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Finite-horizon optimal control of continuous-time stochastic systems with terminal cost of Wasserstein distance

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Abstract— **This study addresses a stochastic optimal control problem for continuous-time systems aimed at steering a probability distribution of the terminal state towards a desired probability distribution. The problem formulation incorporates the Wasserstein distance, a metric of the space of probability measures, in the cost functional. We provide an optimality condition for this optimal control problem in the form of Pontryagin's minimal principle. The condition is obtained by carefully examining the properties of the Wasserstein distance. Consequently, we obtain the optimality condition described by a forward-backward stochastic differential equation and a Kantorovich potential, which appears in optimal transport theory.**

I. INTRODUCTION

This study addresses optimality conditions in the continuous-time stochastic optimal control problem with the cost given by the Wasserstein distance.

Recently, the field of stochastic control theory has focused on control problems that take into account of probability distributions associated with systems [1], [2], [3], [4], [5], [6], [7], [8], [9]. This trend has been invoked by the developments of the optimal transport theory [10], [11], [12], [13] and mean-field game and control theory [14], [15], [16]. The optimal transport theory enables analysis of the space of probability measures geometrically and analytically. Additionally, the mean-field control theory has developed techniques to handle control problems that require explicit considerations of probability distributions, such as control problems of McKean-Vlasov stochastic differential equations (SDEs) [16]. The integration of this trend with control theory has led to the development of numerous control approaches that handle explicitly probability distributions associated with control systems [1], [2], [3], [4], [5], [6], [7]. These methods enable not only the control of covariances, but also the steering of a probability distribution towards a target distribution.

The steering of probability distributions has the potential for various applications. The applications can be found in the control of the autonomous vehicle [17] and the operation of car-sharing services [18]. Additionally, the control of probability distributions is expected to contribute to deep learning. A fundamental issue in deep learning is handling data and its distributions. In particular, the fundamental problem of generative modeling, exemplified by generative adversarial networks [19], [20], normalizing flows [21], diffusion models [22], aims to construct probability distributions to approximate data distributions. Accordingly, the design of neural networks can be viewed as the optimization of probability distributions associated with the networks. Some studies investigate the generative models from the perspective of stochastic control theory [23], [24], [25].

The steering problem of probability distributions can be considered in various settings. In addition to the optimal transport and mean-field control-based approaches, the study [1] investigates the design of feedback controllers to steer a distribution to a steady state distribution. The study [2] also considers the optimal control of the Liouville equation. Moreover, the studies [26], [3], [4], [5], [6], [27] provide approaches to steer probability distributions based on the optimal transport theory. In addition, the studies [17], [7] address the covariance steering. Of these studies, the study [4] addresses the finite-time horizon optimal control problem for steering the terminal distribution of linear systems by introducing a metric of probability distributions, the Wasserstein distance. The study [27] further explores the control problem for discrete-time deterministic nonlinear systems. Incorporating the Wasserstein distances in terminal costs enables the evaluation of the proximity between the probability distribution of a terminal state and a given desired distribution. In deep learning, the Wasserstein distance is used as a loss function and minimized in the learning [20]. Additionally, the network of generative modeling can be implemented as a dynamical system, as demonstrated in the study of neural ordinary differential equations [28]. In this context, determining the weights of such a network can be viewed as finding a control for a dynamical system. Consequently, the minimization of the Wasserstein distance in the deep learning problem can be viewed as the optimal control with the Wasserstein distance. The optimal control with the Wasserstein distance can be expected to provide theoretical foundations for generative modeling. Recently, continuous-time stochastic dynamical systems are increasingly employed in the field [22], [23], [24], [25]. Therefore, it is desirable to address the optimal control problem for continuous-time stochastic systems as well as the nonlinearity. However, the above-mentioned paper [27] do not address the optimal control problem for the continuous-time nonlinear stochastic systems.

