \mathrm{Im}\left(\frac{\partial \phi}{\partial q}^T\right). \tag{46}$$

Proof: The rank of $\frac{\partial \phi}{\partial q}$ equals m so for $v = \frac{\partial \phi}{\partial q}$, $\mathrm{rank}(\mathrm{P}_v) = m$ and $\mathrm{rank}(\tilde{M}) = (n-m)$. This implies that $\mathrm{Ker}(\tilde{M})$ has dimension m. It follows from (33) that for any vectors $x \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$ satisfying $x = \frac{\partial \phi}{\partial q}^T w$:

$$\tilde{M}x = \left(M^{-1}\frac{\partial \phi}{\partial q}^T - M^{-1}\frac{\partial \phi}{\partial q}^T\right) w = 0. \tag{47}$$

The set of all such vectors forms an m-dimensional subspace of $\mathrm{Ker}(\tilde{M})$. This implies:

$$\mathrm{Ker}(\tilde{M}) = \left\{\frac{\partial \phi}{\partial q}^T w : w \in \mathbb{R}^m\right\}. \tag{48}$$

■

Corollary 5.2: For $\tilde{M}$ as defined in (33):

$$\mathrm{Im}(\tilde{M}) = \mathrm{Ker}\left(\frac{\partial \phi}{\partial q}\right). \tag{49}$$

Proof: It is a well known fact from linear algebra that for any symmetric matrix A, the subspaces $\mathrm{Im}(A)$ and $\mathrm{Ker}(A)$ are orthogonal complements of each other. $\tilde{M}$ is symmetric, so:

$$\mathrm{Im}(\tilde{M}) = \mathrm{Ker}(\tilde{M})^{\perp} = \mathrm{Ker}\left(\frac{\partial \phi}{\partial q}\right), \tag{50}$$

where the second equality follows from Lemma 5.2. ■

Lemma 5.3: Given a compact set $\mathcal{D} \subset \mathbb{R}^n$ that contains the origin and $\tilde{M}$ as defined in (33), there exist constants $c_1, c_2 > 0$ such that for $x \in \mathrm{Im}(\tilde{M})$:

$$c_1\|x\| \leq \|\tilde{M}x\| \leq c_2\|x\|. \tag{51}$$

Proof: $\tilde{M}(q)$ is defined such that its rank is constant, and its individual elements are smooth functions of q. As such, the eigenvalues $\lambda_i(q)$ of $\tilde{M}$ are continuous functions of q and are therefore bounded on $\mathcal{D}$. Since $\tilde{M}$ is positive semi-definite, its nonzero eigenvalues are positive over $\mathcal{D}$.

Let $c_1, c_2 > 0$ be constants such that for every nonzero eigenvalue λ_i of $\tilde{M}$, $c_1 \leq \lambda_i(q) \leq c_2$ holds over $\mathcal{D}$. $\mathrm{Im}(\tilde{M})$ is spanned by the eigenvectors of $\tilde{M}$ with nonzero eigenvalues. Thus, for any $x \in \mathrm{Im}(\tilde{M})$ the claim holds. ■

C. Controller Derivation

Define the positive semi-definite matrix $\varPhi = \frac{\partial \phi}{\partial q}^T \frac{\partial \phi}{\partial q}$. It follows from Lemma 5.2 and the symmetry of β (equal to $\tilde{M}$) that $\mathrm{Ker}(\mathrm{P}_\beta) = \mathrm{Im}(\mathrm{P}_\varPhi)$ and $\mathrm{Ker}(\mathrm{P}_\varPhi) = \mathrm{Im}(\mathrm{P}_\beta)$ hold. Thus, P_β and $\mathrm{P}_\varPhi$ satisfy:

$$\mathrm{P}_\beta \cdot \mathrm{P}_\varPhi = \mathrm{P}_\varPhi \cdot \mathrm{P}_\beta = 0 \quad \mathrm{P}_\beta + \mathrm{P}_\varPhi = I, \tag{52}$$

where the second equality follows from P_β and $\mathrm{P}_\varPhi$ having rank $(n-m)$ and m respectively.

