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Using artificial delays for stabilization of linear second-order systems under unknown control directions by extremum seeking controller

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Abstract— **We consider a linear second-order system subject to unknown control directions under the measurements of the position whereas the velocity is not available for measurements. Such system can be stabilized by an extremum seeking (ES) controller using the position and velocity. In this paper, the velocity is approximated via a finite difference leading to a delay-dependent ES controller. By applying the recently proposed time-delay approach to Lie-Brackets-based averaging method, we transform the closed-loop system to a time-delay (neutral type) one, which has a form of perturbed Lie brackets system. The input-to-state stability (ISS) of the time-delay system guarantees the same for the original one. Then, by employing variation of constants formula we derive explicit conditions in terms of simple inequalities for finding the quantitative bounds on the dither period and delay that ensure the regional ISS. An example is provided to illustrate the efficiency of the results.**

I. Introduction

As a powerful real-time model-free optimization method, extremum seeking (ES) has received much attention in the past decades. Since rigorous proofs of local convergence and semi-global convergence were proposed in [1] and [2], respectively, a large number of theoretical developments on ES have emerged in the literature, see [3], [4], [5], [6], [7] and the reviews [8], [9]. The conventional approach to analyze the stability of ES systems depends upon the classical averaging method [10] and Lie-brackets approximation [11], where the trajectory properties of the original and averaged systems were used to ensure the stability of the original system when the small parameters are small enough. That is, the conventional approach presented the qualitative analysis only and cannot provide quantitative bounds on the small parameter preserving the stability.

ES controller has been used in [12], [13] as a stabilizing feedback for systems under unknown control directions. It should be noted that the ES controllers designed in [12], [13] depended upon the full knowledge of the system state. However, in many practical applications only the output is available for measurement. Compared to the observer-based controller with a complicated framework, a simpler static delayed output-feedback is more attractive in the literature, see e.g. [14], [15], [16] and the references therein, where the delayed feedback was obtained via a finite difference approximation.

Recently, a constructive time-delay approach to periodic averaging method was introduced in [17], where efficient upper bounds on the small parameter ensuring the stability and ISS of the original system were found. The time-delay approach to averaging method was then applied to vibrational control [18] as well as to ES [19], [20], [21]. Note that in [21], variation of constants formula was employed, which greatly simplified the results comparatively to Lyapunov-Krasovskii method in [19], [20]. Moreover, a time-delay approach to Lie-brackets-based averaging of affine systems was proposed in [22] with an application to stabilization of linear systems subject to unknown control directions.

In this paper, we consider derivative-dependent ES control of the linear second-order systems subject to unknown control directions, where the derivative is not available for measurements. Under assumption of the stabilizability of the system by a state-feedback that depends on the output and its derivative, a delay-dependent ES controller that stabilizes the system is found using a finite difference approximation of the derivative. Then, based on the recently proposed time-delay approach to Lie-Brackets-based averaging method [22], [23] we transform the closed-loop system into a time-delay (neutral type) one. The ISS of the time-delay system guarantees the ISS of the original one. Following [21], we employ variation of constants formula to derive explicit conditions in terms of simple inequalities. By verifying these conditions, one can find quantitative bounds on the dither period and delay that ensure the regional ISS. Finally, an example illustrates the efficiency of the results.

Notation: Throughout the paper, the superscript T stands for vector/matrix transposition and the notation $P > 0$ ($P \geq 0$), for $P \in \mathbb{R}^{n\times n}$ means that P is symmetric and positive definite (positive semi-definite). The notations $|\cdot|$ and $\|\cdot\|$ refer to the Euclidean vector norm and the induced matrix 2 norm, respectively. Moreover, we use $a\pm b$ to denote $a+b-b$ (not the set $\{a+b, a-b\}$).

II. Problem formulation

We consider a second-order system

$$\begin{aligned}\dot{x}_0(t) &= x_1(t),\\ \dot{x}_1(t) &= \textstyle\sum_{i=0}^{1} a_i(t)x_i(t) + b(t)u(t),\end{aligned} \tag{1}$$

where $x_0(t)$ is the measurement, $u(t) \in \mathbb{R}$ is the control input, and the coefficients $a_i(t)$ $(i = 0, 1)$ and $b(t)$ have the following form

$$a_i(t) = a_{i0} + \Delta a_i(t), \quad b(t) = b_0 + \sqrt{\varepsilon}\Delta b(t). \tag{2}$$

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Here $\varepsilon > 0$ is a small parameter, $a_{00} \geq 0$ and $a_{10} \geq 0$ (implying that A_0 defined in (3) below is non-Hurwitz) are constant, b_0 is a known constant up to its sign, and $\Delta a_i(t)$ and $\Delta b(t)$ denote the time-varying uncertainties.

Denoting

$$A(t) = A_0 + \Delta A(t), \quad B(t) = B_0 + \sqrt{\varepsilon}\Delta B(t),$$

where

$$\begin{array}{ll} A_0 = \begin{bmatrix} 0 & 1 \\ a_{00} & a_{10} \end{bmatrix}, & \Delta A(t) = \begin{bmatrix} 0 & 0 \\ \Delta a_0(t) & \Delta a_1(t) \end{bmatrix}, \\ B_0 = \begin{bmatrix} 0 \\ b_0 \end{bmatrix}, & \Delta B(t) = \begin{bmatrix} 0 \\ \Delta b(t) \end{bmatrix}, \end{array} \tag{3}$$

we present system (1) as

$$\dot{x}(t) = A(t)x(t) + B(t)u(t). \tag{4}$$

A1 Assume that there exist small constants $\Delta a \geq 0$ and $\Delta b \geq 0$ such that

$$\|\Delta A(t)\| \leq \Delta a, \quad |\Delta B(t)| \leq \Delta b \quad \forall t \geq 0 \tag{5}$$

The latter implies

$$\|A(t)\| \leq a \quad \forall t \geq 0, \quad a = \|A_0\| + \Delta a. \tag{6}$$

Since the sign of b_0 is unknown, one cannot design for system (4) a classical PD type stabilizing controller. To tackle the stabilization problem subject to unknown control directions, similar to [12] one may design for system (4) the following bounded ES controller:

$$u(t) = \tfrac{\sqrt{2\pi\alpha}}{\sqrt{\varepsilon}} \cos(\tfrac{2\pi t}{\varepsilon} + k|cx_0(t) + x_1(t)|^2), \tag{7}$$

where $\alpha > 0$ and $k > 0$ are tuning parameters, and the coefficient $c > 0$ is constant. The averaged system that corresponds to system (4), (7) with $\Delta A(t) = 0$ and $\Delta B(t) = 0$ is given by the following Lie Brackets system [5], [6], [12]:

$$\dot{x}_{av}(t) = A_{av}x_{av}(t), \quad x_{av}(t) \in \mathbb{R}^2, \tag{8}$$

where

$$\begin{aligned} A_{av} &= A_0 - \alpha k B_0 B_0^T C^T C \\ &= \begin{bmatrix} 0 & 1 \\ a_{00} - \alpha k b_0^2 c & a_{10} - \alpha k b_0^2 \end{bmatrix} \end{aligned} \tag{9}$$

with $C = [c, 1]$. From (9), it follows that when $c > 0$ there always exist constants α and k leading to Hurwitz A_{av}.

Remark 1: Note that the bounded ES controller using the position information only, i.e.

$$u(t) = \tfrac{\sqrt{2\pi\alpha}}{\sqrt{\varepsilon}} \cos(\tfrac{2\pi t}{\varepsilon} + k|cx_0(t)|^2)$$

cannot stabilize system (4) with $\Delta A(t) = 0$ and $\Delta B(t) = 0$. This is due to that the resulting averaged system is given by (8) with $A_{av} = A_0$. The latter is non-Hurwitz for any α, k and c.