The present work aims to address the finite horizon optimal control problem for continuous-time stochastic systems using the terminal costs given by the Wasserstein distance. To consider the stochasticity and nonlinearity of dynamical systems in the control problem, we focus on the

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control system described by a nonlinear SDE. As the main contribution, this study provides the optimality condition in this problem. We provide the condition for stochastic systems using Pontryagin's minimum principle. In contrast, the previous study [27] has only addressed the problem for deterministic discrete-time systems. In standard deterministic optimal control problems, Pontryagin's minimum principle offers the optimality condition as Hamiltonian systems, a forward-backward differential equation. For standard stochastic optimal control problems, the stochastic Pontryagin's minimum principle has been established based on forward-backward SDEs (FBSDEs) [29]. This study provides the optimality condition for the optimal control problem with the Wasserstein distance based on the stochastic Pontryagin's minimum principle. To obtain the condition for this problem as a form of the stochastic Pontryagin's minimum principle, it is necessary to carefully examine the Wasserstein distance in the terminal costs. Investigating this metric allows us to derive the optimal condition. The minimum principle generally provides foundations for developing numerical methods of optimal control or receding-horizon control strategies. We expect that the minimum principle presented in this study also provides foundations for developing numerical methods for the optimal control of distributions. Moreover, given the connection between the control problem and deep generative modeling, the optimal control of probability distributions can suggest insights into the learnability of generative models.

The rest of this paper is constructed as follows. The following section provides mathematical preliminaries, such as the definition of the Wasserstein distance. Then, section III states the problem of the optimal control using the Wasserstein distance. The following section provides the optimality condition for the optimal control problem and the proof. Finally, we conclude the paper with discussions.

II. Mathematical Preliminaries: Wasserstein Distance

A. Notations

This section provides notations used in this paper and mathematical preliminaries on the Wasserstein distance.

Throughout this paper, $\mathbb{N}$ denotes the set of natural numbers, and we denote by $\mathbb{R}$ and $\mathbb{R}^n$ the set of real numbers and the n-dimensional Euclidean space, respectively. The notation $\|\cdot\|$ denotes the Euclidean norm. The σ-algebra of $\mathbb{R}^n$ is denoted by $\mathcal{B}(\mathbb{R}^n)$, which comprises a measurable space $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Given a sample space Ω, a σ-algebra $\mathcal{F}$ of Ω, the probability measure on the measurable space $(\Omega, \mathcal{F})$, the triplet $(\Omega, \mathcal{F}, \mathbb{P})$ denotes the probability space. Given the space Ω, we denote the probability measure on the space by $\mathcal{P}(\Omega)$. In particular, $\mathcal{P}(\mathbb{R}^n)$ denotes the set of probability measures on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. For a real-valued or vector-valued random variable X defined on the probability space, the notation $\mathbb{E}[X]$ denotes the mathematical expectation of X. When considering stochastic processes, we denote the time variable as $t \in \mathbb{R}_+ := [0, \infty)$, and given a filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$, the quadruplets $\left(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\in\mathbb{R}_+}\right)$ denotes the filtered probability space. Additionally, for a differentiable function $f(x, y)$ with respect to x and y, $\partial_x f(x, y)$ denotes the partial derivative of f with respect to x. When f is a vector-valued function, $\partial_x f(x, y)$ denotes the Jacobian of f.

B. Wasserstein Distance

As stated in the next section, this study explores the optimal control problem for a continuous-time stochastic control system, intending to steer a probability distribution of the terminal state of a control system to a target distribution. To this end, we formulate the problem as an optimal control problem to minimize the proximity between the controlled and desired distributions by introducing a metric of probability distributions. This study employs the Wasserstein distance in the problem, which builds on the optimal transport theory [10], [12]. The following definition considers the metric on the space of probability measures.

Definition 1 (2-Wasserstein distance): For the probability measures $\mu, \nu \in \mathcal{P}(\mathbb{R}^n)$, the following functional W_2 of μ and ν is the Wasserstein distance between μ and ν:

$$W_2(\mu, \nu) = \inf_{\pi\in\Pi(\mu,\nu)} \left[\int_{\mathbb{R}^n\times\mathbb{R}^n} d(x, y)^2 d\pi(x, y)\right]^{1/2}, \quad (1)$$

where $d(x, y) = \|x - y\|$ and $\Pi(\mu, \nu)$ is the set of the coupling of μ and ν defined as the set of probability measures on $\mathbb{R}^n \times \mathbb{R}^n$ whose marginals are μ and ν, respectively. In other words, for $\pi \in \Pi(\mu, \nu)$,

$$\pi(A \times \mathbb{R}^n) = \mu(A), \quad \pi(\mathbb{R}^n \times A) = \nu(A) \quad (2)$$

holds for any $A \in \mathcal{B}(\mathbb{R}^n)$.

The metric can be intuitively interpreted as the cost of transporting the mass of μ into that of ν, where the coupling $\pi \in \Pi(\mu, \nu)$ determines the transport plan between μ and ν. The metric is defined as the minimum transportation costs between μ and ν over all possible couplings of μ and ν. The Wasserstein distance is known as a metric on the space of probability measures, meaning that it satisfies the axiom of metric. In other words, the metric endows the space of probability measures with a metric structure, referred to as the Wasserstein space.

