

Quantification of Distributionally Robust Risk of Cascade of Failures in Platoon of Vehicles

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Abstract—Achieving safety is a critical aspect of attaining autonomy in a platoon of autonomous vehicles. In this paper, we propose a distributionally robust risk framework to investigate cascading failures in platoons. To examine the impact of network connectivity and system dynamics on the emergence of cascading failures, we consider a time-delayed network model of the platoon of vehicles as a benchmark. To study the cascading effects among pairs of vehicles in the platoon, we use the measure of conditional distributionally robust functional. We extend the risk framework to quantify cascading failures by utilizing a bi-variate normal distribution. Our work establishes closed-form risk formulas that illustrate the effects of time-delay, noise statistics, underlying communication graph, and sets of soft failures. The insights gained from our research can be applied to design safe platoons that are robust to the risk of cascading failures. We validate our results through extensive simulations.

I. INTRODUCTION

Networked systems are omnipresent and play a vital role in various engineering applications, ranging from power networks to platoon of autonomous cars. However, the performance of network dynamical systems is often affected by communication issues and external disturbances. These disturbances can steer the system from an optimal state of operation towards suboptimal conditions or even complete network failure. This phenomenon is frequently observed in power networks, supply chains, and financial systems, where events leading to system breakdown occur frequently [1, 2, 4, 5, 8].

The intricacy of network systems often comes with fragility, which can result in a complete breakdown of the system. In consensus networks, where a group of agents negotiates and decides on a particular action, the existence of time delays and environmental noise can cause an agent to deviate from the consensus. The presence of uncertainty in these networks has piqued researchers' interest in the risk analysis of complex dynamical systems [18, 19]. Investigating these phenomena is crucial from a system design standpoint, as the failure of one agent may trigger a domino effect throughout the network.

In this paper, we develop a theoretical framework to quantify distributionally robust risk to evaluate the effect of cascading failures in the consensus network. Our objective is

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to develop a risk framework robust to uncertainty in the probability measures characterizing the stochastic disturbance. We also aim to highlight how the failures in one part of the network affects the safety of other components in the network. Quantification of such phenomena offers valuable insight in designing a reliable and robust network that can minimize the risk of failure of the system or allow the system to take preventive measures to nullify such effects.

Our Contributions: Continuing with our previous work on the risk analysis of networked systems [15, 10, 9], this paper is a first step towards developing a distributionally robust cascading risk framework. First, we employ conditional distributionally robust functional to define risk which is later quantified using steady state statistics of time-delayed system of platoon of vehicles. In particular, explicit formula is obtained for distributionally robust cascading risk. We also provide a lower bound for the distributionally robust cascading risk as function of eigenvalues of covariance matrix. Finally, with extensive simulations, we present the cascading risk profile of vehicles in platoon for different scenarios.

II. MATHEMATICAL NOTATION

We denote the non-negative orthant of the Euclidean space \mathbb{R}^n by \mathbb{R}_+^n , its set of standard Euclidean basis by $\{e_1, \dots, e_n\}$, and the vector of all ones by $\mathbf{1}_n = [1, \dots, 1]^T$. The indicator function of a set A is represented by $\mathbf{1}_{[A]}$. We utilize the notation $\tilde{e}_i = e_{i+1} - e_i$ for all $i \in \{1, \dots, n-1\}$. We reserve the notation S_+^n , S_{++}^n to denote the cone of symmetric positive semidefinite and symmetric positive definite $n \times n$ matrices respectively. For every $X_1, X_2 \in S_{++}^n$, we write $X_2 \preceq X_1$ if and only if $X_1 - X_2 \in S_+^n$. The $n \times n$ identity matrix is denoted by I_n .

Algebraic Graph Theory: A weighted graph is defined by $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \omega)$, where \mathcal{V} is the set of nodes, \mathcal{E} is the set of edges (feedback links), and $\omega : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}_+$ is the weight function that assigns a non-negative number (feedback gain) to every link. Two nodes are directly connected if and only if $(i, j) \in \mathcal{E}$.

Assumption 1. Every graph in this paper is connected. In addition, for every $i, j \in \mathcal{V}$, the following properties hold:

- $\omega(i, j) > 0$ if and only if $(i, j) \in \mathcal{E}$.
- $\omega(i, j) = \omega(j, i)$, i.e., links are undirected.
- $\omega(i, i) = 0$, i.e., links are simple.