It should be pointed out that the bounded ES controller (7) depends on both $x_0(t)$ and $x_1(t)$. Recall from (1) that $x_1(t)$ is the derivative of $x_0(t)$. Differently from [12], we consider that the derivative $x_1(t)$ is not available. To approximate the derivative $x_1(t)$, we employ a finite-difference approximation of the derivative $x_1(t)$ [14], [15]:

$$x_1(t) \approx \tfrac{1}{h\sqrt{\varepsilon}}(x_0(t) - x_0(t - h\sqrt{\varepsilon}))$$

with a constant $h > 0$. By replacing $x_1(t)$ in (7) with its approximation, we have the following delay-dependent bounded ES controller

$$\begin{aligned} u(t) = \tfrac{\sqrt{2\pi\alpha}}{\sqrt{\varepsilon}} \cos(&\tfrac{2\pi t}{\varepsilon} + k|cx_0(t) \\ &+ \tfrac{1}{h\sqrt{\varepsilon}}(x_0(t) - x_0(t - h\sqrt{\varepsilon}))|^2). \end{aligned} \tag{10}$$

We present

$$\begin{aligned} \tfrac{1}{h\sqrt{\varepsilon}}(x_0(t) - x_0(t - h\sqrt{\varepsilon})) &= x_1(t) \\ &- \tfrac{1}{h\sqrt{\varepsilon}} \int_{t-h\sqrt{\varepsilon}}^{t} (s - t + h\sqrt{\varepsilon})\dot{x}_1(s)ds. \end{aligned}$$

Thus, the delay-dependent bounded ES controller (10) is rewritten as

$$u(t) = \tfrac{\sqrt{2\pi\alpha}}{\sqrt{\varepsilon}} \cos(\tfrac{2\pi t}{\varepsilon} + k|Cx(t) + \kappa(t)|^2), \tag{11}$$

where

$$\kappa(t) = -\tfrac{1}{h\sqrt{\varepsilon}} \int_{t-h\sqrt{\varepsilon}}^{t} (s - t + h\sqrt{\varepsilon})\mathcal{I}\dot{x}(s)ds. \tag{12}$$

with $\mathcal{I} = [0, 1]$. Thus,

$$\dot{\kappa}(t) = -\mathcal{I}\dot{x}(t) + \tfrac{1}{h\sqrt{\varepsilon}} \int_{t-h\sqrt{\varepsilon}}^{t} \mathcal{I}\dot{x}(s)ds \tag{13}$$

It is easy to see that $\kappa(t)$ and $\dot{\kappa}(t)$ are, respectively, of the order of $O(h)$ and $O(\frac{1}{\sqrt{\varepsilon}})$ when $\dot{x}(t)$ is of the order of $O(\frac{1}{\sqrt{\varepsilon}})$.

The closed-loop system (4), (11) takes the form

$$\begin{aligned} \dot{x}(t) = A(t)x(t) &+ \tfrac{\sqrt{2\pi\alpha}}{\sqrt{\varepsilon}} B_0 \\ &\times \cos(\tfrac{2\pi t}{\varepsilon} + k|Cx(t) + \kappa(t)|^2) + v(t). \end{aligned} \tag{14}$$

where

$$v(t) = \sqrt{2\pi\alpha}\Delta B(t) \cos(\tfrac{2\pi t}{\varepsilon} + k|Cx(t) + \kappa(t)|^2). \tag{15}$$

III. Main results

In this section, we will first apply the recently proposed time-delay approach to Lie-Brackets-based averaging method [22], [23] that transforms the closed-loop system (14) to a time-delay (neutral type) one, and then derive explicit conditions in terms of simple inequalities via variation of constants formula [21] for finding the quantitative bounds on ε and h that ensure the regional ISS.

A. A time-delay approach to Lie-Brackets-based averaging

Differently from the Lie Brackets averaging method [5], [6], [12], we employ in this paper a time-delay approach to Lie-Brackets-based averaging method [22], [23] for system (14) without any approximations, see Appendix A. The latter allows to transform system (14) to the following time-delay (neutral type) system:

$$\begin{aligned} \dot{z}(t) = &[A_{av} + \Delta A(t)](z(t) - G(t)) \\ &+ \textstyle\sum_{i=1}^{2}(Y_i(t) + Y_{\kappa_i}(t)) + Y_3(t) \\ &+ Y_{\dot{\kappa}}(t) + Y_v(t) + v(t), \quad t \geq \varepsilon + h\sqrt{\varepsilon}, \end{aligned} \tag{16}$$

where $\Delta A(t)$, A_{av} and $v(t)$ are given by (3), (9) and (15), respectively, and

$$\begin{array}{l} z(t) = x(t) + G(t), \\ G(t) = -\frac{\sqrt{2\pi\alpha}}{\varepsilon\sqrt{\varepsilon}} B_0 \int_{t-\varepsilon}^{t} (s-t+\varepsilon) \\ \quad \times \cos(\frac{2\pi s}{\varepsilon} + k|Cx(s)+\kappa(t)|^2) ds, \\ Y_1(t) = \frac{2k\sqrt{2\pi\alpha}}{\varepsilon\sqrt{\varepsilon}} B_0 \int_{t-\varepsilon}^{t}\int_s^t \sin(\frac{2\pi s}{\varepsilon} \\ \quad +k|Cx(\theta)+\kappa(t)|^2) x^T(\theta) C^T C A(\theta) x(\theta) d\theta ds, \\ Y_2(t) = -\frac{8\pi\alpha k^2}{\varepsilon^2} B_0 B_0^T C^T C \int_{t-\varepsilon}^{t}\int_s^t\int_\theta^t \\ \quad \times \cos(\frac{2\pi}{\varepsilon}(s+\theta) + 2k|Cx(\xi)+\kappa(t))|^2) \\ \quad \times x(\theta) x^T(\xi) C^T C \dot{x}(\xi) d\xi d\theta ds, \\ Y_3(t) = -\frac{4\pi\alpha k}{\varepsilon^2} B_0 B_0^T C^T C \int_{t-\varepsilon}^{t}\int_s^t\int_\theta^t \sin(\frac{2\pi s}{\varepsilon} + k|Cx(t) \\ \quad +\kappa(t)|^2) \cos(\frac{2\pi\theta}{\varepsilon} + k|Cx(t)+\kappa(t)|^2) \dot{x}(\xi) d\xi d\theta ds, \\ Y_{\kappa_1}(t) = \frac{2k\sqrt{2\pi\alpha}}{\varepsilon\sqrt{\varepsilon}} B_0 \int_{t-\varepsilon}^{t}\int_s^t \sin(\frac{2\pi s}{\varepsilon} \\ \quad +k|Cx(\theta)+\kappa(t)|^2) \kappa^T(t) C \dot{x}(\theta) d\theta ds, \\ Y_{\kappa_2}(t) = -\frac{8\pi\alpha k^2}{\varepsilon^2} B_0 B_0^T C^T C \int_{t-\varepsilon}^{t}\int_s^t\int_\theta^t \cos(\frac{2\pi}{\varepsilon}(s+\theta) \\ \quad +2k|Cx(\xi)+\kappa(t)|^2) x(\theta) \kappa^T(t) C \dot{x}(\xi) d\xi d\theta ds, \\ Y_{\dot\kappa}(t) = \frac{8\pi\alpha k^2}{\varepsilon^2} B_0 B_0^T C^T C \int_{t-\varepsilon}^{t}\int_s^t\int_\theta^t \sin(\frac{2\pi s}{\varepsilon} \\ \quad +k|Cx(\theta)+\kappa(t)|^2) \sin(\frac{2\pi\theta}{\varepsilon} + k|Cx(\theta) \\ \quad +\kappa(\xi)|^2) x(\theta)(Cx(\theta)+\kappa(\xi))^T \dot\kappa(\xi) d\xi d\theta ds, \\ Y_v(t) = \frac{2k\sqrt{2\pi\alpha}}{\varepsilon\sqrt{\varepsilon}} B_0 \int_{t-\varepsilon}^{t}\int_s^t \sin(\frac{2\pi s}{\varepsilon} \\ \quad +k|Cx(\theta)+\kappa(t)|^2) x^T(\theta) C^T C v(\theta) d\theta ds. \end{array} \tag{17}$$