The definition (1) of the Wasserstein distance can be viewed as an infinite-dimensional linear optimization problem, which leads to a dual problem. The following is used to derive the optimal condition in this study and is commonly referred to as the Kantorovich duality theorem [16, Proposition 5.3].

Theorem 1 ([16]): Given $\mu, \nu \in \mathcal{P}(\mathbb{R}^n)$, the Wasserstein distance (1) between μ and ν can be expressed as

$$W_2(\mu, \nu)^2 = \sup_{(\varphi,\psi)\in\Psi_{d^2}} \left\{\int_{\mathbb{R}^n} \varphi(x) d\mu(x) + \int_{\mathbb{R}^n} \psi(y) d\nu(y)\right\}, \quad (3)$$

where Ψ_{d^2} denotes the set of pairs of bounded continuous functions φ and ψ such that $\varphi(x) + \psi(y) \leq d(x, y)^2$ for $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$. Moreover, let π^* be an element of $\Pi(\mu, \nu)$ that attains the infimum of (1). Then, there exists φ and ψ such that

$$\varphi(x) + \psi(y) = d(x, y)^2 \quad (4)$$

for $x, y \in \mathrm{spt}(\pi^*)$ almost everywhere with respect to π^*, where $\mathrm{spt}(\pi^*)$ denotes the support of π^*

III. Problem Statement: Finite Horizon Optimal Control Problem of Probability Distributions

This section presents the problem statement of this study, which focuses on a control problem of continuous-time stochastic systems. The problem is formulated as the optimal control with the costs given by the Wasserstein distance, which is made for the steering problem of a probability distribution of the terminal state.

The problem of this study concerns a continuous-time stochastic control system given by the Itô SDE:

$$dx_t = f(x_t, u_t)dt + \sigma(x_t)dW_t, \quad x_0 = x \sim \mu_0, \quad (5)$$

where $x_t \in \mathbb{R}^n$ is the state, $u_t \in U \subseteq \mathbb{R}^m$ is a control input, $f : \mathbb{R}^n \times U$, $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times d}$, and W_t is a d-dimensional standard Wiener process. To formulate the problem of this study, we assume that the initial value x_0 is a random variable and the distribution is given by $\mu_0 \in \mathcal{P}(\mathbb{R}^n)$. In the following, we denote the probability distribution of the state x_t at time t by μ_t. Additionally, to simplify the notation, we denote $\{x_t\}_{t\in\mathbb{R}_+}$ as $\boldsymbol{x} = \{x_t\}_{t\in\mathbb{R}_+}$ in the following, when no confusion may arise. Similar notations are used for the control $\boldsymbol{u} = \{u_t\}_{t\in\mathbb{R}_+}$ and other processes. Moreover, we assume that $\boldsymbol{u}$ is progressively measurable.

We consider the finite-horizon optimal control problem for the system (5). The problem aims to steer the terminal distribution of the system state to a desired probability distribution $\mu_d \in \mathcal{P}(\mathbb{R}^n)$. The cost of the optimal control problem takes the form:

$$J(\boldsymbol{u}) = \mathbb{E}\left[\int_0^T L(x_s, u_s)ds\right] + W_2(\mu_T, \mu_d)^2, \quad (6)$$

where $L : \mathbb{R}^n \times U \to \mathbb{R}_+$ and W_2 is the Wasserstein distance. Since the Wasserstein distance is non-negative, $J(\boldsymbol{u})$ is bounded from below. That is, the objective is to minimize the squared Wasserstein distance at the terminal time as well as the running cost by L. Consequently, the following problem is posed:

Problem 1: Given the initial distribution μ_0 of the initial state and the desired distribution μ_d of the state at the terminal time T, consider

$$\inf_{\boldsymbol{u}\in\mathcal{A}} J(\boldsymbol{u}), \quad (7)$$

where $\mathcal{A}$ denotes the set of admissible controls.

This form of the problem was first introduced in [4] for continuous-time stochastic linear systems. Additionally, the previous study by the author [27] discussed the problem of nonlinear discrete-time deterministic systems. This study focuses on the problem of nonlinear continuous-time stochastic systems of (5). The study [27] discussed the control problem for nonlinear systems. However, the dynamics of systems are assumed to be deterministic and discrete time. We extend the results in [27] for continuous-time nonlinear stochastic systems. Additionally, the study [27] considers only the terminal cost of the Wasserstein distance, while this study considers the running and terminal costs.