The Laplacian matrix of \mathcal{G} is a $n \times n$ matrix $L = [l_{ij}]$

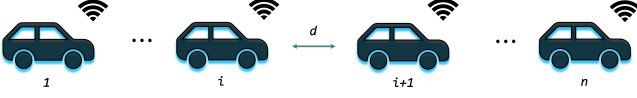


Fig. 1: Schematic of a platoon. Vehicles try to maintain a fixed distance d and achieve the same velocity by updating the feedback using the communication network.

with elements

$$l_{ij} := \begin{cases} -k_{i,j} & \text{if } i \neq j \\ k_{i,1} + \dots + k_{i,n} & \text{if } i = j \end{cases},$$

where $k_{i,j} := \omega(i, j)$. Laplacian matrix of a graph is symmetric and positive semi-definite [16]. Assumption 1 implies the smallest Laplacian eigenvalue is zero with algebraic multiplicity one. The spectrum of L can be ordered as $0 = \lambda_1 < \lambda_2 \leq \dots \leq \lambda_n$. The eigenvector of L corresponding to λ_k is denoted by \mathbf{q}_k . By letting $Q = [\mathbf{q}_1 | \dots | \mathbf{q}_n]$, it follows that $L = Q\Lambda Q^T$ with $\Lambda = \text{diag}[0, \lambda_2, \dots, \lambda_n]$. We normalize the Laplacian eigenvectors such that Q becomes an orthogonal matrix, i.e., $Q^T Q = Q Q^T = I_n$ with $\mathbf{q}_1 = \frac{1}{\sqrt{n}} \mathbf{1}_n$.

Probability Theory: Let $\mathcal{L}^2(\mathbb{R}^q)$ be the set of all \mathbb{R}^q -valued random vectors $\mathbf{z} = [z^{(1)}, \dots, z^{(q)}]^T$ of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with finite second moments. A normal random variable $\mathbf{y} \in \mathbb{R}^q$ with mean $\boldsymbol{\mu} \in \mathbb{R}^q$ and $q \times q$ covariance matrix Σ is represented by $\mathbf{y} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$. The error function $\text{erf}: \mathbb{R} \rightarrow (-1, 1)$ is $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$. We employ standard notation $d\xi_t$ for the formulation of stochastic differential equations.

III. PROBLEM STATEMENT

We consider a system of platoon of n vehicles moving along straight line as shown in figure 1. The vehicles are labeled in descending order such that the n 'th vehicle is the leader of the platoon. We denote the state of the i 'th vehicle by $[x_t^{(i)}; v_t^{(i)}]$, where $x_t^{(i)}$ and $v_t^{(i)}$ are the position and velocity of the i 'th vehicle respectively. The second order dynamics of i 'th vehicle in the platoon is expressed by the following first order linear stochastic differential equations,

$$\begin{aligned} dx_t^{(i)} &= v_t^{(i)} dt \\ dv_t^{(i)} &= u_t^{(i)} dt + g d\xi_t^{(i)}, \end{aligned} \quad (1)$$

where $u_t^{(i)}$ is the control input of the i 'th vehicle. We model the exogenous disturbance from the environment entering the dynamics of the i 'th vehicle using the additive term $g d\xi_t^{(i)}$, where $d\xi_t^{(i)}$ is the differential Brownian motion¹ and $g \in \mathbb{R}_{++}$ is the diffusion coefficient independent of the state and control input of the i 'th vehicle. The diffusion coefficient g which, is a measure of the covariance of the noise entering the vehicles is assumed identical for all the vehicles. We define the vehicles to form a platoon if they

¹The stochastic process $\xi_t^{(i)} \in \mathcal{L}^2(\mathbb{R})$ denotes the real valued Wiener process.

satisfy the following conditions in the steady state: (i) the pairwise difference between the position coordinate of every two consecutive vehicles converges to zero for all $i, j \in \{1, \dots, n\}$; and (ii) all vehicles attain the same constant velocity in the steady state. The control input that satisfies the previous two conditions is determined by the following feedback control law [17]

$$\begin{aligned} u_t^{(i)} &= \sum_{j=1}^n k_{ij} (v_{t-\tau}^j - v_{t-\tau}^i) \\ &+ \beta \sum_{j=1}^n k_{ij} (x_{t-\tau}^j - x_{t-\tau}^i - (d_j - d_i)). \end{aligned} \quad (2)$$

The parameter β is a weighting parameter that assigns relative importance to the velocity and position term in the control law. In order to account for the non-zero time in the communication channel between the vehicles, we assume that the states in control input are delayed. We denote this time delay using the parameter $\tau \in \mathbb{R}_{++}$, which is assumed constant for all the vehicles.