Note that if $x(t)$ is a solution to system (14), then it satisfies the time-delay system (16) with notations (17), where $\dot{x}(t)$ is defined by (14). This implies that if solutions $x(t)$ of the time-delay system (16) for $t \geq \varepsilon + h\sqrt{\varepsilon}$ satisfy some bound (e.g., ISS bound given by (23) below), then the same bound holds for solutions of system (14) for $t \geq \varepsilon + h\sqrt{\varepsilon}$.

Moreover, from (17) it follows that $G(t)$, $Y_i(t)$ $(i = 1,2,3)$, $Y_v(t)$ and $Y_{\dot\kappa}(t)$ are of the order of $O(\sqrt{\varepsilon})$, $Y_{\kappa_1}(t)$ is of the order of $O(h)$, $Y_{\kappa_2}(t)$ is of the order of $O(h\sqrt{\varepsilon})$ provided $\dot{x}(t)$ is of the order of $O(\frac{1}{\sqrt{\varepsilon}})$. Thus, it can be seen that system (16) is a perturbation of the stable averaged system (8). Note that the perturbations in (16) will vanish as $\varepsilon \to 0$ and $h \to 0$. If ε and h increase, the perturbations may ruin the stability of system (16). The objective of this paper is to find the first quantitative bounds on ε and h that ensure the regional ISS.

B. Stability analysis

We are now in a position to derive explicit conditions in terms of simple inequalities for finding the upper bounds on ε and h that ensure the regional ISS of system (14). For the sake of simplicity, we denote

$$\begin{array}{ll} \vartheta_1 = \sqrt{2\pi\alpha}|B_0|, & \vartheta_2 = \sqrt{2\pi\alpha}\Delta b, \\ \vartheta_3 = a\sigma + \sqrt{2\pi\alpha}\Delta b, & \vartheta_4 = |C| + 2. \end{array} \tag{18}$$

Theorem 1: Consider system (1) with notation (2), where b_0 has unknown sign, $a_{00} \geq 0$ and $a_{10} \geq 0$. Let $\alpha > 0$, $k > 0$ and $c > 0$ be such that matrix A_{av} given by (9) is Hurwitz.

(i) Assume that assumption **A1** holds. Given tuning parameters δ and $\Delta a \geq 0$, let there exist $n \times n$ matrix $P > 0$ and and scalar $p \geq 1$, $\lambda > 0$ that satisfy the following inequalities:

$$P - I \geq 0, \quad pI - P \geq 0, \tag{19}$$

$$\Xi = \begin{bmatrix} PA_{av} + A_{av}^T P + 2\delta P + \lambda(\Delta a)^2 I & P \\ P & -\lambda I \end{bmatrix} \leq 0. \tag{20}$$

If additionally, given tuning parameters $\Delta b \geq 0$, $\varepsilon^* > 0$, $h > 0$ and $0 < \sigma_0 < \sigma$, the following inequality

$$\begin{array}{l} p\big[e^{a(\varepsilon^*+h\sqrt{\varepsilon^*})}\big(\sigma_0 + (\vartheta_1 + \sqrt{\varepsilon^*}\vartheta_2)(\sqrt{\varepsilon^*}+h)\big) \\ \quad + \frac{\sqrt{\varepsilon^*}}{2}\vartheta_1 + \frac{1}{\delta}\big(\sqrt{\varepsilon^*}\mu_0 + \varepsilon^*\mu_1 + h(\mu_2 + \sqrt{\varepsilon^*}\mu_3 \\ \quad +\varepsilon^*\mu_4 + \varepsilon^*\sqrt{\varepsilon^*}\mu_5) + \vartheta_2\big)\big]^2 < (\sigma - \frac{\sqrt{\varepsilon^*}}{2}\vartheta_1)^2 \end{array} \tag{21}$$

is valid, where

$$\begin{array}{l} \mu_0 = \frac{1}{2}\vartheta_1(\|A_{av}\| + \Delta a) + k\sigma\vartheta_1\vartheta_3|C|^2 \\ \quad + \frac{1}{3}k\vartheta_1^3|C|^2(1 + 2k\sigma^2\vartheta_4|C|), \\ \mu_1 = \frac{1}{3}k\vartheta_1^2\vartheta_3|C|^2(1 + 2k\sigma^2\vartheta_4|C|), \\ \mu_2 = \frac{1}{2}k\vartheta_1^3|C|, \quad \mu_5 = \frac{1}{3}k^2\sigma\vartheta_1^2\vartheta_3^2\vartheta_4|C|^2, \\ \mu_3 = k\vartheta_1^2|C|(\vartheta_3 + \frac{1}{3}k\sigma\vartheta_1^2\vartheta_4|C|), \\ \mu_4 = \frac{1}{2}k\vartheta_1\vartheta_3|C|(\vartheta_3 + \frac{4}{3}k\sigma\vartheta_1^2\vartheta_4|C|) \end{array} \tag{22}$$

with a defined in (6) and ϑ_i $(i = 1, \ldots, 4)$ defined in (18), then for all $\varepsilon \in (0, \varepsilon^*]$ the solution of (14) starting from the initial condition $\|\phi\|_{C[-h\sqrt{\varepsilon},0]} \leq \sigma_0$ satisfies

$$\begin{array}{l} |x(t)| \leq e^{at}[\|\phi\|_{C[-h\sqrt{\varepsilon},0]} + (\vartheta_1 + \sqrt{\varepsilon}\vartheta_2)(\sqrt{\varepsilon}+h)] \\ \qquad < \sigma, \quad t \in [0, \varepsilon + h\sqrt{\varepsilon}], \\ |x(t)| < \sqrt{p}e^{-\delta(t-\varepsilon-h\sqrt{\varepsilon})}\big[e^{a(\varepsilon+h\sqrt{\varepsilon})}\big(\|\phi\|_{C[-h\sqrt{\varepsilon},0]} \\ \qquad +(\vartheta_1 + \sqrt{\varepsilon}\vartheta_2)(\sqrt{\varepsilon}+h)\big) + \frac{\sqrt{\varepsilon}}{2}\vartheta_1\big] \\ \qquad + \frac{\sqrt{p}}{\delta}[\sqrt{\varepsilon}\mu_0 + \varepsilon\mu_1 + h(\mu_2 + \sqrt{\varepsilon}\mu_3 + \varepsilon\mu_4 \\ \qquad +\varepsilon\sqrt{\varepsilon}\mu_5) + \vartheta_2] + \frac{\sqrt{\varepsilon}}{2}\vartheta_1 < \sigma, \quad t \geq \varepsilon + h\sqrt{\varepsilon}. \end{array} \tag{23}$$

Moreover, for all initial conditions $\|\phi\|_{C[-h\sqrt{\varepsilon},0]} \leq \sigma_0$ the ball

$$\begin{array}{l} \mathcal{X} = \big\{x \in \mathbb{R}^n : |x| \leq \frac{\sqrt{p}}{\delta}[\sqrt{\varepsilon}\mu_0 + \varepsilon\mu_1 \\ \quad +h(\mu_2 + \sqrt{\varepsilon}\mu_3 + \varepsilon\mu_4 + \varepsilon\sqrt{\varepsilon}\mu_5) + \vartheta_2] + \frac{\sqrt{\varepsilon}}{2}\vartheta_1\big\} \end{array} \tag{24}$$

is exponentially attractive with a decay rate δ.