There are two primary motivations for considering Problem 1 with the Wasserstein distance. First, solving Problem 1 enables the approximation of the probability distribution μ_d, with samples of x_T serving as approximations of samples from μ_d. As previously stated, this problem is fundamental in deep learning. Second, Problem 1 aims to contribute to the controllability analysis with respect to probability distributions. In other words, the problem is that we consider if there is a control $\boldsymbol{u}$ of the system (5) that steers the distribution of the state sufficiently close to a given distribution μ_d at time $t = T$. In Problem 1, other functionals, such as the Kullback-Leibler (KL) divergence, may be employed to measure the proximity between distributions instead of the Wasserstein distance. However, the divergence does not provide a metric on the space of probability distributions. The controllability of a Fokker-Planck equation, a partial differential equation describing the evolution of a probability distribution of stochastic systems, for a class of system is addressed in [30]. However, further investigation is necessary. The Wasserstein distance provides a metric on the space of probability distributions. This indicates that measuring the proximity of distributions by the Wasserstein distance would be suitable for the controllability analysis because the metric is expected to play roles similar to the Euclidean metric for finite-dimensional systems. The particular case of $L \equiv 0$ in (6) yields the problem of obtaining the best approximation of the desired distribution by controlling the terminal distribution in terms of the metric, which can serve as a form of controllability analysis.

IV. Main Results: Optimality Conditions

A. Statement of Main Results

This section presents an optimality condition for Problem 1 in the form of Pontryagin's maximum principle for continuous-time stochastic control systems.

To discuss the optimality condition, we make the following assumptions.

Assumption 1: We assume that for control system (5) and cost (6), the elements of functions $f : \mathbb{R}^n \times U \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times d}$, $L : \mathbb{R}^n \times U \to \mathbb{R}_+$ are continuously differentiable in (x, u), x, (x, u), respectively. Additionally, their derivatives $\partial_x f(x, u)$, $\partial_u f(x, u)$, $\partial_x \sigma(x)$, $\partial_x L(x, u)$, and $\partial_u L(x, u)$ are continuous and bounded for $x \in \mathbb{R}^n$ and U. Furthermore, the set of admissible values of control U is convex, and the set of admissible control $\mathcal{A}$ is given by the set of $\boldsymbol{u} = \{u_t\}$ satisfying $u_t \in U$ for $t \in [0, T]$ almost surely. The support of μ_d is $\mathbb{R}^n$.

The main result of this study provides the optimality condition for the continuous-time stochastic control system with the control cost (6). This result extends the optimality condition in [27] for deterministic discrete-time nonlinear systems to the continuous-time nonlinear stochastic system (5). In the following theorem, function $H : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times U \to \mathbb{R}$

is given by

$$H(x,y,z,u) = y^T f(x,u) + L(x,u) + z^T \sigma(x), \quad (8)$$

which is commonly referred to as the Hamiltonian.

Theorem 2: Assume that Assumption 1 holds. Additionally, assume that there exists a control $\boldsymbol{u}^* = \{u_t^*\}_{0\le t\le T} \in \mathcal{A}$ that attains the minimum of control cost $J(\mathbf{u})$ of (6) in a probability space $(\Omega, \mathcal{B}(\Omega), \mathbb{P}, \mathcal{F}_t)$. Then, there exist $\mathcal{F}_t$-adapted processes $\boldsymbol{x}^* = \{x_t^*\}_{0\le t\le T}$, $\boldsymbol{y}^* = \{y_t^*\}_{0\le t\le T}$, $\boldsymbol{z}^* = \{z_t^*\}_{0\le t\le T}$, and a Lipschitz continuous function $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that the following forward-backward SDE and conditions hold:

$$dx_t^* = f(x_t^*, u_t^*)\, dt + \sigma(x_t^*)\, dW_t, \quad (9a)$$

$$dy_t^* = -\partial_x H(x_t^*, y_t^*, z_t^*, u_t^*)\, dt + z_t^* dW_t, \quad (9b)$$