In order to write the dynamics of the platoon, we introduce the following notations. We define a vector of positions, velocities and exogenous noise input as $\mathbf{x}_t = [x_t^{(1)}, \dots, x_t^{(n)}]^T$, $\mathbf{v}_t = [v_t^{(1)}, \dots, v_t^{(n)}]^T$, and $\boldsymbol{\xi}_t = [\xi_t^{(1)}, \dots, \xi_t^{(n)}]^T$ respectively. We define the vector of target distance from the leader of the platoon $\mathbf{y} = [d, 2d, \dots, nd]^T$, where $d \in \mathbb{R}_{++}$. Applying the control law (2) in (1) for all $i = 1, \dots, n$, the closed loop dynamics of the network of n vehicles with graph Laplacian L , can be represented by the following $2n$ first order stochastic delayed differential equations,

$$\begin{aligned} d\mathbf{x}_t &= \mathbf{v}_t dt \\ d\mathbf{v}_t &= -L\mathbf{v}_{t-\tau} dt - \beta L(\mathbf{x}_{t-\tau} - \mathbf{y}) dt + g d\boldsymbol{\xi}_t, \end{aligned} \quad (3)$$

for all $t \geq 0$ and given deterministic initial conditions for \mathbf{x}_t and \mathbf{v}_t for $t \in [-\tau, 0]$. The standard result [11], [7] guarantees that the states $\{(\mathbf{x}_t, \mathbf{v}_t)\}_{t \geq -\tau}$ of stochastic delayed differential equation (3) represent a well defined stochastic process.

The problem is to quantify the distributionally robust risk of a cascade of failures as a function of communication graph Laplacian, time-delay and statistics of noise, conditioned on the fact that at another location, the inter-vehicle distance of consecutive pair of vehicles lies within a certain set. Since the exact knowledge of probability measures characterizing the stochastic disturbance is not known in practice, it becomes important to develop risk models robust to changes in noise statistics. To this end, we develop a distributionally robust systemic risk framework to study the cascade of failures using the steady state statistics of the closed loop system (3). The rest of the paper is organized as follows. In §IV, we state the stability conditions for the platoon, obtain steady state statistics required to quantify risk, and present a formal framework of distributionally robust risk measure. These results help us to quantify the distributionally robust cascading risk for the platoon of vehicles in §V, which

constitutes the main contribution of this work. In §VI, we examine risk profile simulation results for some special graph topologies. The proofs of all theoretical results are provided in [12].

IV. PRELIMINARY RESULTS

In this section, we present the conditions for the stability of platoon, and the steady state statistics of the inter-vehicle distance which is later utilized in the risk analysis. We also introduce a formal framework of distributionally robust cascading risk measure using conditional distributionally robust functionals.

A. Stability of the Deterministic Platoon

We define the convergence of the platoon if the steady-state distance between pair of consecutive vehicles attain the same constant value d , and the velocity of all the vehicles reach the same constant value. These conditions can be equivalently stated as $\lim_{t \rightarrow \infty} |v_t^{(i)} - v_t^{(j)}| = 0$ and $\lim_{t \rightarrow \infty} |x_t^{(i)} - x_t^{(j)} - (i - j)d| = 0$. Using the result in [17] [3], we conclude that the deterministic platoon converges if and only if

$$(\lambda_i \tau, \beta \tau) \in S, \quad (4)$$

where λ_i 's for all $i = 1, \dots, n$ are the eigenvalues of the graph Laplacian, and the set S is as defined below,

$$S = \left\{ (s_1, s_2) \in \mathbb{R}^2 \mid s_1 \in \left(0, \frac{\pi}{2}\right), s_2 \in \left(0, \frac{a}{\tan(a)}\right), \right. \\ \left. \text{for } a \in \left(0, \frac{\pi}{2}\right) \right\}, \text{ the solutions of a } \sin(a) = s_1 \Big\}.$$

The results in this paper are derived when the closed loop dynamics of the platoon is stable, i.e., when (4) holds.