(ii) Given any $\sigma^2 > p\sigma_0^2$, conditions of item (i) is always feasible for small enough $h > 0$, $\varepsilon^* > 0$, $\Delta a > 0$ and $\Delta b > 0$ (meaning that the delay-dependent ES controller (10) exponentially stabilizes (4) with a decay rate $\delta > 0$).

The proof of Theorem 1 is given in Appendix B.

Remark 2: From item (ii) of Theorem 1, it follows that given any initial condition $\|\phi\|_{C[-h\sqrt{\varepsilon},0]} \leq \sigma_0$ one can always find σ for small enough $h > 0$, $\varepsilon^* > 0$, $\Delta a > 0$ and $\Delta b > 0$ such that $\sigma^2 > p\sigma_0^2$ holds subject to p satisfying (19). Therefore, the derived result is semiglobal.

IV. An example

Consider system (1) with

$$\begin{array}{l} a_{00} = a_{10} = 0, \quad b_0 \in \{-1, 1\}, \\ \Delta a_i(t) = \Delta\hat{a}_i, \quad \Delta b(t) = \Delta\hat{b}, \quad t \geq 0 \end{array} \tag{25}$$

under the delay-dependent bounded ES controller (10), where

$$\alpha = 0.01, \quad k = 10, \quad c = 0.5. \tag{26}$$

TABLE I

SOLUTIONS FOR DIFFERENT Δa AND Δb

Δa	Δb	ε^*	h	$h\sqrt{\varepsilon^*}$	UB
0	0	$0.5 \cdot 10^{-7}$	0.0248	$0.5545 \cdot 10^{-5}$	0.5538
0	0	$1.0 \cdot 10^{-7}$	0.0171	$0.5408 \cdot 10^{-5}$	0.5616
0	0	$2.0 \cdot 10^{-7}$	0.0062	$0.2773 \cdot 10^{-5}$	0.5724
0.001	0.001	$0.1 \cdot 10^{-7}$	0.0214	$0.4785 \cdot 10^{-5}$	0.5527
0.001	0.001	$1.0 \cdot 10^{-7}$	0.0137	$0.4332 \cdot 10^{-5}$	0.5605
0.001	0.001	$2.0 \cdot 10^{-7}$	0.0029	$0.1297 \cdot 10^{-5}$	0.5721

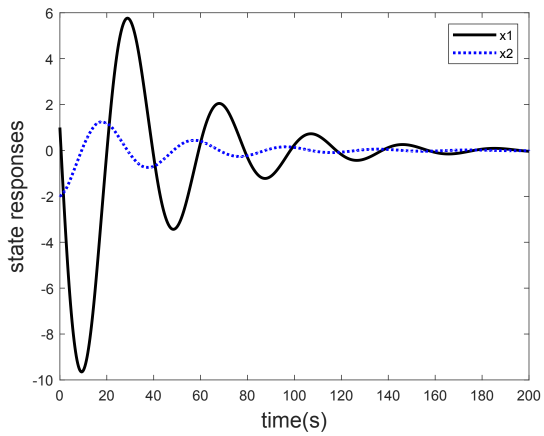

Fig. 1. State response under a delay-dependent bounded ES controller

Thus, we have $\Delta a = \sqrt{\sum_{i=0}^{1}(\Delta \hat{a}_i)^2}$, $\Delta b = \Delta \hat{b}$, $a = 1 + \Delta a$, $|B_0| = 1$ and

$$A_{av} = \begin{bmatrix} 0 & 1 \\ -0.05 & -0.1 \end{bmatrix}. \tag{27}$$

Let the desired decay rate be $\delta = 0.03$.

We verify the inequalities of Theorem 1 with $\sigma_0 = 0.1$, $\sigma = 1$ and different values of Δa and Δb leading to quantitative bounds on ε^* and h (that preserve the ISS for all $\varepsilon \in (0, \varepsilon^*]$) and the resulting ultimate bound (UB), see Table I.

For the numerical simulations, under the initial condition $x(t) = [1, -2]^T$ for $t \le 0$, state responses of system (1), (25) with large uncertainties $\Delta a_0(t) = 0$, $\Delta a_1(t) = 0.01\sin(t)$, $\Delta b(t) = 0.01\cos(t)$ under the delay-dependent bounded ES controller (10), where $\varepsilon = 0.0004$ and $h\sqrt{\varepsilon} = 0.0001$ (that are essentially larger than those in Table I, respectively) is shown in Fig. 1, which confirms our theoretical results illustrating the conservatism.

V. CONCLUSIONS

We have studied stabilization of linear second-order systems under unknown control directions, where a time-delay implementation of derivative-dependent extremum seeking control was presented, and have derived explicit conditions in terms of simple inequalities for finding the quantitative bounds on the dither period and delay that ensure the regional ISS. This was done by applying the recently proposed time-delay approach to Lie-Brackets-based averaging and by employing variation of constants formula. Less conservative results and their extension to higher-order systems may be topics for future research.

APPENDIX A: TRANSFORMATION VIA TIME-DELAY APPROACH

Inspired by [23], we introduce $G(t)$ defined in (17). This G-term provides a simpler stability analysis compared to that via the term $-\frac{\sqrt{2\pi\alpha}}{\varepsilon\sqrt{\varepsilon}}B_0\int_{t-\varepsilon}^{t}(s-t+\varepsilon)\cos(\frac{2\pi s}{\varepsilon}+k|Cx(s)+\kappa(s)|^2)ds$. Thus, we have

$$\begin{aligned}\dot{G}(t) = &-\tfrac{\sqrt{2\pi\alpha}}{\sqrt{\varepsilon}}B_0\cos(\tfrac{2\pi t}{\varepsilon}+k|Cx(t)+\kappa(t)|^2) \\ &+\tfrac{\sqrt{2\pi\alpha}}{\varepsilon\sqrt{\varepsilon}}B_0\int_{t-\varepsilon}^{t}\cos(\tfrac{2\pi s}{\varepsilon}+k|Cx(s)+\kappa(t)|^2)ds.\end{aligned} \tag{28}$$

Using the definition of $z(t)$, via (14) we obtain

$$\begin{aligned}\dot{z}(t) = &A(t)x(t) + \tfrac{\sqrt{2\pi\alpha}}{\varepsilon\sqrt{\varepsilon}}B_0 \\ &\times\int_{t-\varepsilon}^{t}\cos(\tfrac{2\pi s}{\varepsilon}+k|Cx(s)+\kappa(t)|^2)ds + v(t).\end{aligned} \tag{29}$$