$$\partial_u H(x_t^*, y_t^*, z_t^*, u_t^*)(v - u_t^*) \ge 0 \quad (9c)$$

for any $v \in U$ a.s. and

$$x_0^* \sim \mu_0, \quad y_T^* = \partial_x \varphi(x_T^*) \ a.s. \quad (10)$$

The condition in (9) demonstrates that this result is a variation of Pontryagin's maximum principle. The variable y_t^* is commonly referred to as the adjoint variable. The terminal condition in (10) for the adjoint variable is determined by the function φ and x_T^*. As shown in the following proof, the function φ is obtained from the Kantorovich duality theorem of the Wasserstein distance, indicating that φ is the Kantorovich potential of $W_2(\mu_T^*, \mu_d)$ where μ_T^* denotes the distribution of x_T^*. While this result can be seen as a special case of the optimality conditions for general mean-field control problems, for example, [31], this study delves into the specific case where the Wasserstein distance determines the terminal cost. Consequently, this study shows that the gradient of the Kantorovich potential gives the terminal condition of the BSDE.

B. Proof of Theorem 2

This subsection proves Theorem 2. The proof is obtained by integrating Pontryagin's maximum principle for stochastic control problems and the Kantorovich duality theorem. Although the idea is based on the proof of stochastic Pontryagin's principle, the proof requires handling the Wasserstein distance to obtain the terminal condition (10). This study handles this by utilizing the Kantorovich duality theorem.

We first introduce notations before providing the proof. The optimal control $\boldsymbol{u}^* = \{u_t^*\}_{0\le t\le T}$ determines the distribution μ_T^* of the terminal state x_T^* and the squared Wasserstein distance $W_2(\mu_T^*, \mu_d)^2$ in the cost functional (6). According to Theorem 1, there exist Kantorovich potentials φ^* and ψ^* such that the following holds:

$$W_2(\mu_T^*, \mu_d)^2 = \int_{\mathbb{R}^n} \varphi^* d\mu_T^* + \int_{\mathbb{R}^n} \psi^* d\mu_d. \quad (11)$$

Note that, as we assume the support of μ_d to be $\mathbb{R}^n$ in Assumption 1, we can establish the uniqueness of the Kantorovich potentials up to additive constants by applying [12, Proposition 7.18] with minor modifications and that the uniqueness is used in the following proofs. Additionally, we can conclude $\partial_x \varphi$ in Theorem 2 is unique. For an optimal control input $\boldsymbol{u}^*$ and for another control $\hat{\boldsymbol{u}} = \{\hat{u}_t\}_{0\le t\le T} \in \mathcal{A}$, we denote $\delta u_t = \hat{u}_t - u_t^*$ and $\delta \boldsymbol{u} = \{\delta u_t\}_{0\le t\le T}$.

To provide the proof, we utilize the following lemma [32, Lemma 4.7].

Lemma 1 ([32]): Given the control $\boldsymbol{u}^*$, consider the $\mathcal{F}_t$-adapted process $\boldsymbol{V} = \{V_t\}_{0\le t\le T}$ satisfying the SDE

$$dV_t = \{\partial_x f(x_t^*, u_t^*) V_t + \partial_u f(x_t^*, u_t^*) \delta u_t\}\, dt + \partial_x \sigma(x_t^*) V_t dW_t \quad (12)$$

where W_t is the same Wiener process with that in (5) and $V_0 = 0$ a.s. Then, SDE (12) possesses the unique solution, and $\mathbb{E}\left[\sup_{0\le t\le T} |V_t|^p\right] < \infty$ holds for any $p \ge 1$. Moreover, it follows that

$$\lim_{\epsilon\to 0} \mathbb{E} \sup_{0\le t\le T} |V_t^\epsilon|^2 = 0, \quad (13)$$

where V_t^ϵ is given by

$$V_t^\epsilon = \frac{x_t^\epsilon - x_t}{\epsilon} - V_t \text{ for } \epsilon > 0, \quad (14)$$

where x_t^ϵ is a solution of (5) with $\boldsymbol{u}^\epsilon = \{u_t^* + \epsilon \delta u_t\}_{0\le t\le T}$. This lemma shows that $\boldsymbol{V}$ yields a variation process of $\boldsymbol{x}^*$ in the mean square sense.

The next key lemma provides an expression of the Gâteaux derivative of the terminal cost of the Wasserstein distance at $\boldsymbol{u}^*$ with $\delta \boldsymbol{u}$.