B. Steady State Statistics of inter-vehicle Distances

Since the graph Laplacian $L \in S_+^n$, we utilize the spectral decomposition $L = Q\Lambda Q^T$ to decouple (3) using the following coordinate transformation

$$z_t = Q^T(x_t - y), \text{ and } v_t = Q^T v_t. \quad (5)$$

The closed loop dynamics of the platoon in the transformed coordinate system can be written as

$$dz_t = v_t dt \\ dv_t = -\Lambda v_t - \tau dt - \beta \Lambda(z_{t-\tau} - v)dt + gQ^T d\xi_t. \quad (6)$$

The solution of (6) satisfying the stability condition given by (4) can be written as

$$\begin{bmatrix} z_t^{(i)} \\ v_t^{(i)} \end{bmatrix} = \Xi(z_{[-\tau, 0]}^{(i)}; v_{[-\tau, 0]}^{(i)}; \Phi_i(t)) + g \int_0^t \Phi_i(t-s) B_i d\xi_s, \quad (7)$$

where $B_i = [0_{1 \times n}, q_i]^T$, $\Phi_i(t)$ is the principal solution of the deterministic platoon, and both $\Phi_i(t)$ and $\Gamma(\cdot; \cdot; \cdot)$ decay exponentially for all $i \in \{2, \dots, n\}$. The steady state statistics of z_t follows a multivariate normal distribution [14], i.e.,

$$\bar{z} \sim \mathcal{N}(0 \cdot \mathbf{1}_n, \Sigma_z),$$

where $\Sigma_z = \text{diag}\{\sigma_{z_1}^2, \dots, \sigma_{z_n}^2\}$, in which $\sigma_{z_i}^2 = \frac{g^2 \tau^3}{2\pi} f(\lambda_i \tau, \beta \tau)$ and

$$f(s_1, s_2) = \int_{\mathbb{R}} \frac{dr}{(s_1 s_2 - r^2 \cos(r))^2 + r^2 (s_1 - r \sin(r))^2}. \quad (8)$$

Inverting the transformation (5), we have

$$x_t = Qz_t + y. \quad (9)$$

We define the steady state distance vector of the platoon as $\bar{d} \in \mathbb{R}^{n-1}$ [10] such that

$$\bar{d} = D^T Q \bar{z} + d \mathbf{1}_{n-1}, \quad (10)$$

where $D = [\tilde{e}_1 | \dots | \tilde{e}_{n-1}]$. The i 'th element in \bar{d} , i.e., \bar{d}_i denotes the steady state distance between the i 'th and $(i + 1)$ 'th vehicle in the platoon for all $i \in \{1, \dots, n-1\}$. Using the result in [10], the steady state distance vector \bar{d} follows a multivariate normal distribution

$$\bar{d} \sim \mathcal{N}(d \mathbf{1}_{n-1}, \Sigma),$$

such that $\Sigma \in S_{++}^{n-1}$ is given by

$$\Sigma = D^T Q \mathbb{E}[\bar{z} \bar{z}^T] Q^T D, \quad (11)$$

In terms of individual elements, $\Sigma = [\sigma_{ij}]$ is given as

$$\sigma_{ij} = g^2 \frac{\tau^3}{2\pi} \sum_{k=1}^n (\tilde{e}_i^T q_k) (\tilde{e}_j^T q_k) f(\lambda_k \tau, \beta \tau), \quad (12)$$

for all $i, j \in \{1, \dots, n-1\}$ and $(\lambda_k \tau, \beta \tau)$ as defined in (8). For simplicity of notation, we denote the diagonal elements σ_{ii} as σ_i^2 .

C. Systemic Event and Systemic Set

A systemic event is a failure that will potentially lead to an overall malfunction of the network [14, 10]. In probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the set of systemic events of random variable $y : \Omega \rightarrow \mathbb{R}$ is defined as $\{\omega \in \Omega \mid y(\omega) \in \mathcal{W}^*\}$. We define a collections of supersets $\{\mathcal{W}_\delta \mid \delta \in [0, \infty]\}$ of \mathcal{W}^* that satisfy the following conditions for any sequence $\{\delta_n\}_{n=1}^\infty$ with property $\lim_{n \rightarrow \infty} \delta_n = \infty$

- $\mathcal{W}_{\delta_2} \subset \mathcal{W}_{\delta_1}$ when $\delta_1 < \delta_2$
- $\lim_{n \rightarrow \infty} \mathcal{W}_{\delta_n} = \bigcap_{n=1}^\infty \mathcal{W}_{\delta_n} = \mathcal{W}^*$.