Because β is real and symmetric, the finite spectral theorem [16] implies that $\beta = B^T A B$ for A, a real diagonal matrix, and B, a real orthogonal matrix. The rows of B that correspond to the non-zero diagonal entries of A form a basis for $\mathrm{Im}(\beta)$. Thus, if $A^\dagger$ is the matrix obtained by inverting every nonzero diagonal element of A, $\beta^\dagger = B^T A^\dagger B$ defines the pseudoinverse of β. Note that $\beta^\dagger$ satisfies $\mathrm{P}_\beta = \beta^\dagger \beta$.

There exists at least one input u^* for which:

$$\mathrm{P}_\beta(-k_p q - k_d \dot{q} - \alpha) = \mathrm{P}_\beta\, \beta u^* = \beta u^*. \tag{53}$$

With $\beta^\dagger$, define u^* to be the minimum norm solution of (53):

$$u^* = \beta^\dagger(k - \alpha), \tag{54}$$

such that the acceleration error can be expressed as follows:

$$\begin{aligned} e_{\ddot{q}} &= \ddot{q} - k = \ddot{q} - (\mathrm{P}_\beta + \mathrm{P}_\varPhi)(-k_p q - k_d \dot{q}) \\ &= \alpha + \beta u - \mathrm{P}_\beta\, \alpha - \beta u^* + k_p \mathrm{P}_\varPhi\, q \\ &= (I - \mathrm{P}_\beta)\, \alpha + \beta e_u + k_p \mathrm{P}_\varPhi\, q \\ &= \mathrm{P}_\varPhi\, \alpha + \beta e_u + k_p \mathrm{P}_\varPhi\, q. \end{aligned} \tag{55}$$

Note that the third equality in (55) makes use of (53) and $\frac{\partial \phi}{\partial q}\dot{q} = 0$ from (31), while the last equality uses (52).

Combining (31), (38), and (52), we find that:

$$\mathrm{P}_\varPhi\, \alpha = \mathrm{P}_\varPhi\, \tilde{\psi}, \tag{56}$$

where ψ is a continuous function for which each term ψ_i is quadratic with respect to $\dot{q}$. Thus, if we define:

$$\eta = \mathrm{P}_\varPhi(k_p q + \alpha), \tag{57}$$

Lemma 5.1 and (56) tell us that, over any compact set $\mathcal{D}$, the norm of η is bounded by $c\|z\|^2$ for some constant $c > 0$.

Re-express $e_{\ddot{q}}$ as $e_{\ddot{q}} = \eta + \beta e_u$ and construct a dynamic controller, $\dot{u} = -\gamma e_{\ddot{q}} - \rho \varPhi u$ where $\gamma, \rho > 0$. Apply this to (37) to derive the lifted system:

$$\begin{aligned} \ddot{q} &= k + \eta + \beta e_u \\ \dot{e}_u &= -\gamma(\eta + \beta e_u) - \rho \varPhi u - \dot{u}^*. \end{aligned} \tag{58}$$

D. Stability Analysis

Our stability analysis for (58) proceeds in largely the same fashion as before, starting with the following candidate Lyapunov equation:

$$V(z, e_u) = z^T P z^T + \frac{1}{2} e_u^T e_u, \tag{59}$$

the derivative of which is:

$$\begin{aligned} \dot{V}(z, e_u) = &-z^T Q z + 2 z^T P \begin{bmatrix} 0 \\ \eta + \beta e_u \end{bmatrix} - e_u \dot{u}^* \\ &+ e_u^T \mathrm{P}_\beta\, (-\gamma \beta e_u) + e_u^T \mathrm{P}_\varPhi\, (-\gamma \eta - \rho \varPhi e_u), \end{aligned} \tag{60}$$

where we've used (52) along with $\mathrm{P}_\beta\, \eta = 0$ and $\mathrm{P}_\varPhi\, u^* = 0$.