Lemma 2: Consider the state processes $\boldsymbol{x}^*$ given by (9a) and $\boldsymbol{x}^\epsilon = \{x_t^\epsilon\}_{0\le t\le T}$ obtained by (5) with $\boldsymbol{u}^\epsilon = \{u_t^* + \epsilon \delta u_t\}_{0\le t\le T}$. Then, given the Kantorovich potential φ^* in (11), there exists a derivative $\partial_x \varphi^*$ almost everywhere on $\mathbb{R}^n$. Moreover,

$$\frac{d}{d\epsilon} W_2(\mu_T^\epsilon, \mu_d)^2 \mid_{\epsilon=0} = \mathbb{E}\left[\partial_x \varphi^*(x_T^*) V_T\right] \quad (15)$$

holds where μ_T^ϵ denotes the distribution of the terminal state of x_T^ϵ.

Proof: We show the almost everywhere existence of $\partial_x \varphi^*$. First, Theorem 1 ensures the existence of functions φ^* and ψ^* such that (11) holds. As seen in the proof of [12, Theorem 1.17], φ^* and ψ^* are Lipschitz functions. Additionally, Rademacher's theorem [33, Theorem 3.1] implies that φ^* and ψ^* possess the derivatives almost everywhere on $\mathbb{R}^n$. Accordingly, $\partial_x \varphi^*$ exists almost everywhere.

We next show the equation (15). First, we consider the Wasserstein distance between μ_T^ϵ and μ_d, where μ_T^ϵ is the distribution of x_T^ϵ with the solution $\boldsymbol{x}^\epsilon$ of (5) with the control $\boldsymbol{u}^\epsilon = \{u_t^* + \epsilon \delta u_t\}_{0\le t\le T}$. It follows from Theorem 1 that, likewise the Kantorovich potentials φ^* and ψ^* for the optimal state process x_t^*, we obtain the Kantorovich potentials φ^ϵ and ψ^ϵ for the Wasserstein distance between μ_T^ϵ and μ_d such that

$$W_2(\mu_T^\epsilon, \mu_d)^2 = \int_{\mathbb{R}^n} \varphi^\epsilon d\mu_T^\epsilon + \int_{\mathbb{R}^n} \psi^\epsilon d\mu_d. \quad (16)$$

In the following, we show that the derivative of (15) is obtained by showing

$$\liminf_{\epsilon\to 0} \frac{W_2(\mu_T^\epsilon,\mu_d)^2 - W_2(\mu_T^*,\mu_d)^2}{\epsilon} = \limsup_{\epsilon\to 0} \frac{W_2(\mu_T^\epsilon,\mu_d)^2 - W_2(\mu_T^*,\mu_d)^2}{\epsilon}. \tag{17}$$

We first evaluate the limit inferior in (17). Remembering that φ^* and ψ^* are the Kantorovich potentials for $W_2(\mu_T^*,\mu_d)^2$, which are obtained by taking the supremum over the functions of φ and ψ, it follows from Theorem 1 that

$$W_2(\mu_T^\epsilon,\mu_d)^2 \geq \int_{\mathbb{R}^n} \varphi^* d\mu_T^\epsilon + \int_{\mathbb{R}^n} \psi^* d\mu_d. \tag{18}$$

Recalling that μ_T^* and μ_T^ϵ are distributions of x_T^* and x_T^ϵ, respectively, and noting by x_d the random variable whose distribution is μ_d, we have

$$\begin{aligned} W_2(\mu_T^\epsilon,\mu_d)^2 - W_2(\mu_T^*,\mu_d)^2 &\geq \{\mathbb{E}\left[\varphi^*(x_T^\epsilon)\right] + \mathbb{E}\left[\psi^*(x_d)\right]\} \\ &\quad - \{\mathbb{E}\left[\varphi^*(x_T^*)\right] + \mathbb{E}\left[\psi^*(x_d)\right]\} \\ &= \mathbb{E}\left[\varphi^*(x_T^\epsilon) - \varphi^*(x_T^*)\right]. \end{aligned} \tag{19}$$

Noting that $x_T^\epsilon = x_T^* + \epsilon\,(V_T + V_T^\epsilon)$ with V_T and V_t^ϵ given by (12) and (14), respectively, we obtain that

$$\begin{aligned} &\mathbb{E}\left[\varphi^*(x_T^\epsilon) - \varphi^*(x_T^*)\right] \\ &= \mathbb{E}\left[\varphi^*(x_T^* + \epsilon(V_T + V_T^\epsilon)) - \varphi^*(x_T^*)\right] \\ &= \mathbb{E}\left[\int_0^1 \frac{d}{d\lambda}\varphi^*(x_T^* + \lambda\epsilon(V_T + V_T^\epsilon))d\lambda\right] \\ &= \epsilon\mathbb{E}\left[\int_0^1 \partial_x\varphi^*(x_T^* + \lambda\epsilon(V_T + V_T^\epsilon))(V_T + V_T^\epsilon)d\lambda\right]. \end{aligned} \tag{20}$$