Suppose that the knowledge of a soft failure is given a priori such that the severity of the corresponding event can be represented by $\delta = \inf\{\delta \mid y \in \mathcal{W}_\delta\}$, where the value of δ implies how dangerously the current state y is close to the systemic event. Using the systemic level sets to represent the existing soft failures allows for a more systematic and rigorous risk assessment and management approach. By defining different risk levels, one can evaluate the trade-offs between system performance and safety and choose an appropriate level of risk based on the specific application and requirements.

We denote the sigma-algebra generated by a particular superset \mathcal{W}_{δ_i} as $\mathcal{G}_i \subseteq \mathcal{F}$ such that $\mathcal{G}_i = \{\phi, \mathcal{W}_{\delta_i}, \mathcal{W}_{\delta_i}^c, \Omega\}$, where $\mathcal{W}_{\delta_i}^c$ is the complement of the set \mathcal{W}_{δ_i} .

D. Distributionally Robust Risk Measure

To develop risk measure robust to uncertainty in probability distribution of a random variable, we quantify a set of probability distributions using the notion of ambiguity set as defined in 1.

Definition 1. *Ambiguity Set of Probability Measures [13]:* Let (Ω, \mathcal{F}) be measurable space. We define ambiguity set \mathcal{M} as a nonempty set of probability measures (distributions) on (Ω, \mathcal{F}) contained in a ball of a specified radius centered at a reference probability measure.

For exposition of our next result, we introduce the input and the output ambiguity set to characterize the set of probability measures of exogenous input noise vector and steady state inter-vehicle distance vector of the system respectively.

Consider the system

$$\begin{aligned} d\tilde{x}_t &= \tilde{v}_t dt \\ d\tilde{v}_t &= -L\tilde{v}_{t-\tau} dt - \beta(\tilde{L}\tilde{x}_{t-\tau} - y)dt + E d\xi_t. \end{aligned} \quad (13)$$

The system governed by (13) may arise if we consider communication noise in the control law. (13) reduces to (3) for $E = gI_n$.

Since the input noise vector and steady state inter-vehicle distance vector of the systems governed by (3) and (13) are normally distributed, we consider ambiguity set of multivariate normal distribution. Furthermore, since there is no ambiguity in mean of input noise which is assumed to be zero, we quantify the ambiguity set in terms of covariance matrix.

In the next result, we quantify ambiguity set of output \mathcal{M}_{out} given the input noise ambiguity set \mathcal{M}_{in} . We assume the input noise has multivariate normal distribution with mean zero and uncertain covariance matrix $\tilde{\Gamma} \in S_{++}^{n-1}$. As a result, the steady state inter-vehicle distance vector has multivariate normal distribution with mean $d\mathbf{1}_{n-1}$ and uncertain covariance matrix $\tilde{\Sigma} \in S_{++}^{n-1}$.

Lemma 1. *For the system given by (13), with the input ambiguity set*

$$\mathcal{M}_{in} = \left\{ \tilde{\Gamma} \mid (1 - \epsilon)\tilde{\Gamma}_0 \preceq \tilde{\Gamma} \preceq (1 + \epsilon)\tilde{\Gamma}_0 \right\}, \quad (14)$$

we can quantify the output ambiguity set as

$$\mathcal{M}_{out} = \left\{ \tilde{\Sigma} \mid \tilde{\Sigma}_0(1 - \epsilon) \preceq \tilde{\Sigma} \preceq \tilde{\Sigma}_0(1 + \epsilon) \right\}, \quad (15)$$

where $\tilde{\Gamma}_0 \in S_{++}^{n-1}$ is the reference covariance matrix of the input noise, $\tilde{\Sigma}_0 \in S_{++}^{n-1}$ is the output covariance matrix corresponding to $\tilde{\Gamma}_0$ and $\epsilon \in (0, 1)$ is a measure of radius of the ambiguity set.

In order to define the distributionally robust risk measure, we introduce the definition of conditional distributionally robust functional.

Definition 2. *Conditional Distributionally Robust Functional[13]:* Let (Ω, \mathcal{F}) be measurable space and \mathcal{M} be an

ambiguity set on (Ω, \mathcal{F}) . For a random variable $Z : \Omega \rightarrow \mathbb{R}$, we define conditional distributionally robust functional as

$$\mathfrak{R} := \inf_{\mathbb{P} \in \mathcal{M}} \mathbb{E}_{\mathbb{P}|G_i}[Z], \quad (16)$$

where $\mathbb{E}_{\mathbb{P}|G_i}[Z]$ is the conditional expectation of Z given G_i .