Assume there exists a compact subset $\mathcal{D} \subset \mathbb{R}^{3n}$ containing the origin and define the following constant:

$$d_1 = \max_{(z, e_u) \in \mathcal{D}} \{k_p, k_d, \|2P\|, \|\beta\|\}. \tag{61}$$

Then we can derive the following bound for $\dot{V}$:

$$\begin{aligned} \dot{V}(z, e_u) \leq &-\lambda_Q \|z\|^2 + d_1^2 \|z\| \|\eta\| + d_1^2 \|z\| \|\mathrm{P}_\beta\, e_u\| \\ &+ \|e_u\| \|\dot{u}^*\| + \gamma d_1 \|\mathrm{P}_\varPhi\, e_u\| \|\eta\| \\ &- \gamma \lambda_\beta \|\mathrm{P}_\beta\, e_u\|^2 - \rho \lambda_\varPhi \|\mathrm{P}_\varPhi\, e_u\|^2, \end{aligned} \tag{62}$$

where $\lambda_Q = \lambda_{\min}(Q)$, and $\lambda_\beta, \lambda_\varPhi$ are lower bounds on the nonzero eigenvalues of β and $\varPhi$ respectively.

Because the pseudoinverse operation is differentiable [17], u^* is a differentiable function of our system's state. Using a decomposition analogous to that performed in (14), we can differentiate (53) and show that $\beta \dot{u}^*$ is upper bounded by an expression $d_2(\|z\| + \|e_u\|)$ for some $d_2 > 0$ when (z, e_u) is restricted to $\mathcal{D}$. The same then holds for $\dot{u}^*$ (for a different constant). Also, recall from our derivation of η in (57) that $\|\eta\| \leq d_3 \|z\|^2$ holds over $\mathcal{D}$ for some $d_3 > 0$.

Let $d = \max\{d_1, d_1^2, d_2, d_3\}$ and further relax (62) to:

$$\begin{aligned} \dot{V}(z, e_u) \leq &(d\|z\| - \lambda_Q)\|z\|^2 + 2d\|z\| \|\mathrm{P}_\beta\, e_u\| \\ &+ (d - \gamma \lambda_\beta)\|\mathrm{P}_\beta\, e_u\|^2 + (d - \rho \lambda_\varPhi)\|\mathrm{P}_\varPhi\, e_u\|^2 \\ &+ d(1 + \gamma\|z\|)\|z\| \|\mathrm{P}_\varPhi\, e_u\|. \end{aligned} \tag{63}$$

Theorem 5.2: There exist $\gamma > 0$ and $\rho > 0$ rendering the origin of the system in (58) locally asymptotically stable.

Proof: Define V as in (59) with derivative bound (63) and choose $\mathcal{D}$ to be a sublevel set of V such that $\|z\| < \lambda_Q$ holds over $\mathcal{D}$. Pick $\gamma > 0$ large enough that the expression:

$$\frac{1}{2}(d\|z\| - \lambda_Q)\|z\|^2 + 2d\|z\| \|\mathrm{P}_\beta\, e_u\| - \gamma \lambda_\beta \|\mathrm{P}_\beta\, e_u\|^2, \tag{64}$$

becomes negative-definite on $\mathcal{D}$. With γ fixed, pick $\rho > 0$ large enough that the expression:

$$\begin{aligned} &\frac{1}{2}(d\|z\| - \lambda_Q)\|z\|^2 + d(1 + \gamma\|z\|)\|z\| \|\mathrm{P}_\varPhi\, e_u\| \\ &\quad + (d - \rho \lambda_\varPhi)\|\mathrm{P}_\varPhi\, e_u\|^2, \end{aligned} \tag{65}$$

becomes negative-definite on $\mathcal{D}$. This is possible because $\gamma\|z\|$ is bounded on $\mathcal{D}$. For our chosen γ and ρ, it follows from (63) that $\dot{V}$ is negative-definite on $\mathcal{D}$, proving the claim. ∎

As in the unconstrained case, if we use a state estimator to derive $\hat{q}$ and $\hat{\dot{q}}$ then we are only able to compute an approximation of the acceleration error $\hat{e}_{\ddot{q}} = e_{\ddot{q}} + \epsilon$, where ϵ denotes the estimation error with respect to $e_{\ddot{q}}$. This induces the following closed-loop dynamics:

$$\begin{aligned} \ddot{q} &= k + \eta + \beta e_u \\ \dot{e}_u &= -\gamma\left(\eta + \beta e_u + \epsilon\right) - \rho\Phi u - \dot{u}^*. \end{aligned} \tag{66}$$

Theorem 5.3: There exist $\gamma > 0$ and $\rho > 0$ rendering the system in (66) LISS with respect to the estimation error ϵ.