The second equality is obtained by the fundamental theorem of calculus. The last equality is obtained because the Kantorovich potential φ^* is differentiable almost everywhere, as shown above and $\partial_x\varphi^*$ is used in the integral. According to equation (20),

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}\mathbb{E}\left[\varphi^*(x_T^\epsilon) - \varphi^*(x_T^*)\right] = \mathbb{E}\left[\partial_x\varphi^*(x_T^*)V_T\right], \tag{21}$$

where the right-hand side of (21) is obtained by taking the limit of $\epsilon \to 0$ in the last expression of (20) and using the definition of V_t^ϵ and its property (13). Therefore, (19) and (21) imply that

$$\liminf_{\epsilon\to 0} \frac{W_2(\mu_T^\epsilon,\mu_d)^2 - W_2(\mu_T^*,\mu_d)^2}{\epsilon} \geq \mathbb{E}\left[\partial_x\varphi^*(x_T^*)V_t\right], \tag{22}$$

which provides the evaluation of the limit inferior in (17).

We next evaluate the limit superior in (17). For the limit inferior, there exists a sequence of $\{\epsilon_n\}_{n\in\mathbb{N}}$ such that $\epsilon_n \to 0$ as $n \to \infty$ and the following holds:

$$\limsup_{\epsilon\to 0} \frac{W_2(\mu_T^\epsilon,\mu_d)^2 - W_2(\mu_T^*,\mu_d)^2}{\epsilon} = \lim_{n\to\infty} \frac{W_2(\mu_T^{\epsilon_n},\mu_d)^2 - W_2(\mu_T^*,\mu_d)^2}{\epsilon_n}. \tag{23}$$

Then, each ϵ_n determines the state process $\boldsymbol{x}^{\epsilon_n} = \{x_t^{\epsilon_n}\}_{0\leq t\leq T}$ given similarly to $\boldsymbol{x}_t^\epsilon$ and $\mu_T^{\epsilon_n}$ given as the distribution of the terminal state $x_T^{\epsilon_n}$. Like the above, there exist Kantorovich potentials φ^{ϵ_n} and ψ^{ϵ_n} such that

$$W_2(\mu_T^{\epsilon_n},\mu_d)^2 = \mathbb{E}\left[\varphi^{\epsilon_n}(x_T^{\epsilon_n})\right] + \mathbb{E}\left[\psi^{\epsilon_n}(x_d)\right]. \tag{24}$$

Given the Kantorovich potentials φ^{ϵ_n} and ψ^{ϵ_n} for $\mu_T^{\epsilon_n}$ and μ_d, it follows from the dual expression in Theorem 1 that for the optimal state process $\boldsymbol{x}^*$,

$$W_2(\mu_T^*,\mu_d)^2 \geq \mathbb{E}\left[\varphi^{\epsilon_n}(x_T^*)\right] + \mathbb{E}\left[\psi^{\epsilon_n}(x_d)\right] \tag{25}$$

holds. Therefore, (24) and (25) imply that

$$\begin{aligned} &W_2(\mu_T^{\epsilon_n},\mu_d)^2 - W_2(\mu_T^*,\mu_d)^2 \\ &\leq \mathbb{E}\left[\varphi^{\epsilon_n}(x_T^{\epsilon_n})\right] + \mathbb{E}\left[\psi^{\epsilon_n}(x_d)\right] \\ &\quad - \{\mathbb{E}\left[\varphi^{\epsilon_n}(x_T^*)\right] + \mathbb{E}\left[\psi^{\epsilon_n}(x_d)\right]\} \\ &= \mathbb{E}\left[\varphi^{\epsilon_n}(x_T^{\epsilon_n})\right] - \mathbb{E}\left[\varphi^{\epsilon_n}(x_d)\right]. \end{aligned} \tag{26}$$

A similar discussion to obtain (21) shows

$$\lim_{n\to 0} \frac{1}{\epsilon_n}\mathbb{E}\left[\varphi^{\epsilon_n}(x_T^{\epsilon_n}) - \varphi^{\epsilon_n}(x_T^*)\right] = \mathbb{E}\left[\partial_x\varphi^*(x_T^*)V_T\right], \tag{27}$$

which implies

$$\lim_{n\to\infty} \frac{W_2(\mu_T^{\epsilon_n},\mu_d)^2 - W_2(\mu_T^*,\mu_d)^2}{\epsilon} \leq \mathbb{E}\left[\partial_x\varphi^*(x_T^*)V_T\right]. \tag{28}$$

Finally, (22), (23), and (28) prove (17) and yield (15), which completes the proof. ■

Finally, we provide the proof of Theorem 2. Using Lemma 2, we can prove the theorem in a typical way of standard stochastic optimal control problems [29], [32], [16]. Therefore, we show only the outline of the proof.