We now introduce our distributionally robust risk measure for cascading failures. Motivated by [6, 13], we define our dimensionless distributionally robust cascading risk measure as

$$\mathcal{R} := \frac{\kappa}{\inf_{\mathbb{P} \in \mathcal{M}} \mathbb{E}_{\mathbb{P}|G_i}[Z]} + \iota, \quad (17)$$

where κ has same dimension as Z and ι is a dimensionless constant. We interpret (17) as risk of the random variable Z given G_i .

V. DISTRIBUTIONALLY ROBUST CASCADING RISK

In this section, we provide an explicit expression to quantify the distributionally robust cascading risk of soft failures in the platoon of vehicles for a specific type of ambiguity set.

A. Quantification of Distributionally Robust Cascading Risk

For the platoon system (3), let us assume that vehicles are labelled as $1, \dots, n$ in accordance with 1. The main scenario assumes that the inter-vehicle distance of the i 'th pair lies in a certain set $0 \leq \bar{d}_i \leq d$ and we study the risk for the j 'th pair conditioned on \bar{d}_i . To quantify the cascading risk of soft failures, we consider that in the steady state, the inter-vehicle distance of the i 'th pair in the platoon lies in the level set

$$\{\bar{d}_i \in \mathcal{W}_{\delta_i}\}, \text{ where } \mathcal{W}_{\delta_i} = \left(-\infty, \frac{d}{\delta_i + c} \right),$$

and $\delta_i \in \mathbb{R}_+$ can be interpreted as a measure of safety of the i 'th pair of vehicles. For notational simplicity, we denote $d_i^* = \frac{d}{\delta_i + c}$, where $d_i^* \in \mathbb{R}_+$.

For the exposition of our next results, we introduce the ambiguity set that quantifies the ambiguity in the diffusion coefficient g . Let us assume that we have an estimate of g given by g_0 and g is specified by the set $(1 - \epsilon)g_0^2 \leq g^2 \leq (1 + \epsilon)g_0^2$. Then using (12), we conclude that

$$(1 - \epsilon)\sigma_{i,0}^2 \leq \sigma_i^2 \leq (1 + \epsilon)\sigma_{i,0}^2, \quad (18)$$

where $\sigma_{i,0}^2$ are the diagonal elements of covariance matrix corresponding to g_0 for all $i \in \{1, \dots, n-1\}$. We denote the ambiguity set (18) by \mathcal{M}_g .

Theorem 1. *Suppose that platoon (3) reaches the steady state, and the inter-vehicle distance of the i 'th pair, \bar{d}_i belongs to the level set \mathcal{W}_{δ_i} . For $i, j \in \{1, \dots, n-1\}$ and $i \neq j$, the conditional expected value of inter-vehicle distance of j 'th pair, \bar{d}_j given \bar{d}_i can be expressed as*

$$\mathbb{E}_{\mathbb{P}}[\bar{d}_j | \bar{d}_i \in \mathcal{W}_{\delta_i}] = d - \sqrt{\frac{2}{\pi}} \frac{\rho_{ji} \sigma_j e^{-\frac{(d_i^* - d)^2}{2\sigma_i^2}}}{1 + \operatorname{erf}\left(\frac{d_i^* - d}{\sqrt{2}\sigma_i}\right)},$$

where $\rho_{ji} = \frac{\sigma_{ji}}{\sigma_i \sigma_j}$ is the correlation between the inter-vehicle distance of the i 'th and the j 'th pair of vehicles.

The above theorem addresses the conditional expectation of inter-vehicle distance of j 'th pair given that the inter-vehicle distance of the i 'th pair is less than $d_i^* \in \mathbb{R}_+$. When there is no correlation between i 'th and j 'th pair of vehicles, the conditional expected value is the expected value with respect to the trivial σ -algebra, which equals d as the random vector \vec{d} is normally distributed with mean $d\mathbf{1}_{n-1}$. This can also be concluded from the Theorem 1 by setting $\rho_{ji} = 0$.