By subtracting a zero term $\frac{\sqrt{2\pi\alpha}}{\varepsilon\sqrt{\varepsilon}}B_0\int_{t-\varepsilon}^{t}\cos(\frac{2\pi s}{\varepsilon}+k|Cx(t)+\kappa(t)|^2)ds$, we present the last term on the right-hand side of (29) as

$$\begin{aligned}&\tfrac{\sqrt{2\pi\alpha}}{\varepsilon\sqrt{\varepsilon}}B_0\int_{t-\varepsilon}^{t}\cos(\tfrac{2\pi s}{\varepsilon}+k|Cx(s)+\kappa(t)|^2)ds \\ &= \tfrac{\sqrt{2\pi\alpha}}{\varepsilon\sqrt{\varepsilon}}B_0\int_{t-\varepsilon}^{t}[\cos(\tfrac{2\pi s}{\varepsilon}+k|Cx(s)+\kappa(t)|^2) \\ &\quad -\cos(\tfrac{2\pi s}{\varepsilon}+k|Cx(t)+\kappa(t)|^2)]ds \\ &= Y_{\kappa_1}(t) + \tfrac{2k\sqrt{2\pi\alpha}}{\varepsilon\sqrt{\varepsilon}}B_0\int_{t-\varepsilon}^{t}\int_{s}^{t}\sin(\tfrac{2\pi s}{\varepsilon} \\ &\quad +k|Cx(\theta)+\kappa(t)|^2)x^T(\theta)C^TC\dot{x}(\theta)d\theta ds \\ &= Y_1(t) + Y_{\kappa_1}(t) + Y_v(t) + \tfrac{4\pi\alpha k}{\varepsilon^2}B_0B_0^TC^TC \\ &\quad \times\int_{t-\varepsilon}^{t}\int_{s}^{t}\sin(\tfrac{2\pi s}{\varepsilon}+k|Cx(\theta)+\kappa(t)|^2) \\ &\quad \times\cos(\tfrac{2\pi\theta}{\varepsilon}+k|Cx(\theta)+\kappa(\theta)|^2)x(\theta)d\theta ds,\end{aligned} \tag{30}$$

where in the last equality we substituted the right-hand side of (14) for $\dot{x}(t)$ and used the fact $x^T(\theta)C^TCB_0 = B_0^TC^TCx(\theta) \in \mathbb{R}$. Here $Y_1(t)$, $Y_{\kappa_1}(t)$ and $Y_v(t)$ are defined in (17).

Taking into account the following facts:

$$\begin{aligned}&\cos(\tfrac{2\pi\theta}{\varepsilon}+k|Cx(\theta)+\kappa(\theta)|^2) \\ &= \cos(\tfrac{2\pi\theta}{\varepsilon}+k|Cx(\theta)+\kappa(\theta)|^2) \\ &\quad \pm\cos(\tfrac{2\pi\theta}{\varepsilon}+k|Cx(\theta)+\kappa(t)|^2) \\ &= \cos(\tfrac{2\pi\theta}{\varepsilon}+k|Cx(\theta)+\kappa(t)|^2) \\ &\quad +2k\int_{\theta}^{t}\sin(\tfrac{2\pi\theta}{\varepsilon}+k|Cx(\theta)+\kappa(\xi)|^2) \\ &\quad \times(Cx(\theta)+\kappa(\xi))^T\dot{\kappa}(\xi)d\xi,\end{aligned} \tag{31}$$

$$\begin{aligned}&\sin(\tfrac{2\pi s}{\varepsilon}+k|Cx(\theta)+\kappa(t)|^2) \\ &\quad \times\cos(\tfrac{2\pi\theta}{\varepsilon}+k|Cx(\theta)+\kappa(t)|^2) \\ &= \sin(\tfrac{2\pi s}{\varepsilon}+k|Cx(\theta)+\kappa(t)|^2)\cos(\tfrac{2\pi\theta}{\varepsilon} \\ &\quad +k|Cx(\theta)+\kappa(t)|^2) \pm \sin(\tfrac{2\pi s}{\varepsilon}+k|Cx(t) \\ &\quad +\kappa(t)|^2)\cos(\tfrac{2\pi\theta}{\varepsilon}+k|Cx(t)+\kappa(t)|^2) \\ &= \sin(\tfrac{2\pi s}{\varepsilon}+k|Cx(t)+\kappa(t)|^2)\cos(\tfrac{2\pi\theta}{\varepsilon}+k \\ &\quad \times|Cx(t)+\kappa(t)|^2) - 2k\int_{\theta}^{t}\cos(\tfrac{2\pi}{\varepsilon}(s+\theta)+2k \\ &\quad \times|Cx(\xi)+\kappa(t)|^2)(Cx(\xi)+\kappa(t))^TC\dot{x}(\xi)d\xi,\end{aligned} \tag{32}$$

$$x(\theta) = x(\theta) \pm x(t) = x(t) - \int_{\theta}^{t}\dot{x}(\xi)d\xi, \tag{33}$$

$$\begin{aligned}&\tfrac{4\pi}{\varepsilon^2}\int_{t-\varepsilon}^{t}\int_{s}^{t}\sin(\tfrac{2\pi s}{\varepsilon}+k|Cx(t)+\kappa(t)|^2) \\ &\quad \times\cos(\tfrac{2\pi\theta}{\varepsilon}+k|Cx(t)+\kappa(t)|^2)d\theta ds \\ &= \tfrac{2}{\varepsilon}\int_{t-\varepsilon}^{t}\sin(\tfrac{2\pi s}{\varepsilon}+k|Cx(t)+\kappa(t)|^2) \\ &\quad \times[\sin(\tfrac{2\pi t}{\varepsilon}+k|Cx(t)+\kappa(t)|^2) \\ &\quad -\sin(\tfrac{2\pi s}{\varepsilon}+k|Cx(t)+\kappa(t)|^2)]ds \\ &= -\tfrac{2}{\varepsilon}\int_{t-\varepsilon}^{t}\sin^2(\tfrac{2\pi s}{\varepsilon}+k|Cx(t)+\kappa(t)|^2)ds \\ &= -1,\end{aligned} \tag{34}$$

and using the notations given by (17), we have

$$\begin{aligned}
&\tfrac{4\pi\alpha k}{\varepsilon^2}B_0B_0^TC^TC\int_{t-\varepsilon}^t\int_s^t\sin(\tfrac{2\pi s}{\varepsilon}+k|Cx(\theta)\\
&\quad+\kappa(t)|^2\cos(\tfrac{2\pi\theta}{\varepsilon}+k|Cx(\theta)+\kappa(\theta)|^2)x(\theta)d\theta ds\\
&\overset{(31)}{=}Y_{\dot\kappa}(t)+\tfrac{4\pi\alpha k}{\varepsilon^2}B_0B_0^TC^TC\int_{t-\varepsilon}^t\int_s^t\sin(\tfrac{2\pi s}{\varepsilon}+k|Cx(\theta)\\
&\quad+\kappa(t)|^2)\cos(\tfrac{2\pi\theta}{\varepsilon}+k|Cx(\theta)+\kappa(t)|^2)x(\theta)d\theta ds\\
&\overset{(32)}{=}Y_2(t)+Y_{\kappa_2}(t)+Y_{\dot\kappa}(t)+\tfrac{4\pi\alpha k}{\varepsilon^2}B_0B_0^TC^TC\\
&\quad\times\int_{t-\varepsilon}^t\int_s^t\sin(\tfrac{2\pi s}{\varepsilon}+k|Cx(t)+\kappa(t)|^2)\\
&\quad\times\cos(\tfrac{2\pi\theta}{\varepsilon}+k|Cx(t)+\kappa(t)|^2)x(\theta)d\theta ds\\
&\overset{(33)}{=}Y_2(t)+Y_3(t)+Y_{\kappa_2}(t)+Y_{\dot\kappa}(t)+\tfrac{4\pi\alpha k}{\varepsilon^2}B_0B_0^T\\
&\quad\times C^TC\int_{t-\varepsilon}^t\int_s^t\sin(\tfrac{2\pi s}{\varepsilon}+k|Cx(t)+\kappa(t))|^2)\\
&\quad\times\cos(\tfrac{2\pi\theta}{\varepsilon}+k|Cx(t)+\kappa(t)|^2)x(t)d\theta ds\\
&\overset{(34)}{=}-\alpha kB_0B_0^TC^TCx(t)+Y_2(t)+Y_3(t)+Y_{\kappa_2}(t)+Y_{\dot\kappa}(t).
\end{aligned}\tag{35}$$

Substituting (35) into (30) and further into (29), we transform (14) to the following system

$$\begin{aligned}
\dot z(t)=&[A_{av}+\Delta A(t)]x(t)+\textstyle\sum_{i=1}^2(Y_i(t)+Y_{\kappa_i}(t))\\
&+Y_3(t)+Y_{\dot\kappa}(t)+Y_v(t)+v(t),\quad t\ge\varepsilon+h\sqrt{\varepsilon},
\end{aligned}$$

where A_{av} is given by (9). The latter together with $x(t)=z(t)-G(t)$ yields system (16).