Proof: First, we show that the Gâteaux derivative of the cost function becomes

$$\begin{aligned} &\lim_{\epsilon\to 0} \frac{J(\boldsymbol{u}^* + \epsilon\delta\boldsymbol{u}) - J(\boldsymbol{u}^*)}{\epsilon} \\ &= \mathbb{E}\left[\int_0^T (\partial_x L(x_s^*,u_s^*)V_s + \partial_u L(x_s^*,u_s^*)\delta u_s)\,ds\right] \\ &\quad + \mathbb{E}\left[\partial_x\varphi^*(x_T^*)V_T\right]. \end{aligned} \tag{29}$$

The first term on the right-hand side of (29) is obtained similarly to [32, Lemma 4.8], and the second term is obtained by Lemma 2. Then, given the optimal control $\boldsymbol{u}^*$ and the optimal process $\boldsymbol{x}^*$, consider the backward stochastic differential equation (BSDE) of (9b). Under Assumption 1, this BSDE possesses a unique solution thanks to [32, Theorem 2.2]. Observing that for the Hamiltonian of (8)

$$\partial_x H(x,y,z,u) = \partial_x f(x,u)y + \partial_x L(x,u) + \partial_x\sigma(x)z, \tag{30}$$

we express the BSDE (9b) as

$$dy_t^* = -\{\partial_x f(x_t^*,u_t^*)y_t^* + \partial_x L(x_t,u_t) + \partial_x\sigma(x_t^*)z_t^*\}\,dt + z_t^* dW_t \tag{31}$$

with $y_T^* = \partial_x\varphi^*(x_T^*)$ a.s. Then, using the duality relation of the variational process $\boldsymbol{V}$ of (12) and the adjoint process $\boldsymbol{y}_T^*$ as shown in [32, Lemma 4.10], y_t^* and V_t satisfies

$$\mathbb{E}\left[y_T^* V_T\right] = \mathbb{E}\left[\int_0^T (y_s^*\partial_u L(x_s^*,u_s^*)\delta u_s - \partial_x L(x_s^*,u_s^*)V_s)\,ds\right] \tag{32}$$

Noting that $\mathbb{E}\left[y_T^* V_T\right] = \mathbb{E}\left[\partial_x \varphi^*(x_T) V_T\right]$ due to the terminal condition of (9b), we obtain from (29) and (32) that

$$\begin{aligned} &\lim_{\epsilon\to 0} \frac{J(\boldsymbol{u}^* + \epsilon\delta\boldsymbol{u}) - J(\boldsymbol{u}^*)}{\epsilon} \\ &= \mathbb{E}\left[\int_0^T \left\{\partial_u L(x_s, u_s) + y_s^* \partial_u f(x_s^*, u_s^*)\right\} \delta u_s ds\right] \\ &= \mathbb{E}\left[\int_0^T \partial_u H(x_s, y_s, z_s, u_s) \delta u_s ds\right]. \end{aligned} \tag{33}$$

The last equality follows from the definition of the Hamiltonian H in (8). The control $\boldsymbol{u}^*$ is optimal, which implies $J(\boldsymbol{u}^* + \epsilon\delta\boldsymbol{u}) \geq J(\boldsymbol{u}^*)$. This necessitates that (9c) holds in (33). This completes the proof. ■

V. Conclusion

In this paper, we addressed a stochastic optimal control problem for continuous-time systems with the terminal cost of the Wasserstein distance. This extends the results of the previous study [27] for discrete-time nonlinear deterministic systems. We provide a necessary condition of the optimality in the control problem in the form of Pontryagin's minimum principle. This was achieved by carefully examining the differentiability of the Wasserstein distance.

Future work includes investigating the existence conditions of the optimal control and developing efficient numerical algorithms and its theoretical investigations. Additionally, we will study the optimal control problem discussed in this study from the perspective of the dynamic programming approach in mean-field control theory [34].

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