For the exposition of our next result, we introduce the notation

$$h(\epsilon) = d - \sqrt{\frac{2}{\pi}} \frac{\rho_{ji} \sigma_{j,0} \sqrt{1 + \epsilon} e^{-\frac{(d^* - d)^2}{2\sigma_{i,0}^2(1+\epsilon)}}}{\left(1 + \operatorname{erf}\left(\frac{(d^* - d)}{\sqrt{2(1+\epsilon)}\sigma_{i,0}}\right)\right)}. \quad (19)$$

In order to quantify the cascading risk for the system of platoon (3), we choose $\kappa = d$ and $\iota = -1$ such that the distributionally robust cascading risk of the j 'th pair conditioned on i 'th pair is

$$\mathcal{R}^{ji} := \frac{d}{\inf_{\mathcal{M}_g} \mathbb{E}_{\mathbb{P}}[\bar{d}_j | \bar{d}_i \in \mathcal{W}_{\delta_i}]} - 1. \quad (20)$$

Theorem 2. *The conditional expected value of inter-vehicle distance derived in Theorem 1 is a monotonic increasing function of g for $\rho < 0$ and monotonic decreasing function of g for $\rho > 0$ and for the ambiguity set (18), the distributionally robust cascading risk defined in (20) is given by*

$$\mathcal{R}^{ji} := \begin{cases} \frac{d}{h(-\epsilon)} - 1, & \text{if } \rho_{ji} < 0 \\ \frac{d}{h(\epsilon)} - 1, & \text{if } \rho_{ji} > 0 \\ 0, & \text{if } \rho_{ji} = 0, \end{cases}$$

where $h(\cdot)$ is as defined in (19).

VI. CASE STUDIES

We discuss the case studies for the system of platoon (3) for the communication network governed by the complete graph, the path graph and the p -cycle graphs. We present various cases depending on the location of the i 'th pair in the platoon for different collection of supersets \mathcal{W}_{δ_i} . We set the number of agents $n = 50$, the steady state inter-vehicle distance $d = 2$, and $c = 1$ for all simulations. We set $\epsilon = 0.2$ for simulations in Fig. 2, which shows the risk profile as a function of δ_i and Fig 4, where we show the risk profile for different locations of the i 'th pair in the platoon. The Fig 3 shows the risk profile for different values of ϵ .

1) Complete Graph: We set $g_0 = 10, \beta = 1$ and $\tau = 0.02$. As evident from Fig 2, the cascading effect is limited to the immediate neighbour of the location of the i 'th pair. This is due to the result of the covariance matrix for the complete graph being a tri-diagonal matrix [10]. Hence, the correlation ρ_{ji} for all pair j such that $|j - i| > 1$ equals 0. This risk being zero for all pair j for which $\rho_{ji} = 0$ is also supported

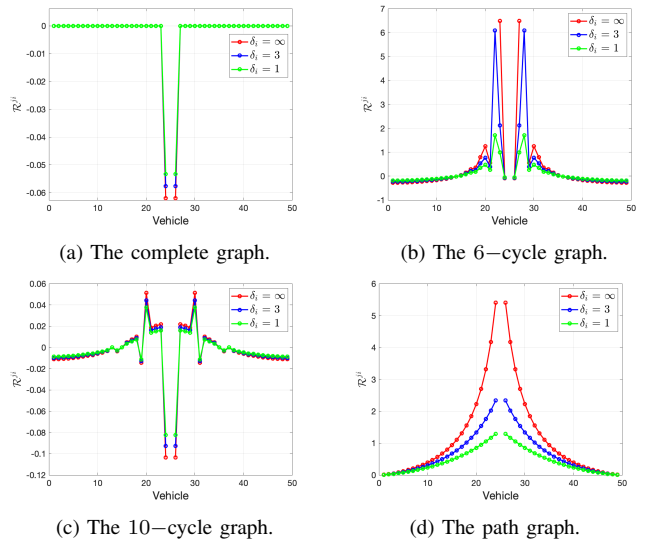


Fig. 2: The distributionally robust cascading risk profile for different value of δ_i for various graph topologies.

by the Theorem 1. We also notice that the risk profile for the complete graph shifts depending upon the location of the i 'th pair as shown in Fig. 4.

2) Path Graph: We set $g_0 = 0.25, \beta = 4$ and $\tau = 0.05$. We find that the cascading risk profile decays as the distance between the j 'th pair and the i 'th pair increases as shown in Fig. 2 and Fig. 4. The magnitude of the risk depends upon the location of i 'th pair as shown in 4. The location of the i 'th pair in the middle leads to a higher risk to neighbouring pair of vehicles compared to location of i 'th pair at the start or the end of the platoon.