APPENDIX B: PROOF OF THEOREM 1

(i) First, we assume as in [20], [22] that

$$|x(t)|<\sigma\quad\forall t\ge 0\tag{36}$$

holds for solutions of system (14). Denote $x_t(\theta)=x(t+\theta)$, $\theta\in[-h\sqrt{\varepsilon},0]$. From (14), it follows that

$$x_t(\theta)=\begin{cases}\phi(t+\theta),\quad t+\theta<0,\\ \phi(0)+\int_0^{t+\theta}[A(s)x(s)+\frac{\sqrt{2\pi\alpha}}{\sqrt{\varepsilon}}B_0\cos(\frac{2\pi s}{\varepsilon}\\ \quad+k|Cx(s)+\kappa(s)|^2)+v(s)]ds,\quad t+\theta\ge 0.\end{cases}$$

The latter together with (5), (6) and (18) implies

$$\begin{aligned}
\|x_t\|_{C[-h\sqrt{\varepsilon},0]}\le&\|\phi\|_{C[-h\sqrt{\varepsilon},0]}\\
&+(\tfrac{1}{\sqrt{\varepsilon}}\vartheta_1+\vartheta_2)t+a\int_0^t|x(s)|ds\\
\le&\|\phi\|_{C[-h\sqrt{\varepsilon},0]}+(\vartheta_1+\sqrt{\varepsilon}\vartheta_2)(\sqrt{\varepsilon}+h)\\
&+a\int_0^t\|x_s\|_{C[-h\sqrt{\varepsilon},0]}ds,\quad t\in[0,\varepsilon+h\sqrt{\varepsilon}],
\end{aligned}$$

which by Gronwall's inequality yields

$$\begin{aligned}
|x(t)|\le&\|x_t\|_{C[-h\sqrt{\varepsilon},0]}\le e^{at}[\|\phi\|_{C[-h\sqrt{\varepsilon},0]}\\
&+(\vartheta_1+\sqrt{\varepsilon}\vartheta_2)(\sqrt{\varepsilon}+h)],\quad t\in[0,\varepsilon+h\sqrt{\varepsilon}].
\end{aligned}\tag{37}$$

Then under the initial condition $\|\phi\|_{C[-h\sqrt{\varepsilon},0]}\le\sigma_0$, inequality (23) follows from (37) since (21) implies

$$e^{a(\varepsilon+h\sqrt{\varepsilon})}[\sigma_0+(\vartheta_1+\sqrt{\varepsilon}\vartheta_2)(\sqrt{\varepsilon}+h)]<\sigma$$

for all $\varepsilon\in(0,\varepsilon^*]$ and $t\in[0,\varepsilon+h\sqrt{\varepsilon}]$.

We next prove the first inequality of (23). The solution of system (16) is given by

$$\begin{aligned}
z(t)=&e^{\int_{\varepsilon+h\sqrt{\varepsilon}}^t(A_{av}+\Delta A(\theta))d\theta}z(\varepsilon+h\sqrt{\varepsilon})\\
&+\int_{\varepsilon+h\sqrt{\varepsilon}}^te^{\int_s^t(A_{av}+\Delta A(\theta))d\theta}[-(A_{av}+\Delta A(s))\\
&\times G(s)+\textstyle\sum_{i=1}^2(Y_i(s)+Y_{\kappa_i}(s))+Y_3(s)\\
&+Y_{\dot\kappa}(s))+Y_v(s)+v(s)]ds,\quad t\ge\varepsilon+h\sqrt{\varepsilon}
\end{aligned}$$

leading to

$$\begin{aligned}
|z(t)|\le&\|e^{\int_{\varepsilon+h\sqrt{\varepsilon}}^t(A_{av}+\Delta A(\theta))d\theta}\||z(\varepsilon+h\sqrt{\varepsilon})|\\
&+\int_{\varepsilon+h\sqrt{\varepsilon}}^t\|e^{\int_s^t(A_{av}+\Delta A(\theta))d\theta}\|\big[|(A_{av}+\Delta A(s))G(s)|\\
&+\textstyle\sum_{i=1}^2(|Y_i(s)|+|Y_{\kappa_i}(s)|)+|Y_3(s)|\\
&+|Y_{\dot\kappa}(s)|+|v(s)|+|Y_v(s)|\big]ds,\quad t\ge\varepsilon+h\sqrt{\varepsilon}.
\end{aligned}\tag{38}$$

From (5), (6), (14), (18) and (36), it follows that

$$\begin{aligned}
|\dot x(t)|=&|A(t)x(t)+\tfrac{\sqrt{2\pi\alpha}}{\sqrt{\varepsilon}}B_0\\
&\times\cos(\tfrac{2\pi t}{\varepsilon}+k|Cx(t)+\kappa(t)|^2)+v(t)|\\
<&a\sigma+\tfrac{\sqrt{2\pi\alpha}}{\sqrt{\varepsilon}}(|B_0|+\sqrt{\varepsilon}\Delta b)\\
=&\tfrac{1}{\sqrt{\varepsilon}}\vartheta_1+\vartheta_3,\quad t\ge 0.
\end{aligned}\tag{39}$$

Based on (12), (13), (15) and (39), we obtain for all $t\ge h\sqrt{\varepsilon}$

$$\begin{aligned}
|\kappa(t)|=&\tfrac{1}{h\sqrt{\varepsilon}}\big|\int_{t-h\sqrt{\varepsilon}}^t(s-t+h\sqrt{\varepsilon})\mathcal{I}\dot x(s)ds\big|\\
<&\tfrac12h(\vartheta_1+\sqrt{\varepsilon}\vartheta_3),
\end{aligned}\tag{40}$$

$$\begin{aligned}
|\dot\kappa(t)|\le&\big|\mathcal{I}\dot x(t)\big|+\tfrac{1}{h\sqrt{\varepsilon}}\big|\int_{t-h\sqrt{\varepsilon}}^t\mathcal{I}\dot x(s)ds\big|\\
<&\tfrac{2}{\sqrt{\varepsilon}}\vartheta_1+2\vartheta_3,
\end{aligned}\tag{41}$$

$$\begin{aligned}
|v(t)|=&\sqrt{2\pi\alpha}\big|\Delta B(t)\cos(\tfrac{2\pi t}{\varepsilon}+k|Cx(t)+\kappa(t)|^2)\big|\\
\le&\vartheta_2.
\end{aligned}\tag{42}$$