Proof: Define V as in (59). We know from Theorem 5.2 that if $\epsilon = 0$, there exists a bounded sublevel set $\mathcal{D}$ of V and controller parameters γ and ρ such that $\dot{V}$ becomes negative-definite over $\overline{\mathcal{D}}$. Since $\dot{V}$ is quadratic with respect to (z, e_u), we can loosen its upper bound to $-a\|(z, e_u)\|^2$ for some $a > 0$. Thus, for $\epsilon \neq 0$, $\dot{V}$ satisfies:

$$\dot{V}(z, e_u, \epsilon) \leq -a\|(z, e_u)\|^2 + \gamma\|(z, e_u)\|\|\epsilon\|. \tag{67}$$

Pick $0 < \theta < 1$ so that this new upper bound equals:

$$-a(1-\theta)\|(z, e_u)\|^2 - a\theta\|(z, e_u)\|^2 + \gamma\|(z, e_u)\|\|\epsilon\|. \tag{68}$$

It follows that $\dot{V}$ is negative-definite for:

$$\|(z, e_u)\| \geq \frac{\gamma}{a\theta}\|\epsilon\|. \tag{69}$$

Choose any $r > 0$ for which $\mathcal{B}_r \subset \mathcal{D}$ and let $r_\epsilon = \frac{a\theta}{\gamma} r$. If $\sup_{t\geq 0}\|\epsilon(t)\| \leq r_\epsilon$, then $\dot{V}$ is negative-definite on the boundary of $\mathcal{D}$. Thus, $\mathcal{D}$ is invariant and (69) indicates that our system is ISS within $\mathcal{D}$. ∎

VI. EXPERIMENTAL RESULTS

We performed experimental validation using version 2.3.2 of the MuJoCo physics engine [18]. We chose a fully-actuated triple-pendulum for our plant. Each plot in Figure 1 shows the step response of the pendulum's pinned joint for different controllers. The solid lines correspond to a standard computed-torque controller using a perfect model of the dynamics. The dotted lines show closed-loop responses under model-free control for different values of γ.

In the right-hand plot, notice that the unmodeled torque limits significantly impact the performance of the computed-torque controller. The model-free controller does a much better job of accommodating these additional dynamics.

VII. FUTURE WORK

Our immediate goal is to relax our assumption that the constraint equations are known. This would significantly broaden the scope of application of our controller, making it suitable for trajectory tracking in uncertain environments and across changing contact regimes. Next, we would like to extend our constrained controller to a larger class of systems, for example partially feedback-linearizable (PFBL) systems. It is straightforward to derive a model-free controller that stabilizes an output, $h(q)$, as long as that output is a smooth function of just the position, q. It is not clear, however, how one should go about handling nonholonomic constraints or the zero-dynamics inherent to PFBL systems.

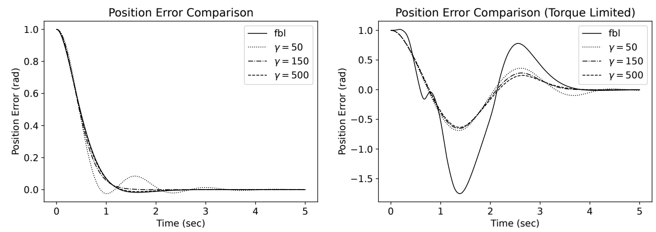

Fig. 1. Left: Position errors. Right: Torque-limited position errors. Comparison of the model-free controller for various values of γ against a feedback-linearizing (fbl) controller.

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