2) p -Cycle Graph: We set $g_0 = 4, \beta = 2$ and $\tau = 0.01$. We present the cascading risk profile for $p = 6$ and $p = 10$. As expected, the risk profile converges towards the profile for the complete graph as p increases. The location of the i 'th pair plays a very important role here as we can see in Fig 2 and Fig 4. When the i 'th pair is in middle, we get a symmetric profile but when the i 'th pair lies at either end, the cascading risk profile is more pronounced for vehicles at the start and the end of the platoon while the vehicles in the middle of the platoon are relatively safe.

The Fig. 3 outlines the importance of considering the ambiguity set of probability measures to quantify the risk as the steady state statistics of the random variable may be different for different probability measures in the ambiguity set. The monotonicity of the conditional expected value with ϵ as stated in Theorem 2 is evident in Fig 3.

VII. CONCLUSION

This work is our first step towards developing a distributionally robust risk framework for the analysis of cascade of failures (e.g., collision or soft failures) in networked dynamical systems. We obtain explicit formulas to quantify ambiguity set and distributionally robust cascading risk. Both our theoretical results and simulations show how the change in steady state statistics of the input noise affects

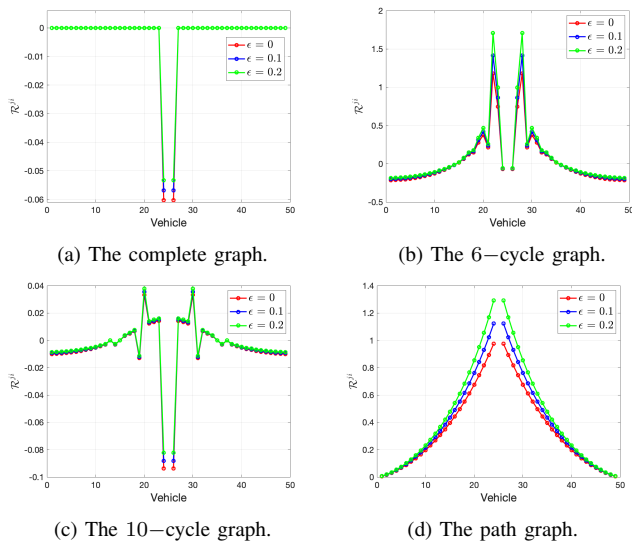


Fig. 3: The cascading risk profile conditioned on different value of ϵ for various graph topologies.

the risk profile of vehicles in the platoon emphasizing the importance of the ambiguity set of probability measures in risk calculation.

Some interesting extensions of the current risk framework includes developing closed form distributionally robust cascading risk formulas by conditioning distributionally robust functional on multiple pair of inter-vehicle distance. Our future work also focuses on extending this framework for a wider range of ambiguity set of probability measures. Lastly, we would like to explore how network parameters such as connectivity and effective resistance impose a fundamental limit on minimum achievable risk for the networked system.

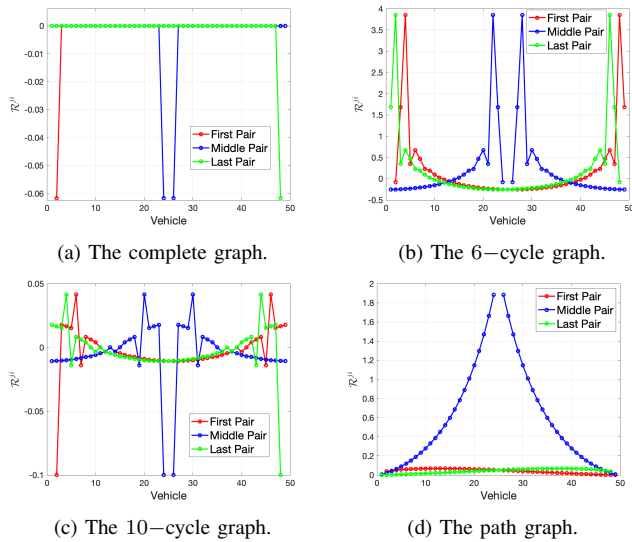


Fig. 4: The cascading risk profile at various location of the i 'th pair for various graph topologies for $\delta_i = 3$.

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