Then, by using (5), (6), (17), (18), (36) and (39)-(42) we obtain for all $t\ge\varepsilon+h\sqrt{\varepsilon}$

$$\begin{aligned}
|(A_{av}+\Delta A(t))G(t)|=&\tfrac{\sqrt{2\pi\alpha}}{\varepsilon\sqrt{\varepsilon}}\big|\int_{t-\varepsilon}^t(s-t+\varepsilon)\\
&\times(A_{av}+\Delta A(t))B_0\cos(\tfrac{2\pi s}{\varepsilon}+k|Cx(s)+\kappa(t)|^2)ds\big|\\
\le&\tfrac{\sqrt{2\pi\alpha}}{\varepsilon\sqrt{\varepsilon}}(\|A_{av}\|+\Delta a)|B_0|\int_{t-\varepsilon}^t(s-t+\varepsilon)ds\\
=&\tfrac{\sqrt{\varepsilon}}{2}\vartheta_1(\|A_{av}\|+\Delta a),
\end{aligned}\tag{43}$$

$$\begin{aligned}
|Y_1(t)|=&\tfrac{2k\sqrt{2\pi\alpha}}{\varepsilon\sqrt{\varepsilon}}\big|B_0\int_{t-\varepsilon}^t\int_s^t\sin(\tfrac{2\pi s}{\varepsilon}\\
&+k|Cx(\theta)+\kappa(t)|^2)x^T(\theta)C^TCA(\theta)x(\theta)d\theta ds\big|\\
\le&\tfrac{2ka\sigma^2\sqrt{2\pi\alpha}}{\varepsilon\sqrt{\varepsilon}}|B_0||C|^2\int_{t-\varepsilon}^t\int_s^td\theta ds\\
=&\sqrt{\varepsilon}ka\sigma^2\vartheta_1|C|^2,
\end{aligned}\tag{44}$$

$$\begin{aligned}
|Y_2(t)|=&\tfrac{8\pi\alpha k^2}{\varepsilon^2}\big|B_0B_0^TC^TC\int_{t-\varepsilon}^t\int_s^t\int_\theta^t\cos(\tfrac{2\pi}{\varepsilon}(s+\theta)\\
&+2k|Cx(\xi)+\kappa(t)|^2)x(\theta)x^T(\xi)C^TC\dot x(\xi)d\xi d\theta ds\big|\\
<&\tfrac{8\pi\alpha k^2}{\varepsilon^2\sqrt{\varepsilon}}\sigma^2(\vartheta_1+\sqrt{\varepsilon}\vartheta_3)|B_0|^2|C|^4\int_{t-\varepsilon}^t\int_s^t\int_\theta^td\xi d\theta ds\\
=&\tfrac{2\sqrt{\varepsilon}}{3}k^2\sigma^2\vartheta_1^2(\vartheta_1+\sqrt{\varepsilon}\vartheta_3)|C|^4,
\end{aligned}\tag{45}$$

$$\begin{aligned}
|Y_3(t)|=&\tfrac{4\pi\alpha k}{\varepsilon^2}\big|B_0B_0^TC^TC\int_{t-\varepsilon}^t\int_s^t\int_\theta^t\sin(\tfrac{2\pi s}{\varepsilon}+k|Cx(t)\\
&+\kappa(t)|^2)\cos(\tfrac{2\pi\theta}{\varepsilon}+k|Cx(t)+\kappa(t)|^2)\dot x(\xi)d\xi d\theta ds\big|\\
<&\tfrac{4\pi\alpha k}{\varepsilon^2\sqrt{\varepsilon}}(\vartheta_1+\sqrt{\varepsilon}\vartheta_3)|B_0|^2|C|^2\int_{t-\varepsilon}^t\int_s^t\int_\theta^td\xi d\theta ds\\
=&\tfrac{\sqrt{\varepsilon}}{3}k\vartheta_1^2(\vartheta_1+\sqrt{\varepsilon}\vartheta_3)|C|^2,
\end{aligned}\tag{46}$$

$$\begin{aligned}
|Y_{\kappa_1}(t)|=&\tfrac{2k\sqrt{2\pi\alpha}}{\varepsilon\sqrt{\varepsilon}}\big|B_0\int_{t-\varepsilon}^t\int_s^t\sin(\tfrac{2\pi s}{\varepsilon}\\
&+k|Cx(\theta)+\kappa(t)|^2)\kappa^T(t)C\dot x(\theta)d\theta ds\big|\\
<&\tfrac{k\sqrt{2\pi\alpha}}{\varepsilon^2}h(\vartheta_1+\sqrt{\varepsilon}\vartheta_3)^2|B_0||C|\int_{t-\varepsilon}^t\int_s^td\theta ds\\
=&\tfrac{h}{2}k\vartheta_1(\vartheta_1+\sqrt{\varepsilon}\vartheta_3)^2|C|,
\end{aligned}\tag{47}$$

$$\begin{aligned}
|Y_{\kappa_2}(t)|=&\tfrac{8\pi\alpha k^2}{\varepsilon^2}\big|B_0B_0^TC^TC\int_{t-\varepsilon}^t\int_s^t\int_\theta^t\cos(\tfrac{2\pi}{\varepsilon}(s+\theta)\\
&+2k|Cx(\xi)+\kappa(t)|^2)x(\theta)\kappa^T(t)C\dot x(\xi)d\xi\big|\\
<&\tfrac{4\pi\alpha k^2}{\varepsilon^2\sqrt{\varepsilon}}\sigma h(\vartheta_1+\sqrt{\varepsilon}\vartheta_3)^2|B_0|^2|C|^3\int_{t-\varepsilon}^t\int_s^t\int_\theta^td\xi d\theta ds\\
=&\tfrac{h\sqrt{\varepsilon}}{3}k^2\sigma\vartheta_1^2(\vartheta_1+\sqrt{\varepsilon}\vartheta_3)^2|C|^3,
\end{aligned}\tag{48}$$

$$
\begin{aligned}
&|Y_{\dot\kappa}(t)| = \tfrac{8\pi\alpha k^2}{\varepsilon^2}\big|B_0B_0^TC^TC\int_{t-\varepsilon}^t\int_s^t\int_\theta^t \sin(\tfrac{2\pi s}{\varepsilon}\\
&\quad +k|Cx(\theta)+\kappa(t))|^2)\sin(\tfrac{2\pi\theta}{\varepsilon}+k|Cx(\theta)\\
&\quad +\kappa(\xi)|^2)x(\theta)(Cx(\theta)+\kappa(\xi))^T\dot\kappa(\xi)d\xi d\theta ds\big|\\
&< \tfrac{8\pi\alpha k^2}{\varepsilon^2\sqrt{\varepsilon}}\sigma(2\sigma|C|+h(\vartheta_1+\sqrt{\varepsilon}\vartheta_3))(\vartheta_1+\sqrt{\varepsilon}\vartheta_3)\\
&\quad \times|B_0|^2|C|^2\int_{t-\varepsilon}^t\int_s^t\int_\theta^t d\xi d\theta ds\\
&= \tfrac{2\sqrt{\varepsilon}}{3}k^2\sigma\vartheta_1^2(2\sigma|C|+h(\vartheta_1+\sqrt{\varepsilon}\vartheta_3))(\vartheta_1+\sqrt{\varepsilon}\vartheta_3)|C|^2,
\end{aligned} \tag{49}
$$

$$
\begin{aligned}
&|Y_v(t)| = \tfrac{2k\sqrt{2\pi\alpha}}{\varepsilon\sqrt{\varepsilon}}\big|\int_{t-\varepsilon}^t\int_s^t \sin(\tfrac{2\pi s}{\varepsilon}+k|Cx(\theta)\\
&\quad +\kappa(t)|^2)B_0x^T(\theta)C^TCv(\theta)d\theta ds\big|\\
&< \tfrac{2k\sqrt{2\pi\alpha}}{\varepsilon\sqrt{\varepsilon}}\sigma\vartheta_2|B_0||C|^2\int_{t-\varepsilon}^t\int_s^t d\theta ds = \sqrt{\varepsilon}k\sigma\vartheta_1\vartheta_2|C|^2.
\end{aligned} \tag{50}
$$

By using (38), (42)-(50), we obtain

$$
\begin{aligned}
&|z(t)| < \|e^{\int_{\varepsilon+h\sqrt{\varepsilon}}^t(A_{av}+\Delta A(\theta))d\theta}\||z(\varepsilon+h\sqrt{\varepsilon})|\\
&\quad +[\sqrt{\varepsilon}\mu_0+\varepsilon\mu_1+h(\mu_2+\sqrt{\varepsilon}\mu_3+\varepsilon\mu_4+\varepsilon\sqrt{\varepsilon}\mu_5)\\
&\quad +\vartheta_2]\int_{\varepsilon+h\sqrt{\varepsilon}}^t\|e^{\int_s^t(A_{av}+\Delta A(\theta))d\theta}\|ds, \quad t\ge\varepsilon+h\sqrt{\varepsilon},
\end{aligned} \tag{51}
$$

where μ_i $(i=0,\dots,5)$ are given by (22). Assuming as in [21], [23] that there exist scalars $\delta>0$ and $p>1$ satisfying

$$
\begin{aligned}
&\|e^{\int_{\varepsilon+h\sqrt{\varepsilon}}^t(A_{av}+\Delta A(\theta))d\theta}\| \le \sqrt{p}e^{-\delta(t-s)}\\
&\qquad\qquad \forall t\ge s\ge\varepsilon+h\sqrt{\varepsilon}.
\end{aligned} \tag{52}
$$

From (51) and (52), we obtain for $t\ge\varepsilon+h\sqrt{\varepsilon}$

$$
\begin{aligned}
&|z(t)| < \sqrt{p}e^{-\delta(t-\varepsilon-h\sqrt{\varepsilon})}|z(\varepsilon+h\sqrt{\varepsilon})|\\
&\quad +[\sqrt{\varepsilon}\mu_0+\varepsilon\mu_1+h(\mu_2+\sqrt{\varepsilon}\mu_3+\varepsilon\mu_4\\
&\quad +\varepsilon\sqrt{\varepsilon}\mu_5)+\vartheta_2]\int_{\varepsilon+h\sqrt{\varepsilon}}^t\sqrt{p}e^{-\delta(t-s)}ds\\
&\le \sqrt{p}e^{-\delta(t-\varepsilon-h\sqrt{\varepsilon})}|z(\varepsilon+h\sqrt{\varepsilon})|+\tfrac{\sqrt{p}}{\delta}[\sqrt{\varepsilon}\mu_0\\
&\quad +\varepsilon\mu_1+h(\mu_2+\sqrt{\varepsilon}\mu_3+\varepsilon\mu_4+\varepsilon\sqrt{\varepsilon}\mu_5)+\vartheta_2],
\end{aligned}
$$

Moreover, the following holds for $t\ge\varepsilon+h\sqrt{\varepsilon}$:

$$
\begin{aligned}
|x(t)| &= |z(t)-G(t)| \le |z(t)|+|G(t)|\\
&\le |z(t)|+\tfrac{\sqrt{\varepsilon}}{2}\vartheta_1,
\end{aligned} \tag{53}
$$

$$
\begin{aligned}
|z(t)| &= |x(t)+G(t)| \le |x(t)|+|G(t)|\\
&\le |x(t)|+\tfrac{\sqrt{\varepsilon}}{2}\vartheta_1.
\end{aligned} \tag{54}
$$

Thus, we arrive at

$$
\begin{aligned}
&|x(t)| \overset{(53)}{<} \sqrt{p}e^{-\delta(t-\varepsilon-h\sqrt{\varepsilon})}|z(\varepsilon+h\sqrt{\varepsilon})|+\tfrac{\sqrt{p}}{\delta}[\sqrt{\varepsilon}\mu_0\\
&\quad +\varepsilon\mu_1+h(\mu_2+\sqrt{\varepsilon}\mu_3+\varepsilon\mu_4+\varepsilon\sqrt{\varepsilon}\mu_5)+\vartheta_2]+\tfrac{\sqrt{\varepsilon}}{2}\vartheta_1\\
&\overset{(54)}{<} \sqrt{p}e^{-\delta(t-\varepsilon-h\sqrt{\varepsilon})}(|x(\varepsilon+h\sqrt{\varepsilon})|+\tfrac{\sqrt{\varepsilon}}{2}\vartheta_1)+\tfrac{\sqrt{p}}{\delta}[\sqrt{\varepsilon}\mu_0\\
&\quad +\varepsilon\mu_1+h(\mu_2+\sqrt{\varepsilon}\mu_3+\varepsilon\mu_4+\varepsilon\sqrt{\varepsilon}\mu_5)+\vartheta_2]+\tfrac{\sqrt{\varepsilon}}{2}\vartheta_1\\
&\overset{(37)}{<} \sqrt{p}e^{-\delta(t-\varepsilon-h\sqrt{\varepsilon})}\big[e^{a(\varepsilon+h\sqrt{\varepsilon})}\big(\|\phi\|_{C[-h\sqrt{\varepsilon},0]}+(\vartheta_1\\
&\quad +\sqrt{\varepsilon}\vartheta_2)(\sqrt{\varepsilon}+h)\big)+\tfrac{\sqrt{\varepsilon}}{2}\vartheta_1\big]+\tfrac{\sqrt{p}}{\delta}[\sqrt{\varepsilon}\mu_0+\varepsilon\mu_1+h(\mu_2\\
&\quad +\sqrt{\varepsilon}\mu_3+\varepsilon\mu_4+\varepsilon\sqrt{\varepsilon}\mu_5)+\vartheta_2]+\tfrac{\sqrt{\varepsilon}}{2}\vartheta_1, \quad t\ge\varepsilon+h\sqrt{\varepsilon}.
\end{aligned}
$$

This implies the second inequality of (23) for all $\varepsilon\in(0,\varepsilon^*]$ if under the initial condition $\|\phi\|_{C[-h\sqrt{\varepsilon},0]}\le\sigma_0$ the following holds

$$
\begin{aligned}
&\sqrt{p}\big[e^{-\delta(t-\varepsilon^*-h\sqrt{\varepsilon^*})}\big(e^{a(\varepsilon^*+h\sqrt{\varepsilon^*})}\big(\sigma_0+(\vartheta_1+\sqrt{\varepsilon^*}\vartheta_2)\\
&\quad \times(\sqrt{\varepsilon^*}+h)\big)+\tfrac{\sqrt{\varepsilon^*}}{2}\vartheta_1\big)+\tfrac{1}{\delta}(\sqrt{\varepsilon^*}\mu_0+\varepsilon^*\mu_1+h(\mu_2\\
&\quad +\sqrt{\varepsilon^*}\mu_3+\varepsilon^*\mu_4+\varepsilon^*\sqrt{\varepsilon^*}\mu_5)+\vartheta_2)\big] < \sigma-\tfrac{\sqrt{\varepsilon^*}}{2}\vartheta_1.
\end{aligned}
$$

The latter, by squaring both sides, is equivalent to (21).

Finally, by following arguments of [20], [23], it can be proved that inequalities (19) and (20) guarantee (52) whereas inequality (21) results in (36).

(ii) The proof of the feasibility of inequalities (19)-(21) is similar to Remark 2 in [21]. This completes the proof.

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